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NEUTRON SUBTHRESHOLD STATES. A GAMOW-SIEGERT STATE APPROACH

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The neutron subthreshold state topic is approached in terms of Gamow-Siegert state. The Gamow-Siegert state is described in R- matrix theory by the channel equation $1-R_{nn}L_n=0$ relating decay channel logarithmic derivative L_n to R- matrix element R_{nn}. The equation for multichannel systems implies replacement of channel R-matrix element R_{nn} by its reduced counterpart $\mathscr{R}_{nn}$. This approach results in Heisenberg's S-matrix formula for a bound state as well as its generalization in case of multichannel systems. The dependence of reaction process on decay channel parameters, as threshold, results into renormalization of reduced width. The renormalization factor is R- matrix compression factor, *ie* wave function normalization over internal region to normalization over all space including decay channel. Both threshold compression factor and multichannel couplings imply a shift to threshold (origin) of the subthreshold pole.

Key words: Gamow-Siegert state, subthreshold state.

1. INTRODUCTION

The threshold states are of intrinsec interest due to their peculiar properties related to threshold, see (Baz, Zeldovich and Perelomov, 1971). Moreover they are of interest in nuclear astrophysics, in study of cold stellar cycles, see (Rolfs and Rodney, 1988). The connection between subthreshold states and spectroscopic asymptotic normalization coefficients was treated in K- and R-matrix terms by (Mukhamedzhanov and Tribble 1999). In this work the subject of subthreshold state and of its basic properties is approached in terms of Gamow-Siegert state.

Gamow firstly (1928) introduced the concept of complex-energy eigenstate in order to describe α decay in a quasistationary formalism (*eg* in Baz, Zeldovich and Perelomov, 1971). Thereafter Siegert (1939) introduced a class of solutions of Schroedinger equation satisfying regularity conditions at origin and *out* wave boundary conditions at infinity. The solution a “pure *out* wave at infinity” corresponds to state's “radioactive” decay. The quasistationary state is a pure outgoing wave; the outgoing wave at infinity corresponds to the quasistationary state decay (*eg* Sitenko, 1990). The Siegert solution results in a discrete set of complex momenta k_λ which coincide with poles of collision matrix in complex *k*-plane. Pure positive imaginary momenta represent bound states; pure negative imaginary ones represent virtual states and poles lying close but below real positive *k*-axis represent quasistationary or resonant states. The Siegert approach provides an unitary description of bound and quasistationary states.

The Gamow-Siegert state was/is research topic in quasistationary formalism, see (Baz, Zeldovich and Perelomov 1971), or Green function formalism, see (Kukulin, Krasnopolsky and Horacek, 1989), or Rigged Hilbert Space formulation (Hernandez, Jauregui and Mondragon, 2003; Michel, Nazarewicz, Ploszajczak and Vertse, 2009). The Gamow-Siegert state can be described also in R-matrix formalism by a “decay channel equation”, relating channel R-matrix element and channel logarithmic derivative, (Comisel, Hategan and Ionescu, 2012).

2. ON GAMOW-SIEGERT STATE CONNECTION TO R- AND REDUCED COLLISION- MATRIX

The channel equation $R_{nn}^{-1} = S_n$ defines the bound state in R- matrix theory: a bound state appears at that energy at which the internal R_{nn}^{-1} and external S_n logarithmic derivatives do match, (Lane and Thomas, 1958). The channel logarithmic derivative S_n at negative energy becomes boundary condition at channel radius for a bound state.

The logarithmic derivative of outgoing wave L_n is the corresponding, at positive energy, of the shift function S_n defined for negative energy. The logarithmic derivative of internal wave function R_{nn}^{-1} has to equal, at channel radius, the logarithmic derivative of outgoing wave L_n; this condition corresponds to a quasistationary state.

The quasistationary state boundary condition is that of *out* wave at state energy ε_λ, not at prescribed energy. The complex energy pole ε_λ is root of the implicit equation $1 - R_{nn}(\mathscr{E}_\lambda)L_n(\mathscr{E}_\lambda) = 0$. As in the case of bound states, the equation yields a set of eigenenergies which now are complex. The energy and decay width of quasistationry state are both constants. Thus the Siegert equation $1 - R_{nn}L_n(\mathscr{E}_\lambda) = 0$ defines both the bound state (below threshold) or quasistationary state (above threshold). The bound or quasistationary state is not more described by a R-matrix pole but rather by a channel state equation.

