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PERIODIC SOLUTIONS OF DAMPED DUFFING-TYPE EQUATIONS WITH SINGULARITY

Shengjun LI [1,2], Fang-fang LIAO [3], Hailong ZHU [4]

[1] Central South University, School of Mathematics and Statistics, Changsha, Hunan, 410083, China
[2] Hainan University, College of Information Sciences and Technology, Haikou, 570228, China
[3] Nanjing College of Information Technology, Department of Quality Education, Nanjing, 210046, China
[4] Anhui University of Finance and Economics, School of Statistics and Applied Mathematics, Bengbu, 233030, China
Corresponding anthor: Shengjun Li, E-mail: shjli626@126.com

Abstract. We consider a second order equation of Duffing type. By applying Mawhin's continuation theorem and a relationship between the periodic and the Dirichlet boundary value problems for second order ordinary differential equations, we prove that the given equation has at least one positive periodic solution when the singular forces exhibits certain some strong force condition near the origin and with some semilinear growth near infinity. Recent results in the literature are generalized and complemented.

Key words: periodic solution, singular equations, Mawhin's continuation theorem.

1. INTRODUCTION

This paper is devoted to the existence of positive *T*-periodic solutions for the Duffing-type equation

$$x'' + Cx' + g(t,x) = 0, \tag{1.1}$$

where $C \in R$ is a constant, $g \in C(R \times (0,+\infty), R)$ is *T*-periodic with the first variable, and exhibits a repulsive singularity at $x = 0$. Following the notion in [6], we say that (1.1) has a repulsive singularity at $x = 0$ if $\lim_{x\to 0^+} g(t,x) = -\infty$ uniformly in $t \in R$, whereas (1.1) has an attractive singularity at $x = 0$ if $\lim_{x\to 0^+} g(t,x) = +\infty$ uniformly in $t \in R$.

As is well known that the Duffing equation named after the German electrical engineer Georg Duffing in 1918, has been widely used in physics, economics, engineering, and many other physical phenomena. An important question is whether this equation can support periodic solutions. Hence, it has been extensively investigated by numerous researchers in recent years. See for example [10, 13, 14, 19] and the references therein. According to the growth speed of *g*, (1.1) can be classified into the following three cases, (S_1) Superlinear case: $g(t,s)/s \to +\infty$, as $|s| \to +\infty$; (S_2) Superlinear case: $0 < k \le g(t,s)/s \le K < +\infty$, as $|s| \to +\infty$; (S_3) Superlinear case: $g(t,s)/s \to 0$, as $|s| \to +\infty$.
When $C = 0$, Eq. (1.1) becomes

$$x'' + g(t,x) = 0 \quad ; \quad x > 0 \tag{1.2}$$

we recall the following results. Let $g(t,x) = g(x) - e(t)$, where $g \in C(R_+, R)$ and $e \in C(R,R)$ is *T*-periodic satisfies the following strong force condition at $x = 0$,

$$\lim_{x\to 0^+} g(x) = -\infty \text{ and } \lim_{x\to 0^+} \int_1^x g(s)\mathrm{d}s = +\infty$$

with g is superlinear at $x \to +\infty$, $\lim_{x\to+\infty}\frac{g(x)}{x}=+\infty$, Fonda, Manásevich, and Zanolin [5] used the Poincaré - Birkhoff theorem to obtain the existence of positive periodic solutions, including all subharmonics. Similarity, when $g(t,x)$ is superlinear at $x \to +\infty$ and satisfies the following strong force condition at $x=0$: there are positive constants c, c', υ such that $\upsilon \ge 1$ and

$$cx^{-\upsilon} \le -g(t,x) \le c'x^{-\upsilon} \tag{1.3}$$

for all t and all x sufficiently small, del Pino and Manásevich proved in [2] the existence of infinitely many periodic solutions to (1.2) .

