THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 18, Number 2/2017, pp. 116–123

ON A CLASS OF NEUMANN PROBLEMS ON KLEIN SURFACES

Monica ROŞIU

University of Craiova, Department of Mathematics, 200585 Craiova, Romania
E-mail: monica_rosiu@yahoo.com

Abstract. We obtain an integral representation for the solution of the Poisson equation with Neumann boundary condition on a compact Klein surface.

Key words: Klein surface, Neumann function.

1. INTRODUCTION

Any compact Klein surface can be completed by the doubling process to a symmetric closed Riemann surface. We obtain harmonic functions on a Klein surface by adding together a pair of harmonic functions of the symmetric Riemann surface whose singularities lie at conjugate points. We construct an analogue of a Neumann function on a Klein surface and express the solution of the Neumann problem for the Poisson equation on a Klein surface in terms of this basic functional. We present an application to the Möbius strip.

2. PRELIMINARIES

A compact Klein surface is a pair (X, A), consisting of a compact surface X and a maximal dianalytic atlas A on X. Given a compact Klein surface X, its canonical (Riemann) double cover X_C admits a fixed point free symmetry k, such that X is dianalytically equivalent with $X_C/\langle k\rangle$, where $\langle k\rangle$ is the group generated by k with respect to the usual composition of functions. Conversely, given a pair (X_C, k) consisting of a Riemann surface X_C and a symmetry k, the orbit space $X_C/\langle k\rangle$ admits a unique structure of Klein surface, such that $\pi : X_C \to X_C/\langle k\rangle$ is a morphism of Klein surfaces. The pair (X_C, k) is called a symmetric compact Riemann surface (see [4, 7]). Forwards, we identify X with the orbit space $X_C/\langle k\rangle$.

Let $\operatorname{Aut}(H)$ be the group of automorphisms of the upper half plane H and $\operatorname{Aut}^+(H)$ be the subgroup of orientation preserving elements in $\operatorname{Aut}(H)$. A discrete subgroup Γ of $\operatorname{Aut}(H)$ it is a NEC group if the quotient H/Γ is compact. A NEC group Γ is said to be a Fuchsian group if $\Gamma \subseteq \operatorname{Aut}^+(H)$. Let $\Gamma^+ = \Gamma \cap \operatorname{Aut}^+(H)$ be the canonical Fuchsian subgroup of Γ.

The next theorem associates a surface NEC group with a compact Klein surface X (see [4]).

THEOREM 2.1. *Let X be a compact Klein surface of algebraic genus $g \geq 2$. Then there exists a surface NEC group Γ such that X and H/Γ are isomorphic as Klein surfaces. Moreover, the double cover X_C is isomorphic with H/Γ^+.*

A common realization of a compact Klein surface X is a region in the complex plane bounded by a finite number of analytic Jordan curves (see [7]).

A set G is called symmetric if $k(G) = G$. A function f defined on a symmetric set is called a symmetric function if $f = f \circ k$.

A symmetric metric on X_C is defined by $d\sigma(z)=\frac{1}{2}\left(|dz|+|dw|\right)$, where $w=k(z)$, $z\in X_C$. Then two symmetric curves have the same length with respect to the metric $d\sigma$ (see [6]). The induced metric $d\Sigma$ on X is defined by

$$d\Sigma\left(\tilde{z}\right)=d\sigma(z)=d\sigma\left(k(z)\right),\ z\in X_C,\ \tilde{z}=\pi(z).$$

Let γ be a σ- rectifiable Jordan curve, parameterized in terms of the arc σ- length. Therefore, γ: $z=z(s)=x(s)+iy(s)$, $s\in[0,l]$, where l is the σ- length of γ. We consider the corresponding unit normal vector $n_\sigma=\left(\frac{dy}{d\sigma},-\frac{dx}{d\sigma}\right)$.

The Klein surfaces X and $X_C/\langle k\rangle$ are dianalytically equivalent, therefore we can identify $\{z,k(z)\}$ with $\tilde{z}=\pi(z)$. Thus Klein surfaces have a lot of applications in quantum physics, chemistry and biology which correspond to similar applications for symmetric Riemann surfaces (see [2, 3, 8]).

