THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 18, Number 2/2017, pp. 166–173

NEW EXACT AND EXPLICIT SOLUTIONS OF THE ZAKHAROV EQUATIONS AND GENERALIZED ZAKHAROV EQUATIONS BY THE QUOTIENT TRIGONOMETRIC FUNCTION EXPANSION METHOD

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Abstract: In this work, by using a simple transformation technique, we have show that the nonlinear wave equations: Zakharov equations and generalized Zakharov equations can be reduced to the same family of Duffing or double-well Duffing equations. Then by means of quotient trigonometric function expansion method, many kinds of exact and explicit solutions of this family of equations are obtained in a unified way. These solutions include periodic and non-periodic exact solutions.

Key words: quotient trigonometric function expansion method (QTFEM), nonlinear partial differential equations (NPDE), exact solutions, Zakharov equations (ZE), generalized Zakharov equations(GZE).

1. INTRODUCTION

NPDE are widely used as models to describe complex physical phenomena in different domains of science as fluid dynamics, solid state physics, heat transfer, vibrations and so on. An exact solution for nonlinear systems is frequently scarce at least at the present state of knowledge. New and innovative approaches capable to solve nonlinear dynamical system should be known. But all methods used in study of the NPDE have their own advantages and deficiencies.

The present paper is motivated by the desire to use the QTFEM for solving some coupled nonlinear systems of two equations in terms of two unknowns u (complex function) and v (real function). In order to decouple the system, we will consider that $u = X(\varphi)e^{i\Psi}$ and $v = Y(\square)$ and a suitable ansatz for function X. It should be emphasized that the basic step for solving the coupled equations lies in making a proper transformation, in order to obtain the implicit relation between these two unknown functions X and Y as functions of another unknown φ. In this way, the system will be decoupled and the equation in X can be solved by QTFEM.

Here, we consider two coupled NPDE. The first of these is ZE, which is used especially in discussions of the evolution of Langmuir turbulence when strong turbulence effects are considered. These equations take into account a simplified model involving fluid concepts. In one dimension, ZE may be written as [1].

$$\mathrm{i}u_t + \Delta u = \alpha uv \tag{1}$$

$$v_{tt} - \Delta \mathrm{v} = \Delta(|u|^2), \tag{2}$$

where α is a known constant. Eq. (1) describes the evolution of the envelope of the high-frequency electric field with the nonlinearity included through a term involving a density fluctuation and the Eq. (2) describes the plasma density measured from its equilibrium value. The Zakharov theory is connected to a more general theory for nonlinear plasma processes. Also kinetic energy is used in [2] to rederive the ZE. The system can be derived from a hydrodynamic description of the plasma. Plasma can be modeled by a superposition of several changed compressible gasses. Beyond of these cases we can look at particular regimes, long-wave regimes or as a high-frequency limit of Klein-Gordon waves systems [3]. The ZE may be modified in various ways to take into account of electromagnetic effects or the magnetization of the plasma. Colin and Colin [4] proposed a generalization of the Zakharov equations (GZE) in the form.

$$\mathrm{i}u_t + \Delta u - \beta |u|^2 = \alpha uv \tag{3}$$

$$v_{tt} - \Delta v = \Delta(|u|^2) \tag{4}$$

where β is a known constant.

In last years, seeking exact solutions of NPDE has been an important role in the study of nonlinear wave phenomena arising in mathematical physics. The wave phenomena are observed in elastic media, plasma physic, optical fibers, fluid dynamics, biology, hydrodynamics, etc. If these exact solutions are known then can help one to understand within wheels of the intricated physical phenomena modeled by the nonlinear evolution equations.

Recently scientists were concerned about finding exact solutions and some efficient procedures were developed for solving the ZE and GZE such as: a conservative difference scheme presented by Chang at al. [5] for the initial boundary value problem. Then Yomba [6] and Yan [7] used the general projective Riccati equations. Variational method is applied by Zhong [8] to find solitary wave solution of GZE. The first integral method is proposed by Achab [9] and by Sun *et al*. [10]. Extended F-expansion method for GZE is employed by Wang and Li [11]. The so-called generalized $(\frac{G'}{G})$ expansion method is modified to obtain new travelling wave solutions for GZE by Zedan [12] and by Khan *et al*. [13]. Other methods can be mentioned as bifurcation method [13], Jacobi elliptic function expansion [14] and so on [15–17].