When $g(t,x)$ is semilinear at $x \to +\infty$, del Pino, Manásevich, and Montero [3] proved the existence of at least one positive T-periodic solution of (1.2) if $g(t,x)$ satisfies (1.3) near $x=0$, and the following nonresonance conditions at $x=+\infty$: there is an integer $k \ge 0$ and a small constant $\varepsilon > 0$ such that

$$\left(\frac{k\pi}{T}\right)^2 + \varepsilon \le \frac{g(t,x)}{x} \le \left(\frac{(k+1)\pi}{T}\right)^2 - \varepsilon \tag{1.4}$$

for all t and all $x \gg 1$.

The generalization of [3] was done for equation of the type of $x'' + g(x) = e(t)$ in [14]. Assume that $g(x)$ satisfies

$$\lim_{x\to 0^+} g(x) = -\infty \text{, and } \lim_{x\to 0^+} G(x) = +\infty, \quad (G(x) = \int_1^x g(s)\mathrm{d}s)$$

and

$$\left(\frac{k\pi}{T}\right)^2 + \varepsilon \le \frac{g(x)}{x} \le \left(\frac{(k+1)\pi}{T}\right)^2 - \varepsilon, \tag{1.5}$$

for all t and all $x \gg 1$. Wang [14] used the phase-plane analysis methods proved that Eq.(1.1) has at least one positive *T*-periodic solution.

We note that conditions (1.4) and (1.5) are the standard uniform nonresonance conditions with respect to the Dirichlet boundary condition, not with respect to the periodic boundary condition. For example, $x'' + \lambda x = x^{-3} + h(t)$, where $\lambda > 0$ and $h \in C(R,R)$ is 2π-periodic. Nonresonance holds when $\lambda \ne (k/2)^2$, $k = 1,2,\ldots$, i.e., λ is not an eigenvalue of the Dirichlet problem.

Some classical tools have been used in the literature to study singular equations. These classical tools include the degree theory [7, 15, 18], the method of upper and lower solutions [12], Schauder's fixed point theorem [1], some fixed point theorems in cones for completely continuous operators [8, 16, 17]. It seems the periodic boundary value problem for singular differential equations is closely related to the Dirichlet boundary value problem. A relationship between periodic and Dirichlet boundary value problems for second-order differential equations with singularities is establish in [21]. As mentioned above, this paper is mainly motivated by the recent papers [3, 14]. The result is obtained using Mawhin's continuation theorem and such a relationship between the periodic boundary value problem and the Dirichlet boundary value problem, thanks to a priori estimates on the solutions of a suitable family of problems. Compared with [3,14], the main novelty in the paper is represented by the conditions at infinity, which remind of a situation between the first and the second eigenvalue, but are more general.
Now we present our main result.

THEOREM 1.1. *Let the following assumptions hold.*

(H_1) *There exist a constant* $R_0 > 0$ *and a function* $g_0 \in C((0,\infty),R)$ *such that* $g(t,x) \le -g_0(x)$ *for all* t *and all* $0 < x \le R_0$, *where* g_0 *satisfies the strong force condition, i.e.*

$$\lim_{x\to 0^+} g_0(x) = +\infty \quad \text{and} \quad \lim_{x\to 0^+}\int_1^x g_0(s)\mathrm{d}s = -\infty .$$

(H_2) *There exist T-periodic continuous functions a, b such that*

$$a(t) \le \liminf_{x\to+\infty} \frac{g(t,x)}{x} \le \limsup_{x\to+\infty} \frac{g(t,x)}{x} \le b(t) \tag{1.6}$$

uniformly in $t \in [0,T]$. *Moreover,*

$$\overline{a} > 0 \text{ and } \underline{\lambda}_1(b) > 0 \tag{1.7}$$

here $\overline{a} = \frac{1}{T}\int_0^T a(t)\mathrm{d}t$ *and* $\{\underline{\lambda}_1(q)\}$ *denotes the first anti-periodic eigenvalues of*

$$x'' + (\lambda + q(t))x = 0 \tag{1.8}$$

subject to the anti-periodic boundary conditions.