3. THE NEUMANN PROBLEM ON SYMMETRIC REGIONS

Let Ω be a region of X bounded by a finite number of σ-rectifiable Jordan curves. Given F a continuous real-valued function on Ω and G a continuous real-valued function on $\partial\Omega$, we consider the problem

$$\begin{cases}\Delta U=F \text{ on } \Omega\\ \dfrac{\partial U}{\partial n_\Sigma}=G \text{ on } \partial\Omega\end{cases}. \tag{1}$$

We define $D=\pi^{-1}(\Omega)$, $f=F\circ\pi$ on D and $g=G\circ\pi$ on ∂D. Since k is an antianalytic involution, without fixed points and $\pi\circ k=\pi$, we obtain that D is a symmetric region bounded by a finite number of σ- rectifiable Jordan curves on X_C, f is a symmetric continuous real-valued function on D and g is a symmetric continuous real-valued function on the boundary ∂D.

Because the Klein surface X is dianalytically equivalent with $X_C/\langle k\rangle$, the problem (1) is equivalent with the problem

$$\begin{cases}\Delta u=f \text{ on } D\\ \dfrac{\partial u}{\partial n_\sigma}=g \text{ on } \partial D\end{cases}, \tag{2}$$

where D is a symmetric region.

In this paper we only consider solutions which are in the class $C^2(D)\cap C^1(\partial D)$.

Given u and v two functions, parameterized by $x(s)$ and $y(s)$, where s is the arc σ- length, then the Green's second identity in terms of $d\sigma$ becomes

$$\iint_D (u\Delta v-v\Delta u)dxdy=\int_{\partial D}\left(u\frac{\partial v}{\partial n_\sigma}-v\frac{\partial u}{\partial n_\sigma}\right)d\sigma.$$

Remark 3.1. By the Green's formula for the integral of the Laplacian in terms of $d\sigma$, a necessary condition for the existence of a solution to the problem (2) is

$$\int_{\partial D} g\,d\sigma=\iint_D f\,dxdy.$$

PROPOSITION 3.2. *If the problem* (2) *admits a solution, then it is unique up to an additive constant.*

Proof. Given u_1 and u_2 solutions of the problem (2), if $u = u_1 - u_2$, then u is harmonic on D and $\dfrac{\partial u}{\partial n_\sigma} = 0$ on ∂D. Applying Green's first identity, we get $\iint\limits_D \left(u_x^2 + u_y^2\right) dxdy = 0$, thus u is constant on D.

PROPOSITION 3.3. *The solution of the problem* (2) *is a symmetric function on* D.

Proof. Let u be a solution of the problem (2). We define $\tilde{u} : \overline{D} \to \mathbb{R}$, by $\tilde{u} = \dfrac{1}{2}\left(u + u \circ k\right)$. By hypothesis, $f = f \circ k$ on D, thus $\Delta \tilde{u} = \dfrac{1}{2}\left(f + f \circ k\right) = f$ on D. Also, since $g = g \circ k$ on ∂D, then $\dfrac{\partial \tilde{u}}{\partial n_\sigma} = \dfrac{\partial u}{\partial n_\sigma} = g$ on ∂D. Thus $\tilde{u}$ is also a solution of the problem (2). By Proposition 3.2, there is a constant c such that $\tilde{u} = u + c$ on D. Thus $u \circ k = u + 2c$ on D and using the symmetry of the region D, we obtain $u = u \circ k + 2c$ on D. Hence $c = 0$, that is $u \circ k = u$ on D.

4. THE SYMMETRIC NEUMANN FUNCTION

The next theorem is an analogue of the Cauchy's integral formula for harmonic functions in terms of the metric $d\sigma$.

