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SHALLOW WATER WAVES MODELED BY THE BOUSSINESQ EQUATION HAVING LOGARITHMIC NONLINEARITY

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Abstract. We study shallow water waves that are described by the Boussinesq equation having logarithmic nonlinearity. The traveling wave hypothesis is applied to obtain Gausson solutions. The method of undetermined coefficients also solves the dynamical model. Finally, the conservation laws are computed using the method of multipliers.

Key words: Gaussons, logarithmic nonlinearity, conservation laws.

1. INTRODUCTION

The unique dynamics of shallow water waves is being studied for several decades [1–19]. In fact, there are several models in this context. These are Korteweg-de Vries (KdV) equation, Kawahara equation, Benjamin-Bona-Mahoney (BBM) equation, Peregrine equation, Gardner's equation, regularized long wave (RLW) equation, Green-Naghdy equation and several others. This paper will address one such nonlinear evolution equation (NLEE) that also models shallow water waves. It is the Boussinesq equation (BE).

While most of these models are studied with quadratic, cubic or power law nonlinearities, this paper will address BE with a logarithmic nonlinear form. The study of NLEEs with such new form of nonlinearity has gained popularity during the past few years, see e.g. [13, 14, 17–19]. One advantage of logarithmic-law models is that the solitary waves do not carry radiation or ripples. This primary advantage has particularly made the study of shallow water waves and optical solitons popular with logarithmic-law nonlinearity.

The study of other models such as KdV equation, Kadomtsev-Petviashvili (KP) equation, RLW equation, BBM equation, all of which describe the dynamics of shallow water waves, have already gained sufficient attention during the past couple of years [4, 11, 13, 14, 17–19]. In the context of nonlinear optics, the widely used nonlinear Schrödinger's equation with logarithmic-law nonlinearity has also been extensively studied [5]. It must be noted that these NLEEs with logarithmic form of nonlinearity produce a special kind of solitary wave that are known as Gaussons. It is however unknown, thus far, if these Gaussons retain the basic properties of solitons, such as the elastic property of soliton-soliton interaction, application of inverse scattering transform to secure soliton solutions, existence of infinitely many conservation laws and others.

In this paper we will study the solitary waves of BE that carry logarithmic-law nonlinearity. The traveling wave hypothesis and the method of undetermined coefficients will be applied to retrieve exact Gausson solutions from this nonlinear model. Subsequently, the conserved densities are listed from the Lie symmetry analysis. Finally, the conserved quantities are also given that are obtained from the corresponding densities and Gausson exact solutions.

2. GOVERNING EQUATIONS

We study the shallow water wave dynamics by the aid of BE with logarithmic nonlinearity. The corresponding BE reads as follows [14]:

$$q_{tt} - k^2 q_{xx} + a\left(q \ln q\right)_{xx} + b_1 q_{xxxx} + b_2 q_{xxtt} = 0 . \tag{1}$$

This dynamical model was introduced by Wazwaz [14]. In Eq. (1), $q(x;\, t)$ represents the wave profile, where the independent variables x and t represent spatial and temporal coordinates, respectively. The first two terms in Eq. (1) represents the wave operator. The coefficient of a is the logarithmic nonlinear term. The coefficients of b_1 and b_2 are dispersion terms, where in particular, the coefficient of b_2 gives the spatiotemporal dispersion.

In the past, the BE was studied by several authors but mostly without the spatiotemporal dispersion effect. We will study Eq. (1) in the next couple of Sections by traveling wave hypothesis and the method of undetermined coefficients. This will lead to Gaussons with appropriate constraints, which will guarantee the existence of such solitary wave solutions.

3. TRAVELING WAVE HYPOTHESIS

This Section secures Gaussons or solitary wave solutions to Eq. (1) by the aid of traveling wave hypothesis. These traveling waves are referred to as waves of permanent form that travel undisturbed without any deformation for long distances. The starting hypothesis for such waves is

$$q(x,\ t) = g(x - vt) = g(s), \tag{2}$$

where

$$s = x - vt. \tag{3}$$

In Eq. (2), v represents the speed of the wave and the functional form of g will give the solitary wave solution. Substituting the hypothesis (2) into (1) and integrating twice yields

$$\left(v^2 - k^2\right)g + ag \ln g + \left(b_1 + b_2 v^2\right)g'' = 0, \tag{4}$$

where $g'' = d^2 s / ds^2$. The integration constant is taken to be zero, both times, since the search is for a localized solitary wave solution.