2. THE QUOTIENT TRIGONOMETRIC FUNCTION EXPANSION METHOD

In the present work we propose an approach to search explicit and exact solutions of NPDE in the form of a quotient of a trigonometric function. Also we apply the simplest nonlinear differential equation that has lesser order than the studied equations. We apply our approach to obtain exact solutions of Duffing and double-well Duffing equations.

Let us consider the nonlinear Duffing equations as follows

$$\ddot{X} + AX - BX^3 = 0, \tag{5}$$

where X is the unknown variable, A and B are positive constants and dot means differentiation in respect to t.

In order to solve Eq. (5), the following transformation is needed:

$$\tau = \omega t, \tag{6}$$

where ω is an unknown parameter called the frequency of the system.

Substituting of Eq. (6) into Eq. (5) leads to

$$\omega^2 X'' + AX - BX^3 = 0, \tag{7}$$

where $' = \dfrac{d}{d\tau}$.

Based on the trigonometric function expansion method, Eq. (7) may have the following quotient trigonometric solution [18]:

$$X(\theta) = \frac{a_0 + \sum_{j=1}^{n} (a_j \sin\theta + a_{n+j}\cos\theta)\sin^{j-1}\theta}{b_0 + \sum_{j=1}^{n} (\sin\theta + b_{n+j}\cos\theta)\sin^{j-1}\theta}, \tag{8}$$

where a_i, b_i, $i = 0, 1, 2, \dots n$ are unknown constants at this moment, θ satisfies the following relation

$$\theta' = \sin\theta \tag{9}$$

and n is determined by partially balancing the highest degree nonlinear term and the derivative terms of higher order in Eq. (7). One obtains $n = 1$.

From Eq. (9) we obtain

$$\sin\theta = \pm\operatorname{sech}(\tau + \Psi_0), \tag{10}$$

where τ is given by Eq. (6) and Ψ_0 is constant.

Differentiating Eq. (10) with respect to τ and taking into account Eq. (9), one gets

$$\cos\theta = \pm\tanh(\tau + \Psi_0). \tag{11}$$

The quotient solution (8) takes the following form:

$$X(\theta) = \frac{a_0 + a_1\sin\theta + a_2\cos\theta}{b_0 + b_1\sin\theta + b_2\cos\theta}. \tag{12}$$

In order to get nontrivial solutions of Eq. (7), we will determine the coefficients a_i and b_i ($i = 0, 1, 2$). From Eq. (12), we obtain the following expression:

$$X''(\theta) = \frac{M(\sin\theta, \cos\theta)}{(b_0 + b_1\sin\theta + b_2\cos\theta)^2}. \tag{13}$$

with

$$\begin{aligned} M(\sin\theta, \cos\theta) &= (a_1b_0^2 + a_1b_2^2 - a_0b_0b_1 - a_2b_1b_2)\sin\theta + (a_0b_1^2 - a_1b_0b_1 + 2a_0b_2^2 - \\ &- 2a_2b_0b_2)\sin^2\theta + (a_1b_2{}^2 - a_2b_1b_2 - b_0^2 + 2a_0b_0b_1)\sin^3\theta + (2a_1b_0b_2 - a_2b_0b_1 - \\ &- a_0b_1b_2)\sin\theta\cos\theta + (a_2b_1^2 - a_1b_1b_2 + 2a_0b_0b_2 - 2a_2b_0^2)\sin^2\theta\cos\theta \end{aligned} \tag{14}$$

and

$$X^3(\theta) = \frac{N(\sin\theta, \cos\theta)}{(b_0 + b_1\sin\theta + b_2\cos\theta)^3}, \tag{15}$$

with

$$\begin{aligned} N(\sin\theta, \cos\theta) &= a_0^3 + 3a_0a_2^2 + 3(a_0^2a_1 + a_1a_2^2)\sin\theta + 3(a_0a_1^2 - a_0a_2^2)\sin^2\theta + (a_1^3 - 3a_1a_2^2)\sin^3\theta + \\ &+ (a_2^3 + 3a_0^2a_2)\cos\theta + (3a_1^2a_2 - a_2^3)\sin^2\theta\cos\theta. \end{aligned} \tag{16}$$