In multichanel systems the Siegert equation becomes $1 - \mathscr{R}_{nn}(\mathscr{E}_\lambda)L_n(\mathscr{E}_\lambda) = 0$, where $\mathscr{R}_{nn}$ is reduced R-matrix element. Observe that in the decoupling limit the equation $L_n^{-1} - R_{nn} = 0$ describes either the bound state (below *n*-channel threshold) or quasistationary state (above *n*-channel threshold). The bound or quasistationary state, from closed or open channel, induces resonance in competing open channels of multichannel system; a phenomenon described in terms of reduced or effective operators.

The concept of reduced or effective operator, previously designed for R-matrix, (or K-matrix), can be extended to collision/scattering matrix (Hategan, 1978). The reason is the collision matrix is a primary concept in Scattering Physics, because it is associated with the whole dynamics of scattering process. The reduced collision matrix consists from collision matrix W^0 of 'bare' retained channels (uncoupled to eliminated ones) and from an effective term ΔW representing the effect of eliminated channel(s) on the retained ones. The effective term contains the retained-eliminated channels couplings as well as a term related to dynamics of collision process in the eliminated channel *n*.

The collision matrix in case of *N* channels is function of *N*-channel component of R-matrix, $W_N^0 \approx f(R_N)$. If an unoserved channel $n = N+1$ is coupled to the *N* observed ones, $W_N = W_N^0 + \Delta W_N$, then a reduced term of collision matrix appears, $\Delta W_N \approx f(\mathscr{R}_N)$, and it is function of reduced R- matrix $\mathscr{R}$, actually on Gamow-Siegert equation. The effective term ΔW_N in the collision matrix, describing effect of the eliminated channel *n*, should be valid both below and above *n*-channel threshold. By coupling the eliminated channel to retained ones the reduced collision matrix term becomes proportional to Siegert term $\Delta W_N \sim [L_n^{-1}(\mathscr{E}_\lambda) - \mathscr{R}_{nn}(\mathscr{E}_\lambda)]^{-1}$ (Hategan, Ionescu and Wolter, 2016).

This qualitative approach to Gamow-Siegert states was formalized in terms of Bloch-Lane-Robson (Lane and Robson, 1966; Robson and Lane, 1967) formalism for quantum collisions, see (Comisel, Hategan and Ionescu, 2012).

3. NEUTRON SUBTHRESHOLD STATE

The neutron subthreshold state in atomic nucleus can be either genuine bound state (in an isolated closed channel) or subthreshold resonance (in a closed channel which belongs to a multichannel system). An unitary approach to subthreshold states related to Gamow-Siegert state concept is here presented.

The bound state is described in R-matrix theory (Lane and Thomas, 1958), by equation $L_n = R_{nn}^{-1}$ relating *n*-channel logarithmic derivative L_n to R- matrix element, $R_{nn} = \gamma_{\lambda n}^2/(E_\lambda - E)$. A R-matrix state is transformed in a Siegert state provided boundary conditions are properly modified (Robson and Lane, 1967).

A modified boundary condition B_n implies a level shift (to be included in E_λ), a change in channel logarithmic derivative $L_n \rightarrow L_n - B_n$ but not in $\gamma^2 \lambda n$ reduced decay width provided B_n is constant.

$$1/\gamma_B^2 = -\mathrm{d}(R_B^{-1})/\mathrm{d}E = -\mathrm{d}(R^{-1} - B)/\mathrm{d}E = 1/\gamma^2. \tag{1}$$

