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ON SOME PROPERTIES OF THE *K*-COHERENT STATES

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Abstract. By using the Pochhammer *k*-symbols, and the *k*-hypergeometric functions we build and examine some properties of the *k*-coherent states. Using the diagonal ordering operation technique we demonstrate that these new constructed states satisfy all requirements imposed to a coherent state as stated by Klauder.

Key words: coherent states, *k*-hypergeometric functions, operator theory.

1. INTRODUCTION

After their introduction by Schrödinger [1], the concept of coherent states (CSs) was developed and examined for many physical quantum systems [2]. Generally, a coherent state (CS) is a ket vector labeled by a complex number $z = |z|\exp(\mathrm{i}\varphi)$, with $|z| \le \infty$, $0 \le \varphi \le 2\pi$, whose expansion in the Fock – vectors basis is

$$|z> = \frac{1}{\sqrt{N(|z|^2)}} \sum_n \frac{z^n}{\sqrt{\rho(n)}} |n>, \tag{1.1}$$

where $\rho(n)$ is a positive sequence of real numbers – the structure constants, which determine the internal structure of CSs.

The CSs exist only if the normalization function $N(|z|^2)$ is an analytical function of real variable $|z|^2$, and if satisfy a number of criteria summarized by Klauder: continuity in complex label, normalization, non orthogonality, unity operator resolution with unique positive weight function of the integration measure, temporal stability and action identity [3]. The most general class of CSs are the generalized hypergeometric coherent states (GH-CSs) whose normalization function is a generalized hypergeometric function ${}_pF_q(\{a_i\}_1^p;\ \{b_j\}_1^q; |z|^2)$ [4, 5]:

$$|z> = \frac{1}{\sqrt{{}_pF_q(\{a_i\}_{i=1}^p;\ \{b_j\}_{j=1}^q; |z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_{p,q}(n)}} |n>, \tag{1.2}$$

where we used the following notation for the sequence of numbers $\{a_i\}_{i=1}^p = \{a_1, a_2, ..., a_p\}$ and so on. By particularizing these numbers, as well as the positive integers p and q, we obtain the all known CSs.

The main purpose of the present paper is to build the so called *k*-coherent states (*k*-CSs) defined by using the Pochhammer *k*-symbols, respectively, the generalized *k*-hypergeometric functions and to reveal some of their properties. We concentrate here only on the examination of the coherent states of Barut - Girardello kind (*k*-BG-CSs), defined as the eigenvectors of the lowering operator of the quantum system in consideration [6]. We point out here that the presented approach can be extended also on the mixed (thermal) states, characterized by density operator ρ. Moreover, the building of the *k*-CSs approach can highlight a series of new formulae related to generalized *k*-hypergeometric functions.

2. *K*-MATHEMATICAL GENERALIZATIONS

Over the decades, as an alternative to "classical" algebraic entities were developed a series of generalizations thereof. Some of them have got the name *k*-generalizations and were denoted by the index *k*. As particular cases (for $k=1$), all these generalized entities lead to the corresponding classical entities. For $x\in\mathbb{C}$, $k\in\mathbb{R}$ and $n\in\mathbb{N}^{+}$, the Pochhammer *k*- symbol, denoted by $(x)_{n,k}$, recently introduced by Díaz and Pariguan, is defined as [7]

$$(x)_{n,k}=x(x+k)(x+2k)\ \ldots\ (x+(n-1)k)=\frac{\Gamma_k(x+n\ \ k)}{\Gamma_k(x)}, \tag{2.1}$$

where $\Gamma_k(x)$ is the *k*-gamma function. For $\mathrm{Re}(x)>0$ it is defined by the integral

$$\Gamma_k(x)=\int_0^{\infty}\mathrm{d}t\ \ e^{-\frac{t^k}{k}}\ \ t^{x-1}. \tag{2.2}$$

In other words, the Pochhammer *k*-symbol is defined so that the rate of increase in the number *x* is a positive number k. We insert below only few usual properties [7]:

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right),\ \Gamma(x)=k^{1-x}\Gamma_k(kx) \tag{2.3}$$

$$\Gamma_k(x+k)=x\Gamma_k(x),\ \Gamma_k(k)=1, \tag{2.4}$$

as well as the following connection with the usual Pochhammer symbol $(a)_n=\Gamma(a+n)/\Gamma(a)$:

$$(x)_{n,k}=k^n\left(\frac{x}{k}\right)_n,\ (x)_{0,k}=\left(\frac{x}{k}\right)_0=1,\ (k)_{n,k}=k^n n!. \tag{2.5}$$