Now, substituting Eqs. (12), (13) and (15) into Eq.(7) we obtain a set of an algebraic equations about expansion coefficients a_i and b_i. Setting the coefficients of various $\sin^j\theta$ ($j = 0, 1, 2, 3$) and $\sin^j\theta\cos\theta$ ($j = 0, 1, 2$) as zero, one can obtain the following equations:

$$\begin{gathered} A(a_0b_0^2 + a_0b_2^2 + 2a_2b_0b_2) - B(a_0^3 + 3a_0a_2^2) = 0 \\ \omega^2(a_1b_0^3 + a_1b_2^2 - a_0b_0b_1 - a_2b_1b_2) + A(a_1b_0^2 + a_1b_2^2 + 2a_0b_0b_1 + 2a_2b_1b_2) - \\ - 3B(a_0^2a_1 + a_1a_2^2) = 0 \\ A(a_2b_0^2 + a_2b_2^2 + 2a_0b_0b_2) - B(a_2^3 + 3a_0^2a_2) = 0 \\ \omega^2(a_0b_1^2 - a_1b_0b_1 + 2a_0b_0b_0^2 - 2a_2b_0b_2) + A(a_0b_1^2 - a_0b_2^2 + 2a_1b_0b_1) - \\ - B(a_1^3 - 3a_1a_2^2) = 0 \\ \omega^2(2a_1b_0b_2 - a_2b_0b_1 - a_0b_1b_2) + 2A(a_0b_1b_2 + a_1b_0b_2 + a_2b_0b_1) = 0 \\ \omega^2(a_2b_1^2 - a_1b_1b_2 + 2a_0b_0b_0 - 2a_2b_0^2) + A(a_2b_1^2 - a_2b_2^2 + 2a_1b_1b_2) - \end{gathered} \tag{17}$$

$$-B(3a_1^3a_2 - a_2^3) = 0.$$

By means Maple 15 or Mathematica 8, we can get the following cases:

$$\begin{aligned}
&\text{Case 1:}\quad a_0 = a_2 = b_0 = b_2 = 0,\ \omega = \sqrt{2A}\\
&\text{Case 2:}\quad a_0 = -a_2,\ b_0 = -b_2 = \pm\sqrt{\frac{B}{A}}a_0,\ a_1 = b_1 = 0\\
&\text{Case 3:}\quad a_0 = a_2 = b_1 = b_2 = 0,\ a_1 = \pm\sqrt{\frac{2A}{B}},\ \omega = \mathrm{i}\sqrt{A}, \mathrm{i}^2 = -1\\
&\text{Case 4:}\quad a_0 = a_2,\ a_1 = b_1 = 0,\ b_0 = b_2 = \pm\sqrt{\frac{B}{A}}a_0,\ \omega = \sqrt{2A}\\
&\text{Case 5:}\quad a_2 = b_0 = 0,\ b_1 = \pm\sqrt{\frac{B}{A}(a_1^2 - a_0^2)},\ b_2 = \pm\sqrt{\frac{B}{A}}a_0,\ \omega = \sqrt{2A}, |a_1| > |a_0|\\
&\text{Case 6:}\quad a_1 = a_2 = b_0 = b_1 = 0,\ b_2 = \pm\sqrt{\frac{B}{A}}a_0,\ \omega = \sqrt{\frac{A}{2}}\\
&\text{Case 7:}\quad a_0 = b_0 = 0,\ b_1 = \pm\sqrt{\frac{B}{A}}a_1,\ b_2 = \pm\sqrt{\frac{B}{A}}a_2,\ \omega = \sqrt{2A}\\
&\text{Case 8:}\quad a_2 = b_2 = 0,\ b_0 = \pm\sqrt{\frac{B}{A}}a_0,\ b_1 = \pm\sqrt{\frac{B}{A}}a_1\\
&\text{Case 9:}\quad a_0 = b_2 = 0,\ b_1 = \pm\sqrt{\frac{B}{A}(a_1^2 + a_2^2)},\ b_0 = \pm\sqrt{\frac{B}{A}}a_2,\ \omega = \sqrt{2A}\\
&\text{Case 10:}\quad a_0 = a_1 = b_1 = b_2 = 0,\ b_0 = \pm\sqrt{\frac{B}{A}}a_2,\ \omega = \sqrt{\frac{A}{2}}\\
&\text{Case 11:}\quad a_1 = b_1 = 0,\ b_0 = \pm\sqrt{\frac{B}{A}}a_0,\ b_2 = \pm\sqrt{\frac{B}{A}}a_2\\
&\text{Case 12:}\quad a_2 = -a_0,\ a_1 = b_1 = 0,\ b_2 = -b_0,\ \omega = \mathrm{i}\sqrt{A}\\
&\text{Case 13:}\quad a_1 = b_1 = 0,\ b_2 = \pm\sqrt{\frac{B}{A}}a_0,\ b_0 = \pm\sqrt{\frac{B}{A}}a_2,\ \omega = \sqrt{\frac{A}{2}}\\
&\text{Case 14:}\quad a_1 = 0,\ b_0 = \pm\sqrt{\frac{B}{A}}a_2,\ b_1 = \pm\sqrt{\frac{B}{A}(a_2^2 - a_0^2)},\ b_2 = \pm\sqrt{\frac{B}{A}}a_0,\ \omega = \sqrt{2A},\ |a_2| > |a_0|.
\end{aligned} \tag{18}$$

