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A FAST AND PRECISE NUMERICAL ALGORITHM FOR A CLASS OF VARIABLE–ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The variable-order fractional differential equations appear in modeling diverse physical problems. The main issue we address in this paper concerns an accurate numerical solution of a class of variable-order differential equations. The given problem is transformed into a system of algebraic equations using the so-called operational matrix of variable-order differentiation and the shifted Legende-Gauss-Radau collocation approach. Accordingly, the effort performed in calculations can be reduced. Numerical simulation for a specific problem is presented to demonstrate the computational efficiency and accuracy of the proposed algorithm.

Key words: variable-order fractional differential equations, Legendre polynomials, Caputo fractional derivatives, collocation method.

1. INTRODUCTION

The theory of fractional differential operators generalizes the notion of standard operators of integer orders to fractional orders. Such differential operators emerges naturally as a tool for the description of a broad range of non-classical phenomena in the applied sciences and engineering [1–11]. Recently, it has been demonstrated that in many dynamic processes, the underlying differential operators not only appear as constant fraction, but they also possess a dynamic nature in a sense that their order is variable, which may vary in time and/or space [12–16]. The pioneering work of variable-order operators can be traced to Samko *et al*. [17] by introducing the variable-order integration and Riemann-Liouville derivative. Since the kernel of the variable-order operators has a variable-exponent, analytical solutions to variable-order fractional differential equations (FDEs) are more difficult to obtain, and have not been the focus of much attention [18–20]. In general, finite-difference methods are today the most developed methods for the numerical approximation of variable-order FDEs, see [21, 22]. In the last decade, there are a special attention to propose and develop spectral methods for solving FDEs with both fixed-order and variable-order operators [23–27]. In the same line of thought, Bhrawy and Zaky [28] proposed an accurate spectral collocation method for solving one- and two-dimensional variable-order fractional nonlinear cable equations. Chen *et al*. [29] proposed a numerical method to estimate the variable-order fractional derivatives of an unknown signal in noisy environment. Tavares *et al*. [30] presented a numerical tool to solve variable-order fractional partial differential equations.

In this paper, we consider a general class of variable-order fractional differential equations:

$$ {}_{0}^{C}\mathrm{D}_{x}^{\zeta(x)}u(x) = f(x,u(x),u'(x)), \quad 0 \le x \le h, \qquad u(0) = \mu, \tag{1}$$

where ${}_{0}^{C}\mathrm{D}_{x}^{\zeta(x)}$ denotes the variable-order Caputo fractional derivative, $0 < \zeta(x) < 1$ and μ is a constant.

The aim of this study is to develop a numerical algorithm to improve the accuracy of the numerical solutions of the variable-order fractional initial value problems (1). The proposed algorithm converts the variable-order fractional differential equation (1) into a system of algebraic equations, which simplifies the solution process.

The paper is laid out as follows. In Sec. 2, we begin with some preliminary definitions of fractional calculus and properties of the shifted Legendre polynomials. In Sec. 3, the operational matrix for the variable-order fractional derivative of the shifted Legendre polynomials is derived. In Sec. 4, we develop a collocation scheme to solve the variable-order fractional initial value problems. In Sec. 5, the proposed method is applied to a specific example. Finally, the conclusions are given in Sec. 6.

2. DEFINITIONS AND PROPERTIES

There are several definitions proposed in the literature for variable-order fractional operators. We state some of them. In all results that follow we assume $u(x)=0$ for $x<0$. We recall some definitions of the fractional integrals of variable-order.

1. The left-sided variable-order fractional integral operator is defined as [17],

$$ {}_0I_x^{\zeta(x)}[u] := x \mapsto \frac{1}{\Gamma(\zeta(x))}\int_0^x (x-s)^{\zeta(x)-1}u(s)\mathrm{d}s,\ 0\le x\le h, \tag{2}$$

2. In [31], several definitions are proposed. The first is identical to (2). The next one is

$$ {}_0I_x^{\zeta(x)}[u] := x \mapsto \int_0^x \frac{(x-s)^{\zeta(s)-1}}{\Gamma(\zeta(s))}u(s)\mathrm{d}s,\ 0\le x\le h, \tag{3}$$

3. We state another definition introduced in [31], where it is assumed that ζ is a function of $(x-s)$, i.e.

$$ {}_0I_x^{\zeta(x)}[u] := x \mapsto \int_0^x \frac{(x-s)^{\zeta(x-s)-1}}{\Gamma(\zeta(x-s))}u(s)\mathrm{d}s\ \ 0\le x\le h, \tag{4}$$

where $\mathrm{Re}(\zeta(x))\ge 0$.