Next, multiplying both sides of Eq. (4) by $g' = dg/ds$ and integrating leads to

$$2\left(b_1 + b_2 v^2\right)\left(g'\right)^2 = g^2\left[\left(a - 2v^2 + 2k^2\right) - 2a \ln g\right]. \tag{5}$$

Upon separating variables, one arrives at

$$\frac{s}{\sqrt{2\left(b_1 + b_2 v^2\right)}} = \int \frac{dg}{g\sqrt{a - 2v^2 + 2k^2 - 2a \ln g}} . \tag{6}$$

Performing the integration in Eq. (6), gives the Gausson solution:

$$q(x,\ t) = g(x - vt) = Ae^{-B^2 (x - vt)^2} , \tag{7}$$

where the amplitude A and the inverse width B are

$$A = \exp\left[\frac{a - 2\left(v^2 - k^2\right)}{2a}\right] \tag{8}$$

and

$$B = \frac{1}{2}\sqrt{\frac{a}{b_1 + b_2 v^2}}, \tag{9}$$

respectively. The constraint conditions introduced from amplitude A and width B of the Gausons are

$$a \neq 0 \tag{10}$$

and

$$a\left(b_1 + b_2 v^2\right) > 0, \tag{11}$$

which must be valid at all times for these Gaussons to exist.

4. THE METHOD OF UNDETERMINED COEFFICIENTS

This is an inverse problem approach. In this method a Gausson of the form (7) is assumed for the solution to Eq. (1). Upon substituting (7) into (1) and simplifying leads to

$$\left(v^2 - k^2\right)\left(2\tau^2 - 1\right) + 2\left(b_1 + b_2 v^2\right)B^2\left(4\tau^4 - 12\tau^2 + 3\right) - a\left[\left(1 + \ln A\right) - \tau^2\left(2\ln A + 5\right) + 2\tau^4\right] = 0, \tag{12}$$

where

$$\tau = B(x - vt). \tag{13}$$

From Eq. (12), upon setting to zero the coefficients of linearly independent functions τ^{2m}, where m = 0, 1, 2, one recovers Eqs. (8) and (9), and the speed of Gaussons as

$$v = \sqrt{k^2 + \frac{3a}{2}}, \tag{14}$$

which gives another constraint for these Gaussons to exist. This constraint is

$$2k^2 + 3a > 0. \tag{15}$$

Thus, three necessary constraint conditions for Gaussons to exist are given by Eqs. (10), (11), and (15). It is interesting to note that condition (15) is retrievable from the second integration scheme, namely the application of the method of undetermined coefficients. However, the first two constraints are obtained from the traveling wave solution. Thus, a complete set of integrability criteria are retrieved with the application of two independent integration schemes.

5. CONSERVATION LAWS

No study of NLEEs is complete without addressing the issue of conservation laws. These conserved quantities give a list of properties that stay invariant with shallow water waves flow, unless perturbation terms introduce its adiabatic change, which is not the case here. Many NLEEs have infinitely many conserved quantities. However, the BE with logarithmic-law nonlinearity has four conserved quantities. They will be retrieved with Lie symmetry analysis through the multipliers approach. The details are explained and listed here.

The conserved 1-form

$$T^t Dx + T^x Dt \tag{16}$$

of the differential equation satisfies

$$Q(x,t,q,\ldots)\left[q_{tt} - k^2 q_{xx} + a\left(q \ln q\right)_{xx} + b_1 q_{xxxx} + b_2 q_{xxtt}\right] = D_t T^t + D_x T^x$$

for some differential function Q and it can be shown, therefore, that

$$\varepsilon\Big(Q(x,\ t,\ q,\ ...)\big[q_{tt}-k^2q_{xx}+a\left(q\ln q\right)_{xx}+b_1q_{xxxx}+b_2q_{xxtt}\big]\Big)=0\,,$$

where ε is the Euler operator. The calculations reveal that $Q=1,t,x,xt$. In each case T^t is the conserved density.