The Pochhammer *k*-symbols and implicitly the *k*- gamma functions are used to define the generalized *k*-hypergeometric functions ${}_pF_{q,k}\left(\{(a_i,k)\}_{i=1}^{p};\{(b_j,k)\}_{j=1}^{q};x\right)$ by the formal power series, connected with the usual hypergeometric functions ${}_pF_q\left(\{(a_i/k)_n\}_{i=1}^{p};\{(b_j/k)_n\}_{j=1}^{q};k^{p-q}x\right)$ [7, 8]:

$$\begin{aligned}{}_pF_{q,k}\left(\{(a_i,k)\}_{i=1}^{p};\{(b_j,k)\}_{j=1}^{q};x\right)&=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{j=1}^{q}(b_j)_{n,k}}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{k^{pn}\prod_{i=1}^{p}\left(\frac{a_i}{k}\right)_n}{k^{qn}\prod_{j=1}^{q}\left(\frac{b_j}{k}\right)_{n,}}\frac{x^n}{n!}=\\&={}_pF_q\left(\left\{\left(\frac{a_i}{k}\right)_n\right\}_{i=1}^{p};\left\{\left(\frac{b_j}{k}\right)_n\right\}_{j=1}^{q};k^{p-q}x\right)\end{aligned} \tag{2.6}$$

where we used the notation $\{(a_i,k)\}_{i=1}^{p}=\left\{(a_1)_{n,k},(a_2)_{n,k},\ldots,(a_p)_{n,k}\right\}$ and so on. Particularly, in this paper, we will be interested in the following *k*-hypergeometric function

$${}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ \ \ k)\ ;\ x\right)=\sum_{n=0}^{\infty}\frac{(k)_{n,k}}{(b+k)_{n,k}}\frac{x^n}{n\ !}=\sum_{n=0}^{\infty}\frac{(1)_n}{\left(\frac{b}{k}+1\right)_n}\frac{x^n}{n\ !}={}_1F_1\left(1\ ;\ \frac{b}{k}+1\ ;\ x\right). \tag{2.7}$$

The *s*-order differentiation of *k*-hypergeometric function with respect to their argument is deduced from the similar formula for usual hypergeometric function [9] and using the above properties:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^s {}_1F_{1,\,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ x\right) = \frac{\prod_{m=1}^{s}(k)_{m\,,\,k}}{\prod_{m=1}^{s}(b+k)_{m,\,k}}\ {}_1F_{1,\,k}\left((k+k\ \ s,\ k)\ ;\ (b+k\ \ +k\ \ s,\ k\right. \tag{2.8}$$

Next we will use the following differential operator [10]

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^s f(x) = \sum_{j=1}^{s}\left[\frac{1}{j\ !}\sum_{m=0}^{j}(-1)^{j+m}\ \binom{j}{m}\ m^s\right] x^j\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j f(x) \equiv \sum_{j=1}^{s} S(s,\ j)\ x^j\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j f(x), \tag{2.9}$$

where $S(s,j)$ are Stirling's numbers of the second kind [11], and apply this formula to *k*-hypergeometric function

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^s {}_1F_{1,\,k}\left((k,k);(b+k,k);x\right) =$$

$$= \sum_{j=1}^{s} S(s,j)\frac{\prod_{m=1}^{j}(k)_{m,k}}{\prod_{m=1}^{j}(b+k)_{m,k}} x^j\ {}_1F_{1,\,k}\left((k+kj,k);(b+k+kj,k);x\right). \tag{2.10}$$

This relation will be useful for calculating some expectation values in the k -CSs representation.

3. *K*-COHERENT STATES

Let's two Hermitic operators A_- (lowering) and A_+ (raising), so as to satisfy the equations

$$A_-\,|\,n>=\sqrt{e_n}\ \ |\,n-1>,\ A_+\,|\,n>=\sqrt{e_{n+1}}\ \ |\,n+1>,\ A_+A_-\,|\,n>=e_n\ \ |\,n>, \tag{3.1}$$

and a dimensionless Hamilton operator factorized as $H = A_+A_-$, with the dimensionless eigenvalues e_n .

Particularly, let we choose the dimensionless eigenvalues in the manner that they depend linearly on the principal quantum number n , i.e. $e_n = b+nk$, where $k>0$, $b\in\mathbb{C}\setminus k\,\mathbb{Z}^-$, and $n=0,1,2,\ldots$. This choice corresponds to some quantum systems, e.g. one dimensional harmonic oscillator, pseudoharmonic oscillator, Landau levels and so on. Consequently,

$$A_-\,|\,n>=\sqrt{b+nk}\,|\,n-1>,\ A_+\,|\,n>=\sqrt{b+(n+1)k}\,|\,n+1>, \tag{3.2}$$

$$H\,|\,n>=A_+A_-\,|\,n>=(b+nk)\,|\,n>. \tag{3.3}$$

The *n*-fold repetition of applying the raising operator to the vacuum state $|\,0>$ leads to

$$|\,n> = \frac{1}{\sqrt{(b+k)_{n,\,k}}}\left(A_+\right)^n|\,0>. \tag{3.4}$$

Using the definition of the Pochhammer *k*-symbol, we can write an useful relation

$$(b+k)_{n,\,k} = k^n\left(\frac{b}{k}+1\right)_n = k^n\ \ \Gamma\left(\frac{b}{k}+1+n\right)\left[\Gamma\left(\frac{b}{k}+1\right)\right]^{-1}. \tag{3.5}$$