Then Eq. (1.1) has at least one positive *T*-periodic solution.

Note that when $q \equiv 0$ and $\overline{\lambda}_0(0) = 0$ and $\underline{\lambda}_k(0) = \overline{\lambda}_k(0) = k\pi / T$ for all $k \in N$. These eigenvalues coincide with the constants in conditions (1.4) and (1.5). The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of Mawhin's continuation theorem, we will prove the main results.

2. PRELIMINARIES

Let us first introduce some known results on eigenvalues. Let $q(t)$ be a *T*-periodic potential such that $q \in L^1(R)$. Consider the eigenvalue problems of (1.8) subject to the *T*-periodic boundary condition

$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2.1}$$

or to the anti-T-periodic boundary condition

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{2.2}$$

Denote by $\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_n^D(q) < \cdots$ all eigenvalues of (1.8) with the Dirichlet boundary condition

$$x(0) = x(T) = 0. \tag{2.3}$$

The following are the standard results for eigenvalues. See, e.g Reference [9, 23]. A partial generalization of these results to the one-dimensional *p*-Laplacian with potentials is given in Reference [22].

LEMMA 2.1. *There exist two sequences* $\{\underline{\lambda}_n(q) : n \in N\}$ *and* $\{\overline{\lambda}_n(q) : n \in Z^+\}$ *such that*:

(E_1) [9, Theorem 2.1, see also 22, Theorem A]). λ *is an eigenvalue of* (1.8–2.1) (or (1.8–2.2)) *if and only if* $\lambda = \underline{\lambda}_n(q)$ *for some even* (*or odd*) *integer n*;

(E_2) [9, Theorem 2.1, see also 22, Theorem A]).

$-\infty < \overline{\lambda}_0(q) < \underline{\lambda}_1(q) \le \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \le \overline{\lambda}_2(q) < \cdots < \underline{\lambda}_n(q) \le \overline{\lambda}_n(q) < \cdots$ *and* $\underline{\lambda}_n(q) \to +\infty$, $\overline{\lambda}_n(q) \to +\infty$ *as* $n \to +\infty$;

(E_3) [22, Remark 4.3]. *The comparison results hold for all of these eigenvalues. If* $q_1 \le q_2$ *then* $\underline{\lambda}_n(q_1) \ge \underline{\lambda}_n(q_2)$, $\overline{\lambda}_n(q_1) \ge \overline{\lambda}_n(q_2)$, $\lambda_n^D(q_1) \ge \lambda_n^D(q_2)$ *for any* $n \ge 1$;

(E_4) [22, Theorem 4.3]. *For any* $n \ge 1$,

$\underline{\lambda}_n(q) = \min\{\lambda_n^D(q_{t_0}) : t_0 \in R\}$, $\overline{\lambda}_n\{q\} = \max\{\lambda_n^D(q_{t_0}) : t_0 \in R\}$, *where* $q_{t_0}(t)$ *denotes the translation of* $q(t)$: $q_{t_0}(t) \equiv q(t+t_0)$;

(E_5) [23, Theorem 5.2]. $\overline{\lambda}_n(q)$, $\underline{\lambda}_n(q)$, *and* $\lambda_n^D(q)$ *continuous in* q *in the* L^1*-topology of* $L^1(0,T)$;

(E_6) (Variational characterization). *The first eigenvalue* λ_1^D *can be found as*

$$\lambda_1^D(q) = \inf_{\substack{x \in H_0^1(0,T) \\ x \neq 0}} \frac{\int_0^T (x'^2(t) - q(t)x^2(t))\mathrm{d}t}{\int_0^T x^2(t)\mathrm{d}t}.$$

In particular, $\lambda_1^D(q) \le -\overline{q}$, where $\overline{q}$ is the mean value.

As we know, continuation theorems play an important role in studying the existence of periodic solutions of the second order differential equations. We now introduce theorem given by Mawhin [11].