In the cases 1, 2, 4, 7, 8, 11 and 12 the solutions (12) are constants. In the following we present the solutions:

In case 3:

$$X_{1,2} = \pm\frac{1}{b_0}\sqrt{\frac{2A}{B}}\frac{1}{\cos(\sqrt{A}t + \Psi_0)}. \tag{19}$$

In case 5:

$$X_{3,4} = \pm\sqrt{\frac{A}{B}}\frac{a_0\cosh(\sqrt{A}t + \Psi_0) \pm a_1}{\sqrt{a_1^2 - a_0^2} - a_0\sinh(\sqrt{2A}t + \Psi_0)}, |a_1| \geq |a_0| \tag{20}$$

where Ψ_0, a_0 and a_1 are real parameters.

In case 6:

$$X_{7,8} = \pm\sqrt{\frac{A}{B}}\coth\left(\sqrt{\frac{A}{2}}t + \Psi_0\right). \tag{21}$$

In case 9:

$$X_{9,12} = \pm\sqrt{\frac{A}{B}}\frac{\pm a_1 - a_2 \sinh(\sqrt{2A}t + \Psi_0)}{a_2 \cosh(\sqrt{2A}t + \Psi_0) \pm \sqrt{a_1^2 + a_2^2}}. \tag{22}$$

In case 10:

$$X_{13,14} = \pm\sqrt{\frac{A}{B}}\tanh\left(\sqrt{\frac{A}{2}}t + \Psi_0\right). \tag{23}$$

In case 13 we obtain the same solutions given by Eq. (21).

In the last case:

$$X_{15,18} = \pm\sqrt{\frac{A}{B}}\frac{a_0 \cosh(\sqrt{2A}t + \Psi_0) - a_2 \sinh(\sqrt{2A}t + \Psi_0)}{a_2 \cosh(\sqrt{2A}t + \Psi_0) + a_0 \sinh(\sqrt{2A}t + \Psi_0) \pm \sqrt{a_2^2 - a_0^2}}, |a_2| \geq |a_0| \tag{24}$$

where a_0, a_1, a_2 are real parameters.

Now, we study the nonlinear double-well Duffing equation.

$$\ddot{X} - AX + BX^3 = 0. \tag{25}$$

where A and B are positive constants. Under the transformation(6), Eq. (25) becomes

$$\omega^2 X'' - AX + BX^3 = 0 \tag{26}$$

where ω is the frequency of the system and $' = \dfrac{d}{d\tau}$.

For Eq.(26) we have the same quotient trigonometric solution(12). Substituting Eqs. (12), (13) and (15) into Eq. (26) we obtain a set of algebraic equations:

$$\begin{gathered}
-A(a_0 b_0^2 + a_0 b_2^2 + 2a_2 b_0 b_2) + B(a_0^3 + 3a_0 a_2^2) = 0 \\
\omega^2 (a_1 b_0^2 + a_1 b_2^2 - a_0 b_0 b_1 - a_2 b_1 b_2) - A(a_1 b_0^2 + a_1 b_2^2 + 2a_0 b_0 b_1 + 2a_2 b_1 b_2) + 3B(a_1 a_0^2 + a_1 a_2^2) = 0 \\
-A(a_2 b_0^2 + a_2 b_2^2 + 2a_0 b_0 b_2) + B(a_2^3 + 3a_0^2 a_2) = 0 \\
\omega^2 (a_0 b_1^2 - a_1 b_2 b_1 + 2a_0 b_2^2 - 2a_2 b_0 b_2) - A(a_0 b_1^2 - a_0 b_2^2 + 2a_1 b_0 b_1 + 2a_2 b_1 b_2) - \\
-2a_2 b_0 b_2) + 3B(a_0 a_1^2 - a_0 a_2^2) = 0 \\
\omega^2 (a_0 b_2^2 - a_2 b_1 b_1 - 2a_1 b_0^2 + 2a_0 b_0 b_1) - A(a_1 b_1^2 - a_1 b_2^2 - 2a_2 b_1 b_2) + B(a_1^3 - 3a_1 a_2^2) = 0 \\
\omega^2 (2a_1 b_0 b_2 - a_2 b_0 b_1 - a_0 b_1 b_2) - 2A(a_0 b_1 b_2 + a_1 b_0 b_2 + a_2 b_0 b_2) = 0 \\
\omega^2 (a_2 b_1^2 - a_1 b_1 b_2 + 2a_0 b_0 b_2 - 2a_2 b_0^2) - A(a_2 b_1^2 - a_2 b_2^2 + 2a_1 b_1 b_2) + B(3a_1^2 a_2 - a_2^3) = 0,
\end{gathered} \tag{27}$$

where the parameters A and B are known.

Solving the nonlinear algebraic equations (27), we obtain :

$$X_{1,2} = \pm\sqrt{\frac{2A}{B(b_0^2 - b_2^2)}}\frac{1}{b_0 \cosh(\sqrt{A}t + \Psi_0) - b_2 \sinh(\sqrt{A}t + \Psi_0)}, \ |b_0| > |b_2|. \tag{28}$$

3. EXACT SOLUTIONS OF NPDE BY MEANS OF A SUITABLE TRANSFORMATION AND OF THE QTPEM

Because into Eqs. (1–2) and (3–4) the function u(x, t) is a complex function and v(x, t) is a real function, we assume that these equations has travelling wave solutions in the forms:

$$u(x,t)=X(\varphi)e^{i\Psi},\ v(x,t)=Y(\varphi),\ \varphi=aX-\Omega_1 t,\ \Psi=bx-\Omega_2, \tag{29}$$

where both X(φ) and Y(φ) are real functions, and a, b, Ω_1 and Ω_2 are constants to be determined later. Substituting Eqs.(30) into Eqs.(1) and (2), we have the following ODE for X(φ)and Y(φ):

$$a^2X''-\alpha XY-b^2X+\Omega_2X+i(2ab-\Omega_1)X'=0 \tag{30}$$

$$(\Omega_1^2-a^2)Y''-(X^2)_{xx}=0 \tag{31}$$

where prime meaning differentiation with respect to φ.

From Eq.(31) we obtain $\Omega_1=2ab$ and integrating Eq.(32) twice with respect to φ and taking into account that $(X^2)_{xx}=a_2(X_2)''$ we get

$$(\Omega_1^2-a^2)Y-aX^2=C_1\varphi+C, \tag{32}$$

where C_1 and C are integration constants. Because it is clear that $\lim_{\varphi\to\infty}X(\varphi)=\text{constant}$ and $\lim_{\varphi\to\infty}Y(\varphi)=\text{constant}$, it holds that $C_1 = 0$ and from Eq.(33) yields

$$Y(\varphi)=\frac{C+a^2X^2(\varphi)}{a^2(4b^2-1)}. \tag{33}$$

Substituting Eq.(33) into Eq.(30), it can be shown that

$$X''+\left[\frac{\Omega_2}{a^2}-\frac{b^2}{a^2}-\frac{aC}{a^4(4b^2-1)}\right]X-\frac{\alpha}{a^2(4b^2-1)}X^3=0 \tag{34}$$

In the same way, for Eqs. (3) and (4), one can get

$$X''+\left[\frac{\Omega_2}{a^2}-\frac{b^2}{a^2}-\frac{aC}{a^4(4b^2-1)}\right]X-\left[\frac{\alpha}{a^2(4b^2-1)}+\frac{\beta}{a^2}\right]X^3=0 \tag{35}$$

and the function Y(φ) given by Eq.(33).

We remark that Eqs. (34) and (35) are of the Duffing(or double-well Duffing) type equations, namely Eq.(7) and (26).

At this point, we can establish the following exact travelling solutions of ZE and GZE, taking into consideration Eqs. (29), (34), (35), (19–24) and (28).