The Siegert boundary condition B_n at channel radius is just channel logarithmic derivative evaluated at level's energy $B_n = L_n(E_\lambda)$; it results into $L_n(E) \rightarrow L_n(E) - L_n(E_\lambda)$ which in Thomas approximation (Lane and Thomas, 1958), is $(E - E_\lambda)\mathrm{d}L_n(E_\lambda)/\mathrm{d}E$. This way the Siegert boundary condition results into

$$R_{nn}^{-1} - L_n = \frac{E_\lambda - E}{\gamma_{\lambda n}^2} + (E_\lambda - E)\frac{\mathrm{d}L_n(E_\lambda)}{\mathrm{d}E} = \frac{E_\lambda - E}{\omega_{\lambda n}^2}, \tag{2}$$

where $\omega_{\lambda n}^2 = \beta_{\lambda n}\gamma_{\lambda n}^2$ is renormalized reduced width in terms of R-matrix compression factor $\beta_{\lambda n}$ (Lane, 1970),

$$\beta_{\lambda n} = \frac{1}{1 + \gamma_{\lambda n}^2 \dfrac{\mathrm{d}L_n(E_\lambda)}{\mathrm{d}E}}, \tag{3}$$

with $\beta_{\lambda n} \leq 1$ (as $\mathrm{d}L_n/\mathrm{d}E \geq 0$ at least for neutron case). The β factor is a measure (Lane, 1970), of bound state wave function extension in the channel space from channel radius r_n to outer turning point a_t

$$\beta_{\lambda n} = \frac{\int_0^{r_n} v_\lambda^2 \mathrm{d}r}{\int_0^{r_n} v_\lambda^2 \mathrm{d}r + \int_{r_n}^{a_t} v_\lambda^2 \mathrm{d}r}. \tag{4}$$

The transition amplitude $2ikr_nT_n$ defined by

$$T_n = \frac{1}{R_{nn}^{-1} - L_n} = \frac{\omega_{\lambda n}^2}{E_\lambda - E} \tag{5}$$

becomes for negative energy $E_\lambda = -|E_\lambda| = k_\lambda^2$, $k_\lambda = i\kappa_\lambda$; $\kappa_\lambda > 0$, $|E_\lambda| = \kappa_\lambda^2$,

$$2ikr_nT_n = -2ikr_n \frac{\omega_{\lambda n}^2}{|E_\lambda| + E} = -2ikr_n \frac{\omega_{\lambda n}^2}{(k - i\kappa_\lambda)(k + i\kappa_\lambda)} \tag{6}$$

in limit $k \rightarrow i\kappa_\lambda$

$$2ikr_nT_n \rightarrow -ir_n \frac{\omega_{\lambda n}^2}{k - i\kappa_\lambda}. \tag{7}$$

It is bound state formula in S-matrix theory as derived by Heisenberg (see Perelomov and Zeldovich, 1998, ch 3.3). The bound state pole, located on positive imaginary axis of complex momentum plane, has a residue C^2 called asymptotic normalization constant. This quantity does include both R-matrix residue $\gamma_{\lambda n}^2$ as well as the normalization $\beta_{\lambda n}$ of the wave function tail extending over channel radius.

The bound state from a closed neutron channel embeded in a multichannel system is subject of equation $L_n = \mathcal{R}_{nn}^{-1}$, where reduced R-matrix element $\mathcal{R}_{nn}$ is a complex quantity, $\mathcal{R}_{nn}^{-1} = (E_\lambda - \Delta_\lambda - i\Gamma_\lambda - E)/\gamma_{\lambda n}^2$, with level shift Δ_λ and decay width $\Gamma_\lambda > 0$ originating in coupling of neutron channel to complementay open ones. The Siegert condition implies now analytic continuation to a complex eigenvalue $\mathcal{E}_\lambda = E_\lambda - \Delta_\lambda - i\Gamma_\lambda$. The bound state in closed neutron channel of a multichannel system becomes a complex energy state; it is the neutron subthreshold resonance. As $\Delta_\lambda < |E_\lambda|$ and $E_\lambda < 0$ it results the actual level energy $E_\lambda - \Delta_\lambda < 0$ is still negative. The complex energy $\mathcal{E}_\lambda = \tilde{k}_\lambda^2$ is related to complex momentum $\tilde{k}_\lambda = k_\lambda + i\kappa_\lambda$ *ie* $E_\lambda - \Delta_\lambda = -|E_\lambda - \Delta_\lambda| = k_\lambda^2 - \kappa_\lambda^2 < 0$ and $\Gamma_\lambda = -2k_\lambda\kappa_\lambda > 0$. These relations imply $\kappa_\lambda > 0$ and $k_\lambda < 0$, also $|k_\lambda| < \kappa_\lambda$. Siegert boundary condition, $B_n = L_n(\mathcal{E}_\lambda)$, results in