Definition 2.1. Let $L: X \supset \operatorname{dom} L \to Z$ be a linear operator, and X, Z be real Banach space. L is said to be a Fredholm operator of index zero provided that

(i) $\operatorname{Im} L$ is a closed subset of Z,

(ii) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Set $X = \operatorname{Ker} L \oplus X_1$, $Z = Z_0 \oplus \operatorname{Im} L$. We will also need the projector $P: X \to \operatorname{Ker} L$ and $Q: Z \to Z_0$ are continuous such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Then it follows that $L|_{\operatorname{dom} L \setminus \operatorname{Ker} P}: \operatorname{dom} L \setminus \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_p and $K_{PQ} = K_P(I - Q)$, and suppose that Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq 0$.

Definition 2.2. Let $L: X \supset \operatorname{dom} L \to Z$ be a Fredholm operator of index zero, $N: \Omega \to Z$ be a continuous operator. N is said to be L-compact in Ω if the map $QN: \Omega \to Z$ and $K_{PQ}N: \overline{\Omega} \to X$ are compact.

LEMMA 2.2. [11, Theorem 2.4]. *Let* L *be a Fredholm operator of index zero and let* N *be* L*-compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied*:

(i) $Lx + \tau Nx \neq 0$, *for each* $(x, \tau) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin \operatorname{Im} L$ *for each* $x \in \operatorname{Ker} L \cap \partial\Omega$;

(iii) $D_0(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L) \neq 0$, *where* $Q: Z \to Z$ *is a continuous projector such that* $\operatorname{Ker} Q = \operatorname{Im} L$ *and* D_0 *is the Brouwer degree*.

Then the equation $Lx + Nx = 0$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

We refer the reader to [20] for more details about degree theory.

3. PROOF OF THEOREM 1.1

We will apply Lemma 2.2 to the singular problem (1.1). Let $X = \{x: R \to R$ is C^1 and satisfies $x(t+T) = x(t)\}$, endowed with the C^1-norm, cleary, X is a Banach space, and let $Z = L^1(0,T;R)$ with the L^1-norm. Let $\operatorname{dom} L = \{x \in X: x'$ is absolutely continuous on $R\}$ and $L: \operatorname{dom} L \to Z$ be the operator defined by $(Lx)(t) = x''(t)$, and $N: X \to Z$ by $(Nx)(t) = Cx'(t) + g(t, x(t))$. Then (1.1) is equivalent to the operator equation $Lx + Nx = 0$. Define projectors $P: X \to X$ and $Q: Z \to Z$ by

$$Px = \frac{1}{T}\int_0^T x(s)\mathrm{d}s \quad ; \quad Qz = \frac{1}{T}\int_0^T z(s)\mathrm{d}s. \tag{3.1}$$

It is easy to see that $\operatorname{Ker} L = R$, $\operatorname{Im} L = \{z \in Z: \int_0^T z(t)\mathrm{d}t = 0\}$, $\operatorname{Ker} Q = \operatorname{Im} L$, $\operatorname{Im} P = \operatorname{Ker} L$, and then L is a Fredholm linear mapping with zero index.

Let K denote the inverse of $L|_{\operatorname{Ker} P \cap \operatorname{dom} L}$. Then we have

$$\left[Kz\right](t)=\int_0^T G(t,s)z(s)\mathrm{d}s, \tag{3.2}$$

where $G(t,s)$ is the Green function .

Now we consider the following (homotopy) family of (1.1)

$$x''(t)+\tau(Cx'(t)+g(t,x(t)))=0 \quad , \quad \tau\in(0,1] \tag{3.3}$$

i.e. the operator equation $Lx+\tau Nx=0$.