The Caputo variable-order derivative operator could now be defined simply (as in the case of constant order [32]) as follows,

$$ {}_0^CD_x^{\zeta(x)} := {}_0I_x^{n-q(x,s)}\circ\frac{\mathrm{d}^n}{\mathrm{d}x^n}, \tag{5}$$

where $q(x,s)=\zeta(x), q(x,s)=\zeta(s)$ and $q(x,s)=\zeta(x-s)$, in cases (2–4). Thus, we obtain, respectively:

1. Type **I:** The left Caputo fractional derivative of order $\zeta(x)$

$$ {}_0^C\mathrm{D}_x^{\zeta(x)}u(x) = \frac{1}{\Gamma(n-\zeta(x))}\int_0^x \frac{u^{(n)}(s)ds}{(x-s)^{\zeta(x)-n+1}},\ n-1<\zeta(x)<n; \tag{6}$$

2. Type **II:** The left Caputo fractional derivative of order $\zeta(x)$

$$ {}_0^C\mathrm{D}_x^{\zeta(x)}u(x) = \int_0^x \frac{1}{\Gamma(n-\zeta(s))}\frac{u^{(n)}(s)ds}{(x-s)^{\zeta(s)-n+1}},\ n-1<\zeta(x)<n; \tag{7}$$

3. Type **III:** The left Caputo fractional derivative of order $\zeta(x)$

$$ {}_0^C\mathrm{D}_x^{\zeta(x)}u(x) = \int_0^x \frac{1}{\Gamma(n-\zeta(x-s))}\frac{u^{(n)}(s)ds}{(x-s)^{\zeta(x-s)-n+1}},\quad n-1<\zeta(x)<n. \tag{8}$$

Such definitions have been used by numerous researchers, for example, Coimbra *et al*. [33, 34] used the first type in the modeling of viscous-viscoelastic oscillator; Ingman and Suzdalnitsky [35] employed the second type in the modeling of viscoelastic deformation process. Sun *et al*. [36] made a comparative investigation of integer-order derivative, constant-order fractional derivative and the first two types of variable-order fractional derivatives in characterizing the memory property of systems. However, the differences between the three types in applications are still not clear. There are other definitions of variable-order fractional derivatives. In this paper, we will focus our attention on the first type.

The operator ${}_0^C\mathrm{D}_x^{\zeta(x)}$ satisfies the following property $(1<\zeta(x)<2)$

$$ {}_0^C\mathrm{D}_x^{\zeta(x)} x^{\gamma} = \begin{cases} 0, & \gamma = 0,\ 1, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\zeta(x))}\, x^{\gamma-\zeta(x)}, & \gamma = 2,3,\ldots \end{cases} \tag{9} $$

Next, let us introduce some properties of the shifted Legendre polynomials [37, 38]. It is well-known that the classical Legendre polynomials are defined on –1, 1, by the three-term recurrence relation:

$$ \mathrm{L}_0(z) = 1, \qquad \mathrm{L}_1(x) = x, $$
$$ \mathrm{L}_{j+1}(z) = \frac{2j+1}{j+1} z\mathrm{L}_j(z) - \frac{j}{j+1}\mathrm{L}_{j-1}(z), \quad j \geq 2. $$

Let the shifted Legendre polynomials $\mathrm{L}_j(\frac{2x}{h}-1)$ be denoted by $\mathrm{L}_j^h(x)$. Then $\mathrm{L}_j^h(x)$ can be generated by using the following recurrence relation:

$$ \mathrm{L}_0^h(x) = 1, \qquad \mathrm{L}_1^h(x) = \frac{2x}{h} - 1, $$
$$ \mathrm{L}_{j+1}^h(x) = \frac{(2j+1)(2x-h)}{(j+1)h}\mathrm{L}_j^h(x) - \frac{j}{j+1}\mathrm{L}_{j-1}^h(x), \quad j \geq 2. $$

The orthogonality condition of the shifted Legendre polynomials is

$$ \int_0^h \mathrm{L}_i^h(x)\mathrm{L}_j^h(x)\mathrm{d}x = \rho_j, \tag{10} $$

where $\rho_j = \dfrac{\delta_{ij}h}{2j+1}$.