As it is well-known, the Fock vectors satisfy the completeness relation $\sum_{n=0}^{\infty} | n >< n | = 1$, and by considering the Hermitic conjugate of Eq. (3.4) and substituting both equations in the above relation, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(b+k)_{n,\,k}} (A_+)^n \,|\, 0 >< 0 \,|\, (A_-)^n = 1. \tag{3.6}$$

This relation becomes useful if we appeal to our previously introduced new operator ordering technique, i.e. to the diagonal ordering operation technique (DOOT), denoted by the symbol # # and whose fundamental rules are [12]: *a*) Inside the symbol # #, the order of the operators A_- and A_+ can be permuted like commutable operators, so that finally we obtain an operator function that depends only on the entire powers of the product $A_+ A_-$, i.e. $\#(A_-)^n (A_+)^n \# = \#(A_+)^n (A_-)^n \# = \#(A_+ A_-)^n \#$; *b*) A symbol # # inside another symbol # # can be deleted; *c*) If the integration is convergent, a normally ordered product of operators can be integrated or differentiated (generally, it can be applied any algebraic operations) with respect to the *c*-numbers, according to the usual algebraic rules. Consequently, the *c*-numbers can be taken out from the symbol # #; *d*) the vacuum state projector $| 0 >< 0 |$ in the frame of DOOT generally has the following normal ordered form:

$$| 0 > < 0 | = \# \frac{1}{{}_pF_q\left(\{a_i\}_{i=1}^{p} \;;\; \{b_j\}_{j=1}^{q} \;;\; A_+ A_- \right)} \#. \tag{3.7}$$

By applying the rules *a*) – *c*) on the Eq. (3.7), and using the properties of the Pochhammer *k*-symbols, we obtain that the vacuum operator connected with our pair of operators is

$$| 0 > < 0 | = \# \frac{1}{\sum_{n=0}^{\infty} \frac{(k)_{n,\,k}}{(b+k)_{n,\,k}} \frac{1}{n\;!} \left(\frac{A_+ A_-}{k} \right)^n} \# \;=\; \# \frac{1}{{}_1F_{1,\,k}\left((k,\;k) \;;\; (b+k,\;k) \;;\; \frac{A_+ A_-}{k} \right)} \#, \tag{3.8}$$

where ${}_1F_{1,k}\left((k,k);(b+k,k);\, \frac{A_+ A_-}{k} \right)$ is the Kummer confluent *k*-hypergeometric function [7]. This means that the DOOT rules can be applied not only to systems with infinite [12] or finite (e.g. spin systems [13]) energy spectra, but can be extended also on the *k*-generalized functions.

Now, let us we define the *k*-coherent states in the Barut-Girardello manner (*k*-BG-CSs) $| z >_k$, i.e. as the eigenstates of the lowering operator A_- [6]

$$A_- \,|\, z >_k = z \;|\, z >_k \,. \tag{3.9}$$

In what follows we will demonstrate that *k*-BG-CSs fulfill all Klauder's requirements imposed to any CS [3]. First, the *k*-BG-CSs can be also expanded as a superposition of the energy eigenstates $| n >$, using Eqs. (3.4) and (3.9), as well as the normalization relation:

$$| z >_k = \frac{1}{\sqrt{{}_1F_{1,\,k}\left((k,\;k) \;;\; (b+k,\;k) \;;\; \frac{|z|^2}{k} \right)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(b+k)_{n,\,k}}} \,| n >. \tag{3.10}$$

From the overlap (or scalar product) of two *k*-BG-CSs, i.e.

$${}_k< z \,|\, z' >_k = \frac{{}_1F_{1,k}\left((k,k);(b+k,k);\, \frac{z^* z'}{k} \right)}{\sqrt{{}_1F_{1,k}\left((k,k);(b+k,k);\, \frac{|z|^2}{k} \right)} \sqrt{{}_1F_{1,k}\left((k,k);(b+k,k);\, \frac{|z'|^2}{k} \right)}} \tag{3.11}$$

it can be seen that the k-BG-CSs are *normalizable* but *non orthogonal*.