$$T_n = \frac{1}{\mathscr{R}_{nn}^{-1} - L_n} = \frac{\omega_{\lambda n}^2}{\mathscr{E}_\lambda - E}, \tag{8}$$

with same definitions as before for $\omega_{\lambda n}^2$ and $\beta_{\lambda n}$. In the limit $k \to \tilde{k}_\lambda$ one obtains for transition amplitude

$$2ikr_n T_n \to -ir_n \frac{\omega_{\lambda n}^2}{k - i(\kappa_\lambda - |k_\lambda|)}. \tag{9}$$

The subthreshold resonance pole is not more located on positive imaginary axis (as for the stationary one) but rather is shifted to left of positive imaginary axis in second quadrant of complex momentum plane.

The renormalization factor $\beta_{\lambda n}$ becomes effective near threshold because there the logarithmic derivative is strongly varying. It provides a dip-like shape of spectroscopic factor $\omega_{\lambda n}^2$ centered just at threshold. The multichannel coupling and threshold effects on spectroscopic factors of weakly bound states in exotic nuclei is a topic of interest in Gamow Shell Model (Michel, Nazarewicz, Ploszajczak and Vertse, 2009). The above analytical approach is concerned only with the threshold energy dependence of spectroscopic factors; the multichannel effect on subthreshold pole is approached in next chapter.

4. DYNAMICAL ASPECTS OF SUBTHRESHOLD RESONANCE

A bound or a quasistationary state, originating in an eliminated channel, induces a resonance in open competing channels. Both the width and level shift are determined by channels couplings and by rescattering in open channels. A channel resonance pole, defined by equation $1 - \mathscr{R}_{nn}(\mathscr{E}_\lambda)L_n(\mathscr{E}_\lambda) = 0$ is subject of motion in complex energy plane, both by proximity threshold and by couplings to complementary reaction channels. A broad quasistationary state (from eliminated channel) results in a smaller width resonance which, in addition, is shifted to a lower energy (Hategan and Ionescu, 2014). This result can be analytically proved in terms of complex scattering length *via* its relation to channel reduced R-matrix element.

$$a_1 = a_n - b\Delta_n = a_n + b\Gamma_n \tan\delta_0 . \tag{10}$$

Phenomena developing in threshold channel near zero-energy (bound-unbound transition zone) could be properly described in terms of scattering length (*eg* Drukarev, 1978; Burke, 2011). The scattering length a_n, for neutron *s*-wave, is defined in terms of δ_n scattering phase shift

$$k \cot\delta_n = -1/a_n + 1/2 r_0 k^2 , \tag{11}$$

where k and r_0 are channel wave number and effective radius; the scattering length defines the scattering amplitude just at threshold energy. On the other hand the *s*-wave nuclear scattering phase-shift δ_n is related to R- matrix element R_{nn} and penetration factor P_n through relation $\tan\delta_n = P_n R_{nn} = kr_n R_{nn}$. In zero-energy limit the neutron scattering length and R_{nn} matrix element are related by $a_n = -r_n R_{nn}$ with r_n -channel radius. The relation can be extended to complex scattering phase-shift and reduced R- matrix element too, $\tan\tilde{\delta}_n = P_n \mathscr{R}_{nn}$. The complex scattering length $\tilde{a}_n = a_1 - ia_2 = -r_n \mathscr{R}_{nn}$ components are related to those of reduced R-matrix element, $\mathscr{R}_{nn} = R_{nn} + \Delta_n + i\Gamma_n$, namely $a_1 = -r_n(R_{nn} + \Delta_n)$, and $a_2 = r_n \Gamma_n$. (The imaginary component $\Gamma_n > 0$ is consequence of subunitary value of collision matrix element $|W_{nn}| < 1$. The Optical Model scattering length has also a negative imaginary component too, $-ia_2$, with $a_2 > 0$ due to absorptive component of optical potential.) The complex scattering length, $\tilde{a}_n = a_1 - ia_2$ is an alternative way to take into account channels couplings. The complex scattering length is dependent on coupling and rescattering in open channel.