The explicit analytic form of $\mathrm{L}_j^h(x)$ of degree j is given by [32]

$$ \mathrm{L}_j^h(x) = \sum_{k=0}^{j} \varepsilon_{j,k}^h\, x^k, \tag{11} $$

where

$$ \varepsilon_{j,k}^h = \frac{(-1)^{j+k}(j+k)!}{(j-k)!\,(k!)^2 h^k}, \tag{12} $$

which alternatively may be written in the following matrix form

$$ \Theta_{h,M}(x) = E_h X_M(x), \tag{13} $$

where $\varepsilon_{j,k}^h$ for $j,k = 0,1,\ldots,M$ are the matrix entries of E_h, and

$$ \Theta_{h,M}(x) = [\mathrm{L}_0^h(x), \mathrm{L}_1^h(x)), \ldots, \mathrm{L}_M^h(x)]^T, \quad X_M(x) = [1, x, x^2, \ldots, x^M]^T. \tag{14} $$

Due to the orthogonality property of the shifted Legendre polynomials (10), the matrix E_h is invertible and the vector $X_M(x)$ can be expressed in terms of $\Theta_{h,M}(x)$ as

$$ X_M(x) = E_h^{-1}\Theta_{h,M}(x). \tag{15} $$

We assume $u(x)$ is a square integrable function in $[0,h]$, then it can be expressed in terms of shifted Legendre polynomials as

$$u(x)=\sum_{j=0}^{\infty}c_j \mathrm{L}_j^h(x), \tag{16}$$

from which the coefficients c_j are given by

$$c_j=\frac{1}{\rho_j}\int_0^h u(x)L_j^h \mathrm{d}x,\quad j=0,1,\ldots. \tag{17}$$

If we approximate $u(x)$ by the first $(M+1)$ -terms, then we can write

$$u_M(x)=\sum_{j=0}^{M}c_j \mathrm{L}_j^h(x)=C^T\Theta_{h,M}(x), \tag{18}$$

where the shifted Legendre coefficient vector C is given by $C^T=[c_0,c_1,\ldots,c_M]$.

3. DERIVATION OF THE DIFFERENTIATION MATRIX

The first-order derivative of the shifted Legendre vector $\Theta_{h,M}(x)$ can be expressed by

$$\frac{\mathrm{d}}{\mathrm{d}x}\Theta_{h,M}(x)=D_h^{(1)}\Theta_{h,M}(x), \tag{19}$$

where $D_h^{(1)}$ is the $(M+1)\times(M+1)$ operational matrix of derivative. Otherwise, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}\Theta h,M\left(x\right)=E_h\frac{\mathrm{d}}{\mathrm{d}x}X_M\left(x\right)=E_h\Delta_M X_M\left(x\right)(20)$$

where Δ_M is the $(M+1)\times(M+1)$ operational matrix of derivative of $X_M(x)$ and can be obtained from

$$\Delta_M=(\lambda_{ij})=\begin{cases} j+1, & \text{for } i=j+1,\ j=0,1,\ldots,M,\\ 0, & \text{otherwise.}\end{cases} \tag{21}$$

Now, using (20) and (15), then it is easy to write

$$\frac{\mathrm{d}}{\mathrm{d}x}\Theta_{h,M}(x)=E_h\Delta_M E_h^{-1}\Theta_{h,M}(x)=D_h^{(1)}\Theta_{h,M}(x). \tag{22}$$

Accordingly, we provide

$$D_h^{(1)}=E_h\Lambda_M E_h^{-1}. \tag{23}$$

Repeated use of (22), enables one to write

$$\frac{\mathrm{d}^q}{\mathrm{d}x^q}\Theta_{h,M}(x)=(D_h^{(1)})^q\Theta_{h,M}(x)=D_h^{(q)}\Theta_{h,M}(x),\ \ q=1,2,..., \tag{24}$$

where $q\in\mathrm{N}$ and the superscript in $D_h^{(q)}$, denotes matrix powers.