1. $Q=1$

$$\left(-\frac{1}{2}q_{xxt}b_2-q_t\right)Dx+\left(\frac{1}{2}q_{xtt}b_2+q_{xxx}b_1-q_xk^2+q_xa\ln q+q_xa\right)Dt$$

2. $Q=t$

$$\left(-\frac{1}{2}q_{xxt}tb_2+\frac{1}{6}q_{xx}b_2-q_tt+q\right)Dx+\left(\frac{1}{2}q_{xtt}tb_2+q_{xxx}tb_1-\frac{1}{3}q_{xt}b_2-q_xtk^2+q_xta\ln q+q_xta\right)Dt$$

3. $Q=x$

$$\left(-\frac{1}{2}q_{xxt}xb_2+\frac{1}{3}q_{xt}b_2-q_tx\right)Dx+\begin{pmatrix}\frac{1}{2}q_{xtt}xb_2+qk^2+q_{xxx}xb_1-qa\ln q-\frac{1}{6}q_{tt}b_2-q_{xx}b_1-\\ q_xxk^2+q_xxa\ln q+q_xxa\end{pmatrix}Dt$$

4. $Q=xt$

$$\left(-\frac{1}{2}q_{xxt}txb_2+\frac{1}{3}q_{xt}tb_2+\frac{1}{6}q_{xx}xb_2-q_txt-\frac{1}{3}q_xb_2+xq\right)Dx+$$
$$\begin{pmatrix}\frac{1}{2}q_{xtt}xtb_2+q_xxta\ln q+q_{xxx}xtb_1+q_xxta-\frac{1}{6}q_{tt}tb_2+tqk^2-\frac{1}{3}q_{xt}xb_2-tqa\ln q-q_{xx}tb_1+\\ \frac{1}{3}q_tb_2-q_xtxk^2\end{pmatrix}Dt$$

The corresponding conserved quantities are:

$$I_1=-\int_{-\infty}^{\infty}\left(q_t+q_{xxt}b_2\right)dx=0 \tag{17}$$

$$I_2=\int_{-\infty}^{\infty}\left(q-q_tt+\frac{1}{6}q_{xx}b_2-\frac{1}{2}q_{xxt}tb_2\right)dx=\frac{\sqrt{\pi}A}{2B} \tag{18}$$

$$I_3=-\int_{-\infty}^{\infty}\left(q_tx-\frac{1}{3}q_{xt}b_2+\frac{1}{2}q_{xxt}xb_2\right)dx=-\frac{\sqrt{\pi}vA}{2B} \tag{19}$$

and

$$I_4=\int_{-\infty}^{\infty}\left(xq-\frac{1}{3}q_xb_2-q_txt+\frac{1}{6}q_{xx}xb_2+\frac{1}{3}q_{xt}tb_2-\frac{1}{2}q_{xxt}txb_2\right)dx=\frac{\sqrt{\pi}tvA}{2B} \tag{20}$$

These integrals are evaluated using the Gaussons from Eq. (7). It needs to be noted that for I_4 to be a conserved quantity, one needs to have

$$\frac{dI_4}{dt}=0, \tag{21}$$

which gives $v = 0$. This shows that I_4 will be a conserved quantity for stationary Gaussons only. These four conserved quantities are related to mass, linear momentum, energy and speed of the Gausson, not necessarily in this order.

6. CONCLUSIONS

This paper addressed the Boussinesq equation with logarithmic-law nonlinearity. The traveling wave hypothesis led to Gausson soliton solutions. The method of undetermined coefficients also reveals the same solution. There is a list of constraint conditions on the coefficients of Boussinesq equation that must be obeyed for Gaussons to exist. These conditions emerge from the structure of soliton parameters that fall out from the derivation of these solutions. There are four conservation laws that are retrieved from this shallow water wave model. The multiplier approach yields these laws.

The results of this paper form a strong foundation for further research of this dynamical model. Later, additional integration schemes will be applied to address Boussinesq equation. Some of these are Kudryashov's method, extended trial equation method, tanh method, Lie symmetry analysis and several others. Additionally, perturbation terms will be included such as shoaling, higher order dispersion as well as nonlinear dispersion. Such extended models for Boussinesq equation with logarithmic nonlinearity will be addressed in future and the results will be published elsewhere.

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