LEMMA 3.1. *Under the assumptions as in Theorem 1.1, there exist* $B_1>B_0>0$ *such that for any positive solution* $x(t)$ to (3.3–2.2), *there exists some to* $t_0\in[0,T]$ *such that*

$$B_0<x(t_0)<B_1\,. \tag{3.4}$$

Proof. Let $x(t)$ be a positive solution of (3.3)–(2.2). By condition (H_1), there is $B_0>0$ such that $g(t,s)<0$ for all $0<s<B_0$. Integrating (3.3) from 0 to T, we get

$$\tau\int_0^T g(t,x(t))\mathrm{d}t=-\int_0^T x''(t)\mathrm{d}t-\tau\int_0^T Cx'\mathrm{d}t=0\,.$$

Thus $\int_0^T g(t,x(t))\mathrm{d}t=0$, there exist $t^*\in[0,T]$ such that $x(t^*)>B_0$.

Next, noticing (1.7), we take some constant $\varepsilon_0\in(0,\min\{\overline{a},\underline{\lambda}_1(b)\})$. By condition (H_2) there is $B_1(>B_0)$ large enough such that

$$a(t)-\varepsilon_0\le\frac{g(t,s)}{s}\le b(t)+\varepsilon_0 \tag{3.5}$$

for all t and $s\ge B_1$. We assert that $x(t_*)<B_1$ for some t_*. Once this is proved, combining with the fact $x(t^*)>B_0$, we know that (3.4) is necessarily satisfied for some t_0.

Now we prove the assertion by contradiction. Assume that $x(t)\ge B_1$ for all t. Define $p(t):=\dfrac{g(t,x(t))}{x(t)}$. By (3.5), $a(t)-\varepsilon_0\le p(t)\le b(t)+\varepsilon_0$, for all t. Moreover, $x(t)$ satisfies the following differential equation $x''+\tau(Cx'+p(t)x)=0$ .Write $x=\tilde{x}+\overline{x}$, where $\overline{x}=\dfrac{1}{T}\int_0^T x(t)\mathrm{d}t$, then $\tilde{x}'(t)=x'(t)$ and $\tilde{x}$ satisfies

$$-\tilde{x}''-\tau C\tilde{x}'=\tau(p(t)\tilde{x}+p(t)\overline{x}). \tag{3.6}$$

Integrating the equation (3.6) from 0 to T, we have

$$\int_0^T p(t)\tilde{x}(t)\mathrm{d}t=-\overline{x}\int_0^T p(t)\mathrm{d}t\,. \tag{3.7}$$

Multiplying (3.6) by $\tilde{x}$ and integrate , using the fact that $\int_0^T \tilde{x}'(t)\tilde{x}(t)\mathrm{d}t=0$, we get

$$\begin{aligned}\|\tilde{x}'\|_2^2&=\tau\int_0^T p(t)\tilde{x}^2(t)\mathrm{d}t+\tau\overline{x}\int_0^T p(t)\tilde{x}(t)\mathrm{d}t=\tau\int_0^T p(t)\tilde{x}^2(t)\mathrm{d}t-\tau\overline{x}^2(t)\int_0^T p(t)\mathrm{d}t\le\\&\le\tau\int_0^T p(t)\tilde{x}^2(t)\mathrm{d}t\quad\le\int_0^T p(t)\tilde{x}^2(t)\mathrm{d}t,\end{aligned} \tag{3.8}$$

where (3.7) is used and the last inequality follows from the fact that $\int_0^T p(t)\mathrm{d}t>T(\overline{a}-\varepsilon_0)>0$.

Note that $\tilde{x}(t_0)=0$ for some t_0, $\tilde{x}(t_0+T)=0$, so $\tilde{x}(t)\in H_0^1(t_0,t_0+T)$. We assert that $\tilde{x}\equiv 0$. On the contrary , assume that $\tilde{x}\neq 0$. Now by (3.8), the first Dirichlet eigenvalue

$$\lambda_1^D(p|_{[t_0,t_0+T]})=\inf_{\substack{\varphi\in H_0^1(t_0,t_0+T)\\ \varphi\neq 0}}\frac{\int_{t_0}^{t_0+T}(\varphi'^2(t)-p(t)\varphi^2(t))\mathrm{d}t}{\int_{t_0}^{t_0+T}\varphi^2(t)\mathrm{d}t}\leq 0.$$

So, we conclude that $\underline{\lambda}_1(p)=\min\{\lambda_1^D(p)\}\leq 0$. However, from $p(t)<b(t)+\varepsilon_0$, we have $\underline{\lambda}_1(p)\geq \underline{\lambda}_1(b+\varepsilon_0)=\underline{\lambda}_1(b)-\varepsilon_0>0$ by the choice of ε_0.