In the following theorem we generalize the operational matrix of derivative of shifted Legendre polynomials given in (24) for variable-order fractional derivatives.

Theorem 3.1 *The Caputo variable-order fractional derivative of the shifted Legendre vector* $\Theta_{h,M}(x)$ *is given by*

$${}_0^C\mathrm{D}_x^{\zeta(x)}\Theta_{h,M}(x)=D_{h,\zeta(x)}\Theta_{h,M}(x), \tag{25}$$

where $n-1<\zeta_{\min}<\zeta(x)<\zeta_{\max}<n$ *and* $D_{h,\zeta(x)}$ is an $(M+1)\times(M+1)$ *matrix of the following form*

$$D_{h,\zeta(x)}=E_h B E_h^{-1},$$

where E_h *is defined in* (13) *and* B *is a* $(M+1)\times(M+1)$ *matrix and its elements,* $b_{ij};0\le i,j\le M$ *are given as follows*

$$b_{ij}=\begin{cases}\dfrac{x^{-\zeta(x)}\Gamma(i+1)}{\Gamma(i+1-\zeta(x))}, & \text{for } i=j,\quad j=n,n+1,\ldots,M,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The analytic form of the shifted Legendre polynomials $\mathsf{L}_i^h(x)$ of degree i is given by (11). Using (14) and (15) yields

$$ {}_0^C\mathsf{D}_x^{\zeta(x)}\Theta_{h,M}(x)={}_0^C\mathsf{D}_x^{\zeta(x)}[\mathsf{L}_0^h(x),\mathsf{L}_1^h(x),\ldots,\mathsf{L}_M^h(x)]^T=E_h\,{}_0^C\mathsf{D}_x^{\zeta(x)}X_M(x). \tag{26}$$

This, the above relation together with (9), leads to

$$\begin{aligned}{}_0^C D_x^{\zeta(x)}X_M(x)=\ &[0,\ldots,\frac{\Gamma(n+1)}{\Gamma(n+1-\zeta(x))}x^{n-\zeta(x)},\frac{\Gamma(n+2)}{\Gamma(n+2-\zeta(x))}x^{n+1-\zeta(x)},\ldots,\\ &\frac{\Gamma(M+1)}{\Gamma(M+1-\zeta(x))}x^{M-\zeta(x)}]^T=BX_M(x),\end{aligned} \tag{27}$$

where B is a $(M+1)\times(M+1)$ matrix and has the form

$$B=\begin{pmatrix}0&0&0&\cdots&0\\ 0&\ddots&\vdots&\cdots&0\\ 0&0&\dfrac{x^{-\zeta(x)}\Gamma(n+1)}{\Gamma(n+1-\zeta(x))}&\cdots&0\\ \vdots&\vdots&0&\ddots&0\\ 0&0&0&\ddots&\dfrac{x^{-\zeta(x)}\Gamma(M+1)}{\Gamma(M+1-\zeta(x))}\end{pmatrix}. \tag{28}$$

Employing (26) and (27) yields

$$ {}_0^C\mathsf{D}_x^{\zeta(x)}\Theta_{h,M}(x)=E_h B X_M(x). \tag{29}$$

Since E_h is invertible, then

$$ {}_0^C\mathsf{D}_x^{\zeta(x)}\Theta_{h,M}(x)=E_hBE_h^{-1}E_hX_M(x)=E_hBE_h^{-1}\Theta_{h,M}(x)=D_{h,\zeta(x)}\Theta_{h,M}(x), \tag{30}$$

where $D_{h,\zeta(x)}=E_hBE_h^{-1}$ is an upper triangular matrix, and this proves the theorem.