PROPOSITION 4.1. *Let* D *be a symmetric region bounded by a finite number of* σ*-rectifiable Jordan curves and let* u *be a harmonic function in* D *and continuous on its boundary* ∂D. *Then*

$$u(\zeta) = \frac{1}{2\pi} \int\limits_{\partial D} \left(\upsilon \frac{\partial u}{\partial n_\sigma} - u \frac{\partial \upsilon}{\partial n_\sigma} \right) d\sigma, \tag{3}$$

where ζ *is a fixed point inside* D, $z \in \overline{D}$ *and* $\upsilon(z;\zeta) = -\ln|z - \zeta|$.

Proof. Let C_r be a positively oriented circle of radius r, centered at ζ and let $D_r = D - \Delta_r$, where Δ_r is the closed disk bounded by C_r. Applying Green's second identity over D_r, for the harmonic functions u and $\upsilon = \upsilon(z;\zeta)$, we obtain

$$\int\limits_{\partial D} \left(u \frac{\partial \upsilon}{\partial n_\sigma} - \upsilon \frac{\partial u}{\partial n_\sigma} \right) d\sigma = \int\limits_{C_r} \left(u \frac{\partial \upsilon}{\partial n_\sigma} - \upsilon \frac{\partial u}{\partial n_\sigma} \right) d\sigma .$$

Then, by the compatibility property, $\int\limits_{C_r} \upsilon \dfrac{\partial u}{\partial n_\sigma} d\sigma = -\ln r \int\limits_{C_r} \dfrac{\partial u}{\partial n_\sigma} d\sigma = 0$ and, by the mean value property for harmonic functions, we get $\int\limits_{C_r} u \dfrac{\partial \upsilon}{\partial n_\sigma} d\sigma = -2\pi u(\zeta)$, see [5].

Let ζ be a point inside D. A Neumann function $N_D(z;\zeta)$ for the region D, with singularity at ζ, in terms of the metric $d\sigma$, is the function

$$N_D(z;\zeta) = \upsilon(z;\zeta) - h(z;\zeta), \ z \in D, \ z \neq \zeta,$$

where $h(z;\zeta)$ is a solution of the following Neumann problem in terms of the metric $d\sigma$

$$\begin{cases} \Delta h(z;\zeta)=0,\ z\in D \\ \dfrac{\partial h}{\partial n_\sigma}(z;\zeta)=\dfrac{\partial \upsilon}{\partial n_\sigma}(z;\zeta)+\dfrac{2\pi}{l},\ z\in\partial D \end{cases},$$

where $l=\int\limits_{\partial D} d\sigma$ is the σ- length of ∂D, (see [5]).

Remark 4.2. The boundary value of the normal derivative of the Neumann function is a constant equal to $-\dfrac{2\pi}{l}$, where l is the σ-length of ∂D.

PROPOSITION 4.3. *Let D be a symmetric region bounded by a finite number of σ- rectifiable Jordan curves. If u is harmonic in D, then, up to an additive constant,*

$$u(\zeta)=\frac{1}{2\pi}\int\limits_{\partial D}\frac{\partial u}{\partial n_\sigma}(z)N_D(z;\zeta)d\sigma,\ \zeta\in D. \tag{4}$$

Proof. Let C_r be a positively oriented circle of radius r, centered at ζ and let $D_r=D-\Delta_r$, where Δ_r is the closed disk bounded by C_r. Using Green's second identity, we get

$$\int\limits_{\partial D}\left(u(z)\frac{\partial h}{\partial n_\sigma}(z;\zeta)-h(z;\zeta)\frac{\partial u}{\partial n_\sigma}(z)\right)d\sigma=0, \tag{5}$$

where $\dfrac{\partial h}{\partial n_\sigma}=\dfrac{\partial \upsilon}{\partial n_\sigma}+\dfrac{2\pi}{l}$ on ∂D. Dividing (5) by 2π and adding it to formula (3) it results

$$u(\zeta)=\frac{1}{2\pi}\int\limits_{\partial D}\frac{\partial u}{\partial n_\sigma}(z)N_D(z;\zeta)d\sigma+\frac{1}{l}\int\limits_{\partial D}u(z)d\sigma.$$

Thus u is determined uniquely up to the additive constant $\dfrac{1}{l}\int\limits_{\partial D}u(z)d\sigma$.