We have a contradiction, which shows that $\tilde{x}=0$, thus $x=\overline{x}$. Now it follows from (3.7) that $\overline{x}=0$ and $x\equiv 0$. This contradicts with the fact that x is a positive solution. We finished the proof of the assertion needed.

LEMMA 3.2. *Assume* $\underline{\lambda}_1(b)>0$ *of the equation* $y''+(\lambda+b(t))y=0$, *then*

$$\|y'\|_2^2\geq\int_0^T b(t+t_0)y^2(t)\mathrm{d}t+\lambda_1^D(b_{t_0})\int_0^T y^2(t)\mathrm{d}t.$$

Proof. From Lemma 2.1 in (E_4), we have $\lambda_1^D(b_{t_0})\geq\underline{\lambda}_1(b)>0$ for all $t_0\in R$. Then, by the theory of differential operators [4], the eigenvalues of $y''+(\lambda+b(t+t_0))y=0$ with Dirichlet boundary conditions $y(0)=y(T)=0$ form a sequence $\lambda_1^D(b_{t_0})<\lambda_2^D(b_{t_0})<\cdots$ which tends $+\infty$, and the corresponding eigenfunctions ψ_1, $\psi_2,\cdots$ are an orthonormal base of $L^2(0,T)$. Hence , given $c_i\in R$ and $y\in H_0^1(0,T)$, we can write $y(t)=\sum_{i\geq 1}c_i\psi_i(t)$, and

$$\int_0^T((y'(t))^2-b(t+t_0)y^2(t))\mathrm{d}t=\sum_{i\geq 1}c_i^2\int_0^T((\psi_i'(t))^2-b(t+t_0)\psi_i^2(t))\mathrm{d}t$$
$$=\sum_{i\geq 1}c_i^2\lambda_i^D(b_{t_0})\int_0^T\psi_i^2(t)\mathrm{d}t\geq\lambda_1^D(b_{t_0})\int_0^T y^2(t)\mathrm{d}t.$$

This completes the proof.

LEMMA 3.3. *There exist* $B_2>B_1>0$, $B_3>0$ *such that any positive T-periodic solution* $x(t)$ of (1.1–2.2) *satisfies*

$$\|x\|_\infty<B_2,\ \|x'\|_\infty<B_3.$$

Proof. From condition (H_2) and (3.5), we know that there is $h_0>0$ such that

$$g(t,s)\leq(b(t)+\varepsilon_0)s+h_0 \tag{3.9}$$

for all t and $s>0$.

Multiplying (3.3) by $x(t)$ and then integrating over [0, T], we get

$$\|x'\|_2^2=\tau\int_0^T-(xx''+Cxx')\mathrm{d}t=\tau\int_0^T g(t,x(t))x(t)\mathrm{d}t\leq$$
$$\leq\int_0^T((b(t)+\varepsilon_0)x(t)+h_0)x(t)\mathrm{d}t=\int_0^T b(t)x^2(t)\mathrm{d}t+\varepsilon_0\|x\|_2^2+h_0\|x\|_1. \tag{3.10}$$