4. NUMERICAL SCHEME

We now use the shifted Legendre polynomials [32, 37] as basis functions for the collocation scheme [38–40] together with the operational matrix of the variable-order fractional derivative in order to transform the problem (1) into a problem consisting of a system of algebraic equations.

Now, making use of (18), (19), and (25), enables one to write

$$ {}_0^C\mathsf{D}_x^{\zeta(x)}u(x)=C^TD_{h,\zeta(x)}\Theta_{h,M}(x),\quad u'(x)=C^TD_h^{(1)}\Theta_{h,M}(x),\qquad u(0)=C^T\Theta_{h,M}(0), \tag{31}$$

Employing Eq. (31) in Eq. (1), yields

$$C^T D_{h,\zeta(x)} \Theta_{h,M}(x) = f(x, C^T \Theta_{h,M}(x), C^T D_h^{(1)} \Theta_{h,M}(x)), \qquad C^T \Theta_{h,M}(0) = \mu. \tag{32}$$

Suppose x_i $(0 \le i \le M)$ are the shifted Legende-Gauss-Radau nodes of $\mathsf{L}_M^h(x) - \mathsf{L}_{M+1}^h(x)$. We substitute these nodes in (32); therefore the collocation scheme can be written as:

$$\begin{gathered} C^T D_{h,\zeta(x_i)} \Theta_{h,M}(x_i) = f(x_i, C^T \Theta_{h,M}(x_i), C^T D_h^{(1)} \Theta_{h,M}(x_i)), \\ C^T \Theta_{h,M}(0) = \mu, \qquad 1 \le i \le M. \end{gathered} \tag{33}$$

This constitutes a system of $(M+1)$ algebraic equations in the required Legendre coefficients $c_i, i = 0,1,\ldots,M$, which may be evaluated by employing Newton's iteration method. Consequently, the approximate solution $u_M(x)$ given in (18) can be achieved.

5. NUMERICAL EXAMPLE

In this Section we present a numerical example to illustrate the high accuracy and efficiency of our method proposed in this paper. First, let us define the convergence order (CO) by

$$CO = \frac{\log(error(M_1)/error(M_2))}{\log(M_2/M_1)},$$

where $error(M)$ denotes the error corresponding to polynomial degree M . To show the high efficiency of the algorithm, we examine the CPU time of our method. Here, all the computations are carried out by using Mathematica, version 8.0, on an Lenovo laptop with the configuration: Intel(R) Core(TM) i3-2328M CPU, 2.20 GHz and 4.00 GRAM, with 64 bits operation system. Consider the following variable-order fractional differential equation:

$${}_0^C\mathsf{D}_x^{\zeta(x)} u(x) + 3u'(x) - u(x) = f(x), \quad 0 \le x \le 1 \quad , \quad u(0) = 1, \tag{34}$$

where

$$\begin{aligned} f(x) &= e^x \left(3 - \frac{\Gamma(2-\zeta(x), x)}{\Gamma(1-\zeta(x))} \right), \\ \zeta(x) &= \frac{1+\cos^2(x)}{4}, \end{aligned}$$

and the exact solution is given by $u(x) = e^x$.

We investigate the convergence order and CPU time of our method. The errors for different degrees M of the polynomials and computational time are shown in Table 1. From the results, a high order accuracy is achieved. Moreover, the method is highly efficient.

Table 1

L^{∞} - errors of problem (34) versus *M*,
convergence order, and CPU time (in seconds)

M	L^{∞} - errors	CO	CPU time
4	7.181×10^{-5}		0.298
6	1.105×10^{-7}	15.973	0.313
8	9.833×10^{-11}	24.417	0.421
10	5.790×10^{-14}	33.330	0.448
12	5.668×10^{-16}	25.375	0.522

6. CONCLUSIONS

In this paper, we have proposed a fast and precise algorithm based on Legende-Gauss-Radau collocation technique combined with the associated operational matrices of variable-order fractional derivatives. This algorithm was employed for solving a class of variable-order fractional differential equations. The algorithm has the advantage of transforming the problem into the solution of a system of algebraic equations, which greatly simplifies it. Finally, a numerical example has been presented to demonstrate the efficiency of the proposed algorithm.

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