The complex scattering length, in case of only one open channel *o*, depends on corresponding open channel phase shift δ_0 as

$$a_1 = a_n - b\Delta_n = a_n + b\Gamma_n \tan\delta_0 = a_n + a_2 \tan\delta_0 . \tag{12}$$

Therefore $a_1 > a_n$ provided open channel scattering phase shift $\tan\delta_0 > 0$; also $a_2 = b\Gamma_n$ decreases as effect of coupling and rescattering in open channel.

$$k = \frac{i}{\tilde{a}} = \frac{ia_1 - a_2}{|a_1|^2 + |a_2|^2}. \tag{13}$$

Observe that a bound state $a_1 > 0$ is located in 2-nd quadrant of *k*- plane and a virtual state $a_1 < 0$ in the third one ($a_2 > 0$). (The Optical Model pole is located at $k = i/\tilde{a}$, either in second or third wave number quadrant.) In energy plane, $k^2 = E - i\Gamma$, the energy $E < 0$ is negative ($a_2^2 < a_1^2$), and the width Γ is either positive (originating in a bound state) or negative (originating in a virtual state). As a_1 increases and a_2 decreases one obtains that both $\mathrm{Im}\,k \sim a_1/a_1^2$ and $\mathrm{Re}\,k \sim a_2/a_1^2$ decrease so the pole is shifted to origin (or threshold). It is an analytical demonstration that channels couplings result into shift of the channel state pole to the real axis and into decrease of its decay width.

In literature (Badalyan *et al*., 1982), one reports on "channel coupling pole" observed in numerical experiments for multichannel scattering; a single channel pole may be driven to physical region of the complex energy plane when channel coupling becomes effective. The "channel coupling resonances" and multichannel resonances originating in quasistationary or bound channel states have similar width property.

This approach to channel (also to inner) resonances can be compared to K-matrix formalism for resonances too (Chung *et al*., 1995). There are two types of resonances which differ in dynamical character; they are parametrized, according to K-matrix, in two distinct forms. Resonances can arise from strongly varying K-matrix elements (pole). These "normal resonances" correspond to dynamical sources at the constituent level; in our case they correspond to "compound nucleus" resonances. Resonances can appear also from constant K-matrix element provided the energy variation is supplied by phase space. These "molecular resonances" are assumed to arise from couplings in the reaction channels; in our case the reduced R-matrix element $\mathscr{R}_{nn}$ does include couplings to complementary channels. The "channel resonance", described by channel equation $\mathscr{R}_{nn} - L_n^{-1} = 0$, originates in constant reduced R-matrix element and in energy dependent logarithmic derivative. The energy variation of channel logarithmic derivative is implied in realization of the quasistationary state condition $\mathscr{R}_{nn} = L_n^{-1}$.

5. CONCLUSIONS

The neutron subthreshold states are viewed in this paper as near-threshold Gamow-Siegert states. The Gamow-Siegert state is described by relating channel R-matrix element to channel logarithmic derivative, $1 - R_{nn}L_n = 0$. The extension to multichannel case implies replacement of R_{nn} matrix element by its reduced counterpart $\mathscr{R}_{nn}$. The corresponding term $(1 - \mathscr{R}_{nn}L_n)^{-1}$ describes *n*- channel physics in reduced collision matrix of multichannel system. The Gamow-Siegert condition, $1 - \mathscr{R}_{nn}L_n = 0$, implied in reduced collision matrix, describes the neutron subthreshold resonance including its main spectroscopical properties as threshold compression of the reduced width. A merit of this approach is rederivation of Heisenberg's S-matrix formula for a bound state as well as its generalization to multichannel case.

Dynamical aspects of subthreshold resonance are discussed in terms of (complex) neutron scattering length. A multichannel property of subthreshold resonance is revealed, *ie* the shift to the origin of the Gamow-Siegert pole. The Gamow-Siegert pole of subthreshold state is subject of shift to the origin; physically two mechanisms of compression are involved the R-matrix compresssion factor, effective only near threshold, and direct interaction (channels couplings) compression factor due to channels couplings. The last mechanism could be put into correspondance with "molecular resonances" or with "channel coupling pole" observed in numerical experiments for multichannel scattering: a single channel pole may be driven to physical region of the complex energy plane if channel couplings become effective.

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