Let ζ be a point inside D. Let $N_D^{(k)}(z;\tilde{\zeta})$ be the function defined by

$$N_D^{(k)}(z;\tilde{\zeta})=\frac{1}{2}\left[N_D(z;\zeta)+N_D(z;k(\zeta))\right],\ z\in D\setminus\{\zeta,k(\zeta)\},$$

where $N_D(z;k(\zeta))$ is a Neumann function for the region D, with singularity at $k(\zeta)$ and $\tilde{\zeta}=\{\zeta,k(\zeta)\}$.

From the definition of a Neumann function, it follows that

$$N_D^{(k)}(z;\tilde{\zeta})=\frac{1}{2}\left[\upsilon(z;\zeta)+\upsilon(z;k(\zeta))\right]-h_s(z;\tilde{\zeta}),\ z\neq\zeta,\ z\neq k(\zeta), \tag{6}$$

where h_s is a harmonic function on D and it satisfies

$$\frac{\partial h_s}{\partial n_\sigma}(z;\tilde{\zeta})=\frac{1}{2}\left[\frac{\partial \upsilon}{\partial n_\sigma}(z;\zeta)+\frac{\partial \upsilon}{\partial n_\sigma}(z;k(\zeta))\right]+\frac{2\pi}{l},\ \text{for } z\in\partial D.$$

Therefore, $N_D^{(k)}(z;\tilde{\zeta})$ is a harmonic function of z in $D\setminus\{\zeta,k(\zeta)\}$, with singularities at ζ and $k(\zeta)$ and $\dfrac{\partial N_D^{(k)}}{\partial n_\sigma}(z;\tilde{\zeta})=-\dfrac{2\pi}{l}$, for all z on the boundary ∂D.

PROPOSITION 4.4. *If D is a symmetric region, then the function $N_D^{(k)}(z;\tilde{\zeta})$ is symmetric with respect to z on D i.e. for every $z \in D$,*

$$N_D^{(k)}(z;\tilde{\zeta}) = N_D^{(k)}(k(z);\tilde{\zeta}).$$

Proof. Let $h^*(\cdot;\zeta)$ be a harmonic function in D, such that

$$\frac{\partial h^*}{\partial n_\sigma}(z;\zeta) = -\frac{1}{2}\left(\frac{\partial}{\partial n_\sigma}\ln|z-\zeta| + \frac{\partial}{\partial n_\sigma}\ln|k(z)-\zeta|\right) + \frac{2\pi}{l},\ z \in \partial D.$$

Therefore $\frac{\partial h^*}{\partial n_\sigma}(z;\zeta) = \frac{\partial h^*}{\partial n_\sigma}(k(z);\zeta)$, for every $z \in \partial D$. By Proposition 3.3, $h^*(\cdot;\zeta)$ is a symmetric function. Hence the function

$$M_D^{(k)}(z;\tilde{\zeta}) = \frac{1}{2}\left[\upsilon(z;\zeta) + \upsilon(k(z);\zeta)\right] - h^*(z;\zeta)$$

is a symmetric function, harmonic in $D \setminus \{\zeta, k(\zeta)\}$ and $\frac{\partial M_D^{(k)}}{\partial n_\sigma}(z;\tilde{\zeta}) = -\frac{2\pi}{l}$. So, $N_D^{(k)}(z;\tilde{\zeta})$ and $M_D^{(k)}(z;\tilde{\zeta})$ are solutions of the same Neumann problem, then by Proposition 3.2, there is a constant c such that $N_D^{(k)}(z;\tilde{\zeta}) = M_D^{(k)}(z;\tilde{\zeta}) + c$. Since $M_D^{(k)}(z;\tilde{\zeta})$ is a symmetric function, we obtain that $N_D^{(k)}(z;\tilde{\zeta})$ is also a symmetric function.

The function $N_D^{(k)}(z;\tilde{\zeta})$ is called a symmetric Neumann function for the region D, with singularity at $\tilde{\zeta}$, where $\tilde{\zeta} = \{\zeta, k(\zeta)\}$.

5. THE NEUMANN PROBLEM ON THE DOUBLE COVER

First, we express the solution of the Neumann problem for harmonic functions in terms of $d\sigma$ as a line integral involving the boundary function and a symmetric Neumann function.