Note from Lemma 3.1 that there exists t_0 satisfies $B_0<x(t_0)<B_1$. Let $u(t)=x(t+t_0)-x(t_0)$, then $u\in H_0^1(0,T)$. Thus

$$\int_0^T b(t)x^2(t)\mathrm{d}t=\int_0^T b(t+t_0)x^2(t+t_0)\mathrm{d}t=\int_0^T b(t+t_0)(x^2(t_0)+2x(t_0)u(t)+u^2(t))\mathrm{d}t\leq$$

$$\le B_1^2\|b\|_1+2B_1\|b\|_2\|u\|_2+\int_0^T b(t+t_0)u^2(t)\mathrm{d}t.$$

By the Hölder inequality, the other terms in (3.10) can be estimated as follows :

$$\varepsilon_0\|x\|_2^2\le\varepsilon_0\left(TB_1^2+2B_1T^{\frac{1}{2}}\|u\|_2+\|u\|_2^2\right),\quad h_0\|x\|_1\le h_0\left(TB_1+T^{\frac{1}{2}}\|u\|_2\right).$$

Thus (3.10) reads as

$$\|u'\|_2^2\le A_0+A_1\|u\|_2+\varepsilon_0\|u\|_2^2+\int_0^T b(t+t_0)u^2(t)\mathrm{d}t, \tag{3.11}$$

where $A_0=\varepsilon_0TB_1^2+h_0TB_1+B_1^2\|B\|_1$, $A_1=2\varepsilon_0B_1T^{\frac{1}{2}}+h_0T^{\frac{1}{2}}+2B_1\|b\|_2$ are positive constants.

Now, using $\underline{\lambda}_1(b(t))\|u\|_2^2\le\lambda_1^D(b_{t_0})\|u\|_2^2\le\int_0^T(u'^2(t)-b(t+t_0)u^2(t))\mathrm{d}t,$ we get from (3.11) that $(\underline{\lambda}_1(b(t))-\varepsilon_0)\|u\|_2^2\le A_1\|u\|_2+A_0$.

Consequently, $\|u\|_2<A_2$ for some $A_2>0$. By (3. 11) , one has $\|x'\|_2=\|u'\|_2<A_3$ for some $A_3>0$. From these, for any $t\in[t_0,t_0+T]$, $|x(t)|\le|x(t_0)|+\left|\int_{t_0}^t x'(t)\mathrm{d}t\right|\le B_1+T^{\frac{1}{2}}\|x'\|_2\le B_1+T^{\frac{1}{2}}A_3:=B_2$. Thus $\|x\|_\infty<B_2$ is obtained .

In order to prove $\|x'\|_\infty<B_3$, we write (1.1) as $-x''(t)=H(t):=Cx'(t)+g(t,x(t))$. As $\int_0^T H(t)\mathrm{d}t=0$, thus $\|H(t)\|_1=2\|H^+(t)\|_1$. From (3. 9) we have

$$H^+(t)=\max(H(t),0)\le C|x'(t)|+|b(t)+\varepsilon_0|x(t)+h_0\le C|x'(t)|+C_1,$$

where $\|x\|_\infty<B_2$ is used . As $x'(t_1)=0$ for some t_1, we have

$$\|x'\|_\infty=\max_{0\le t\le T}|x'(t)|=\max_{0\le t\le T}\left|\int_{t_1}^t x''(s)\mathrm{d}s\right|\le\int_0^T|H(s)|\mathrm{d}s=2\int_0^T|H^+(s)|\mathrm{d}s\le$$

$$\le 2\left(CT^{\frac{1}{2}}\|x'\|_2+TC_1\right)\le 2\left(CA_3T^{\frac{1}{2}}+TC_1\right):=B_3.$$

We have proved is that the $W^{2,1}$ norms of x are bounded .
Next , the positive lower estimates $m=\min_{t\in[0,T]}x(t)$ are obtained from (H_1) .