3.1. EXACT SOLUTIONS FOR THE GZE

3.1.a. If into Eq.(34) we consider that $\Omega_1 = 2ab$ and

$$\frac{\Omega_2}{a^2}-\frac{b^2}{a^2}-\frac{aC}{a^4(4b^2-1)}=:A>0,\frac{\alpha}{a^2(4b^2-1)}=:B>0 \tag{36}$$

then exact solutions of ZE obtained through QTPEM are:

$$u_{1,2}(x,t) = \pm\frac{1}{b_0}\sqrt{\frac{2A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\frac{1}{\cos(\sqrt{A}\varphi+\Psi_0)} \tag{37}$$

$$u_{3,6(x,t)} = \pm\sqrt{\frac{A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\frac{a_0\cosh(\sqrt{2A}\varphi+\Psi_0)\pm a_1}{\sqrt{a_1^2-a_0^2}-a_0\sinh(\sqrt{A}\varphi+\Psi_0)} \tag{38}$$

$$u_{7,8}(x,t) = \pm\sqrt{\frac{A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\coth(\sqrt{\frac{A}{2}}\varphi+\Psi_0) \tag{39}$$

$$u_{9,12}(x,t) == \pm\sqrt{\frac{A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\frac{\pm a_1 a_2\sinh(\sqrt{2A}\varphi+\Psi_0)}{a_2\cosh(\sqrt{2A}\varphi+\Psi_0)\pm\sqrt{a_1^2+a_2^2}} \tag{40}$$

$$u_{13,14}(x,t) == \pm\sqrt{\frac{A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\tanh(\sqrt{\frac{A}{2}}\varphi+\Psi_0) \tag{41}$$

$$u_{15,18}(x,t) == \pm\sqrt{\frac{A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\frac{a_0\cosh(\sqrt{2A}\varphi+\Psi_0)-a_2\sinh(\sqrt{2A}\varphi+\Psi_0)}{a_2\cosh(\sqrt{2A}\varphi+\Psi_0)+a_0\sinh(\sqrt{2A}\varphi+\Psi_0)} \tag{42}$$

3.1.b. If

$$\frac{\Omega_2}{a^2}-\frac{b^2}{a^2}-\frac{aC}{a^4(4b^2-1)} =: -A < 0, \frac{\alpha}{a^2(4b^2-1)} =: -B < 0 \tag{43}$$

then the exact solutions of ZE are

$$u_{1,2}(x,t) == \pm\sqrt{\frac{2A}{B}}\mathrm{e}^{\mathrm{i}(bx-\Omega_2 t)}\frac{1}{b_0\cos(\sqrt{A}\varphi+\Psi_0)-b_2\sinh(\sqrt{2A}\varphi+\Psi_0)}, |b_0|>|b_2| \tag{44}$$

where $b_0, b_1, b_2, C, a, b, a_0, a_1, a_2$ are constants and $\varphi = a(x-2bt)$.

3.2. EXACT SOLUTIONS FOR THE GZE

Comparing Eq. (34) corresponding to ZE and Eq.(35) corresponding to GZE, we obtain the same solutions given by Eqs. (37–42) for the same A given by Eq. (37) but B given by the expression

$$\frac{\alpha}{a^2(4b^2-1)}+\frac{\beta}{a^2} =: B > 0 \tag{45}$$

For A given by Eq. (43) and B given by

$$\frac{\alpha}{a^2(4b^2-1)}+\frac{\beta}{a^2} =: -B < 0 \tag{46}$$

we obtain the solution given by Eq.(44) for GZE where $\varphi = a(x-2bt)$. It should be emphasized that the second variable becomes

$$v_{ij}(x,t) = \frac{C}{a^2(4b^2-1)}-\frac{X_{ij}^2}{4b^2-1}. \tag{47}$$

4. CONCLUSIONS

In the present work, by using the quotient trigonometric function expansion method we have been able to obtain in an unified way by means of symbolic computation system Mathematica 8 or Maple 15, many kinds of exact solution of Zakharov equations and generalized Zakharov equations by means of a simple transformation technique. We showed that nonlinear wave equations can be reduced to the same family of Duffing or double-well Duffing equations. It is remarkable that through this transformation, the quantity of computations involved in solving nonlinear partial differential equations is greatly reduced. Our procedure is one of the most effective method to obtain the exact solutions of nonlinear partial differential equations. The results show that our procedure is a powerful mathematical tool for finding periodic and non-periodic exact solutions of some complicated nonlinear partial differential equations. Moreover, the obtained results in this way clearly demonstrated the reliability of the proposed procedure.

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Received December 14, 2015