THEOREM 5.1. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves and g be a symmetric, continuous function on ∂D. If u is harmonic in D and g is its normal derivative on ∂D, then up to an additive constant,*

$$u(\zeta) = \frac{1}{4\pi}\int_{\partial D} g(z)\left[N_D(z;\zeta) + N_D(z;k(\zeta))\right]d\sigma,\ \zeta \in D. \tag{7}$$

Proof. Since k is an involution of D, the function $\frac{u(\zeta)+u(k(\zeta))}{2}$ is a symmetric function on D. By Proposition 4.3, we have

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial D} g(z)N_D(z;\zeta)d\sigma + \frac{1}{l}\int_{\partial D} u(z)d\sigma$$

and

$$u(k(\zeta)) = \frac{1}{2\pi}\int_{\partial D} g(z)N_D(z;k(\zeta))d\sigma + \frac{1}{l}\int_{\partial D} u(z)d\sigma.$$

The symmetry of g implies

$$\frac{u(\zeta)+u(k(\zeta))}{2}=\frac{1}{2\pi}\int_{\partial D} g(z)\frac{N_D(z;\zeta)+N_D(z;k(\zeta))}{2}d\sigma+\frac{1}{l}\int_{\partial D} u(z)d\sigma .$$

By Proposition 3.3, u is a symmetric function on D, then the left side of the last equality is $u(\zeta)$ and we obtain $u(\zeta)=\frac{1}{4\pi}\int_{\partial D} g(z)\left[N_D(z;\zeta)+N_D(z;k(\zeta))\right]d\sigma+\frac{1}{l}\int_{\partial D} u(z)d\sigma$.

Next we find the solution of the Poisson equation with zero boundary values of the normal derivative in terms of $d\sigma$.

THEOREM 5.2. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let f be a symmetric, continuous function on D. There is a unique symmetric function $u\in C^2(D)\cap C^1(\partial D)$, with zero boundary value of the normal derivative, such that $\Delta u = f$ on D. Moreover, for all $\zeta\in D$ we have,*

$$u(\zeta)=-\frac{1}{4\pi}\iint_D f(z)\left[N_D(z;\zeta)+N_D(z;k(\zeta))\right]dxdy\ ,\ z=x+\mathrm{i}y . \tag{8}$$

Proof. By hypothesis, $u(\zeta)=-\frac{1}{2\pi}\iint_D \Delta u(z)N_D(z,\zeta)dxdy$, $\zeta\in D$ (see [5]). The rest of the proof follows by arguments similar to those in the proof of Theorem 5.1.

We conclude with the formula for the solution of the problem (2) on a symmetric region.

THEOREM 5.3. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves, let f be a symmetric, continuous function on D and let g be a symmetric, continuous function on ∂D. If u is a solution of the problem (2), then up to an additive constant*

$$u(\zeta)=-\frac{1}{2\pi}\iint_D f(z)N_D^{(k)}(z;\tilde{\zeta})dxdy+\frac{1}{2\pi}\int_{\partial D} g(z)N_D^{(k)}(z;\tilde{\zeta})d\sigma\ ,\ \zeta\in D . \tag{9}$$

Proof. By definition, $N_D^{(k)}(z;\tilde{\zeta})$ is a symmetric Neumann function for the region D, with the singularities at ζ and $k(\zeta)$. We combine the solution (7) of the Neumann problem for harmonic functions, with the solution (8) of the Poisson equation with zero boundary data.

6. THE NEUMANN PROBLEM ON THE ORBIT SPACE

Let Ω be a region bounded by a finite number of σ- rectifiable Jordan curves. The Klein surface X is the factor manifold of the symmetric Riemann surface X_C with respect to the group $\langle k\rangle$. Then, Ω is obtained from a symmetric region D by identifying the symmetric points.