LEMMA 3.4. *There exists a constant* $B_4\in(0,B_0)$ *such that any positive solution* $x(t)$ *of* (1.1–2.2) *satisfies*

$$x(t)>B_4 \text{ for all } t.$$

Proof. From (H_1), we fix some $A_4\in(0,B_0)$ such that $g(t,s)<-CB_3$ for all t and all $0<s\le A_4$, where B_3 is same as above. Assume now that $m=\min_{t\in[0,T]}x(t)=x(t_2)<A_4$. By Lemma 3.1, $\max_t x(t)>B_0$. Let $t_3>t_2$ be the first time instant such that $x(t)=A_4$. Then for any $t\in[t_2,t_3]$, we have $x(t)\le A_4$ and $|-Cx'(t)|\le CB_3$. Hence, for $t\in[t_2,t_3]$, $x''(t)=-Cx'(t)-g(t,x(t))>CB_3-Cx'(t)\ge 0$. As $x'(t_2)=0,x'(t)>0$ for $t\in(t_2,t_3]$. Therefore, the function $x:[t_2,t_3]\to R$ has an inverse, denoted by ξ.
Now multiplying (1.1) by $x'(t)$ and integrating over $[t_2,t_3]$, we get

$$\int_m^{A_4}-g(\xi(x),x)\mathrm{d}x=\int_{t_2}^{t_3}-g(t,x(t))x'(t)\mathrm{d}t=\int_{t_2}^{t_3}(x''(t)x'(t)+Cx'(t)(Cx'(t))^2)\mathrm{d}t=$$

$$= C^2 \int_{t_2}^{t_3} (x'(t))^2 \mathrm{d}t + \frac{1}{2}(x'(t))^2 \le A_5$$

for some $A_5 > 0$. Using condition (H_1)

$$\int_m^{A_4} -g(\xi(x), x)\mathrm{d}x \ge \int_m^{A_4} g_0(x)\mathrm{d}x \to +\infty \tag{3.12}$$

if $m \to 0^+$. Thus we know from (3.12) that $m > B_4$ for some constant $B_4 > 0$.

Now we give the proof of Theorem 1.1. From (3.5), there exist constant $B_1 > 0$ such that $\frac{1}{T}\int_0^T g(t,x)\mathrm{d}t > 0$ for all $x > B_1$. Let the open bounded in X be

$$\Omega = \left\{ x \in X : E_1 < x(t) < E_2 \ \text{ and } \ |x'(t)| < B_3 \quad \text{for all } t \in [0,T] \right\},$$

where $0 < E_1 < \min\{B_0, B_4\}$, $E_2 > \max\{B_1, B_2\}$. From (3.1), (3.2), it follows that QN and $K(I-Q)N$ are continuous, and $QN(\bar{\Omega})$ is bounded and then $K(I-Q)N(\bar{\Omega})$ is compact for any open bounded $\Omega \subset X$ which means N is L-compact on $\bar{\Omega}$, condition (i) of Lemma 2.2 is satisfied. For any $x \in \mathrm{Ker}L \cap \partial\Omega$, we have $QNx = \frac{1}{T}\int_0^T g(t,x)\mathrm{d}t$ and we obtain $QN(E_1) = \frac{1}{T}\int_0^T g(t,E_1)\mathrm{d}t < 0$, $QN(E_2) = \frac{1}{T}\int_0^T g(t,E_2)\mathrm{d}t > 0$, which implies the condition (ii) of Lemma 2.2 is satisfied . Define

$$H(x,\mu) = \mu x + (1-\mu)QNx = \mu x + (1-\mu)\frac{1}{T}\int_0^T g(t,x)\mathrm{d}t.$$

$H(x,\mu)$ is a homotopic mapping for all $(x,\mu) \in (\partial\Omega \cap \mathrm{Ker}L) \times [0,1]$ and by using homotopic invariance theorem (we refer to [11, Section 2.3] for the notion of homotopic mapping and homotopic invariance theorem) , we have

$$D_0(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L) = D_0(x, \Omega \cap \mathrm{Ker}L) \ne 0.$$

Thus condition (iii) of Lemma 2.2 is also verified. Therefore $Lx + Nx = 0$ has at least one solution in $\bar{\Omega}$, which means (1.1) has at least one positive T-periodic solution.

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