Let $\tilde{\zeta}$ be a point inside Ω. A Neumann function $N_\Omega(\tilde{z};\tilde{\zeta})$ for the region Ω, with singularity at $\tilde{\zeta}$ is defined by

$$N_\Omega(\tilde{z};\tilde{\zeta})=N_D^{(k)}(z;\tilde{\zeta})=N_D^{(k)}(k(z);\tilde{\zeta}),\ \tilde{z}=\pi(z)\in\Omega . \tag{10}$$

Remark 6.1. By Proposition 4.4, it results that $N_\Omega(\tilde{z};\tilde{\zeta})$ is well defined on X .

Thus $N_\Omega\left(\tilde{z};\tilde{\zeta}\right)$ is a harmonic function on $\Omega\setminus\left\{\tilde{\zeta}\right\}$, which has a constant normal derivative $\dfrac{\partial N_\Omega}{\partial n_\Sigma}$ on the boundary $\partial\Omega$ and has a logarithmic pole at the point $\tilde{\zeta}$.

Next we derive the solution of the problem (1) on the region Ω.

THEOREM 6.2. *Let F be the continuous real-valued function on Ω, defined by the relation $f = F\circ\pi$ and let G be the continuous real-valued function on $\partial\Omega$, defined by the relation $g = G\circ\pi$. Then, up to an additive constant, the solution of the problem (1) is the function U defined by the relation $u = U\circ\pi$, where π is the canonical projection of X_C on X and u is the solution (9) of the problem (2) on the symmetric region D.*

Proof. The symmetry of the function f on D, yields

$$\Delta U\left(\tilde{\zeta}\right)=\Delta u(\zeta)=f(\zeta)=f\left(k(\zeta)\right)=F\left(\tilde{\zeta}\right),$$

for all $\tilde{\zeta}\in\Omega$, where $\tilde{\zeta}=\pi(\zeta)$. Also, the symmetry of the function g on ∂D, yields

$$\frac{\partial U}{\partial n_\Sigma}\left(\tilde{\zeta}\right)=\frac{\partial u}{\partial n_\sigma}(\zeta)=g(\zeta)=g\left(k(\zeta)\right)=G\left(\tilde{\zeta}\right),$$

for all $\tilde{\zeta}\in\partial\Omega$. Then, up to an additive constant, the function U defined on Ω by

$$U\left(\tilde{\zeta}\right)=u(\zeta)=u\left(k(\zeta)\right),$$

for all $\tilde{\zeta}$ in Ω, is the solution of the problem (1).

7. NEUMANN FUNCTION FOR THE MÖBIUS STRIP

For the Möbius strip $\overline{M}_R$, the orientable double cover is the annulus

$$\overline{A}_R=\left\{z\in\boldsymbol{C}\,\middle|\,\frac{1}{R}\le|z|\le R\right\},$$

where the points z and $k(z)=-\dfrac{1}{\bar{z}}$ are identified (see [7]). The corresponding symmetric metric $d\sigma=\dfrac{1}{2}\left[1+\dfrac{1}{|z|^2}\right]|dz|$ defines a structure of Riemann surface on $\overline{A}_R$, with respect to which the mapping k is an antianalytic involution, without fixed points. The orbit space $\overline{A}_R/\langle k\rangle$ carries a unique dianalytic structure on $\overline{M}_R$ which makes the canonical projection $\pi:\overline{A}_R\to\overline{M}_R$ dianalytic.

By Theorem 6.2, to solve the Neumann problem on the Möbius strip we need to determine a symmetric Neumann function for A_R.

THEOREM 7.1. *A symmetric Neumann function for A_R is*

$$N_{A_R}^{(k)}\left(z;\tilde{\zeta}\right)=C+\frac{1}{2n}\sum_{n=1}^{\infty}\frac{\rho^n+(-\rho)^{-n}}{R^{n+1}}\cdot\frac{r^n+(-r)^{-n}}{R^{n-1}+(-R)^{-n-1}}\cos n(\theta-\alpha)- \tag{11}$$

$$-\frac{1}{2}\ln\left|\rho e^{i\theta}-re^{i\alpha}\right|-\frac{1}{2}\ln\left|\frac{1}{\rho}e^{i(\theta+\pi)}-re^{i\alpha}\right|,$$

where $\tilde{\zeta}=\left\{\zeta,-\frac{1}{\bar{\zeta}}\right\}$, $\zeta=re^{i\alpha}$, $\frac{1}{R}<r<R$, $z=\rho e^{i\theta}$, $\frac{1}{R}<\rho<R$ *and* C *is an arbitrary constant.*

Proof. A symmetric Neumann function $N_{A_R}^{(k)}\left(z;\tilde{\zeta}\right)$ for A_R with singularities at ζ and $k(\zeta)$ is given by (6), where $D=A_R$. Since

$$\int_{\partial A_R}\frac{\partial h_s}{\partial n_\sigma}\left(z;\tilde{\zeta}\right)d\sigma=\int_{\partial A_R}\frac{\partial \upsilon}{\partial n_\sigma}\left(z;\zeta\right)d\sigma+\frac{2\pi}{l}\int_{\partial A_R}d\sigma=0,$$

the compatibility condition is satisfied.

By Proposition 3.3, it follows that h_s is a symmetric function on A_R. Since the function h_s is also harmonic on A_R, for $z=\rho e^{i\theta}\in A_R$, we have

$$h_s\left(\rho e^{i\theta};\tilde{\zeta}\right)=\alpha_0+\sum_{n=1}^{\infty}\left[\rho^n+(-\rho)^{-n}\right]\left(\alpha_n\cos n\theta+\beta_n\sin n\theta\right), \tag{12}$$

where the coefficients

$$\alpha_n=-\frac{1}{2n}\cdot\frac{r^n+(-r)^n}{R^{n+1}\left[R^{n-1}+(-R)^{-n-1}\right]}\cos n\alpha,\ n\geq 1$$

and

$$\beta_n=-\frac{1}{2n}\cdot\frac{r^n+(-r)^n}{R^{n+1}\left[R^{n-1}+(-R)^{-n-1}\right]}\sin n\alpha,\ n\geq 1$$

are determined from the Fourier expansion of $\Phi(z)=\frac{\partial h_s}{\partial n_\sigma}\left(z;\tilde{\zeta}\right)$ on $|z|=R$. Then plugging (12) in (6) we achieve (11), where $C=-\alpha_0$.

REFERENCES

1. N. ALLING AND N.GREENLEAF, *The Foundation of the Theory of Klein Surfaces,* Lectures Notes in Math, **219** (Springer-Verlag, New-York), 1971.
2. A. ALONSO, A. FERNÁNDEZ AND J. PÉREZ, *Harmonic forms on non-orientable surfaces,* Topics in analysis and its applications, NATO Sci. Ser. II Math. Phys. Chem., **147** (Kluwer Acad. Publ., Dordrecht), pp.31–46, 2004.
3. Y. BOULIGAND, *Geometry and topology of cell membranes*, Geometry in condensed matter physics, Dir. Condensed Matter Phys., **9** (World Sci. Publ., River Edge), NJ, pp. 191–231, 1990.
4. E. BUJALANCE, J.J. ETAYO, J.M. GAMBOA AND G. GROMADZKI, *Automorphisms Groups of Compact Bordered Klein Surfaces, A Combinatorial Approach,* Lecture Notes in Math., **1439** (Springer-Verlag, New-York), 1990.
5. Y. PINCHOVER and J. RUBINBSTEIN, *An Introduction to Partial Differential Equations*, Cambridge University Press, 2005.
6. M. ROȘIU, *Geodesics of quadratic differentials on Klein surfaces*, Electron. J. Differential Equations, **118**, pp. 1–8, 2014.
7. M. SCHIFFER AND D. SPENCER, *Functionals on Finite Riemann Surfaces,* Princeton University Press, Princeton, 1954.
8. M.D. WALBA, C.T. HOMAN, M.R. RICHARDS AND R.C. HALTIWANGER, *Synthesis and cutting "in half" of a molecular Möbius strip-applications of low-dimensional topology in chemistry*, Ser. Knots Everything, **6** (Knots and applications, World Sci. Publ., River Edge), NJ, pp. 427–463, 1995.

Received, September 6, 2016