Using Eq. (3.4), the k-BG-CSs can be written as an operator acting on the vacuum state $|0>$:

$$|z>_k = \frac{1}{\sqrt{{}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{|z|^2}{k}\right)}}\ {}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{z\ A_+}{k}\right)|0>. \tag{3.12}$$

The *continuity in the complex label*, i.e. if $z' \to z$, then $|z'>_k \to |z>_k$, can be demonstrated by calculating the following limit: $\lim\limits_{z'\to z}\left\| |z'>_k - |z>_k \right\|^2 = 2 - \lim\limits_{z'\to z}\left({}_k<z|z'>_k + {}_k<z'|z>_k\right) = 0$.

With the Hermitic conjugate of Eq. (3.12), as well as with Eq. (3.8) for the vacuum operator, we are able to write the k-BG-CSs projector:

$$|z>_k\ {}_k<z| = \#\ \frac{{}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{zA_+}{k}\right)\ {}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{z^*A_-}{k}\right)}{{}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{|z|^2}{k}\right)\ {}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{A_+A_-}{k}\right)}\#. \tag{3.13}$$

We use this relation in order to demonstrate the *resolution of the unity operator*, i.e.

$$\int \mathrm{d}\mu_k(z)\,|z>_k\ {}_k<z| = 1, \tag{3.14}$$

where it is necessary to determine the integration measure $\mathrm{d}\mu_k(z) = (2\pi)^{-1}\,\mathrm{d}\varphi\cdot\mathrm{d}\left(|z|^2\right)\cdot h_k\left(|z|^2\right)$, respectively their weight function $h_k\left(|z|^2\right)$ that must be a positive defined function.

The main steps of the calculations are:

– The function change

$$\tilde{h}_k\left(|z|^2\right) = h_k\left(|z|^2\right)\ \left[{}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{|z|^2}{k}\right)\right]^{-1}; \tag{3.15}$$

– The angular integration

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\ \#\ {}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{zA_+}{k}\right)\ {}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{z^*A_-}{k}\right)\# &= \\ = \#\,{}_1F_{2,k}\left((k,\ k)\ ;\ (b+k,\ k),\ (b+k,\ k)\ ;\ \frac{|z|^2}{k}\frac{A_+A_-}{k}\right)\#;&\end{aligned} \tag{3.16}$$

– The resolution of the following Stieltjes moment problem [14]:

$$\int_0^\infty \mathrm{d}\left(|z|^2\right)\ \tilde{h}_k\left(|z|^2\right)\ \left(\frac{|z|^2}{k}\right)^n = (b+k)_{n,k} = \left[\Gamma\left(\frac{b}{k}+1\right)\right]^{-1} k^n\ \Gamma\left(\frac{b}{k}+1+n\right); \tag{3.17}$$

– The index change $n = s-1$, and the solution is [15]

$$\tilde{h}_k\left(|z|^2\right) = \frac{1}{k\,\Gamma\left(\frac{b}{k}+1\right)}G^{1,0}_{0,1}\left(\frac{|z|^2}{k}\middle|\frac{b}{k}\right) = \frac{1}{k\,\Gamma\left(\frac{b}{k}+1\right)}e^{-\frac{|z|^2}{k}}\left(\frac{|z|^2}{k}\right)^{\frac{b}{k}} \tag{3.18}$$

so that, finally, the integration measure becomes

$$\mathrm{d}_k\mu(z) = \left[\Gamma\left(\frac{b}{k}+1\right)\right]^{-1}\frac{\mathrm{d}\varphi}{2\pi}\ \mathrm{d}\left(\frac{|z|^2}{k}\right)\ e^{-\frac{|z|^2}{k}}\left(\frac{|z|^2}{k}\right)^{\frac{b}{k}}\ {}_1F_{1,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \frac{|z|^2}{k}\right) \tag{3.19}$$

from which it can see that the weight function is positive definite.

Temporal stability relative to the time evolution operator $\mathbb{U}(t) = \exp(-\mathrm{i}\,H\,t)$ can be expressed as

$$\mathbb{U}(t)\ |z>_k = e^{-\mathrm{i}Ht}\,|z>_k = e^{-\mathrm{i}bt}\ |z\ \ e^{-\mathrm{i}kt}>_k = e^{-\mathrm{i}bt}\ ||z|\ e^{-\mathrm{i}(\varphi+kt)}\ >_k \equiv |\ z(t)>_k, \tag{3.20}$$

which shows that, during the time, the k-BG-CSs remain coherent.

To show *the action identity* (or, the lower symbol of Hamiltonian H), we use the following recurrence relation of the Pochhammer k-symbols

$$\frac{1}{(b+k)_{n,k}}(b+nk) = \frac{1}{(b+k)_{n-1,k}} \tag{3.21}$$

change the summation index $n' = n-1$, eliminate the unphysical term with $n' = -1$, and we obtain:

$${}_k\!<z\,|\,H\,|\,z>_k \equiv <H>_{z,\,k} = |\,z\,|^2\,. \tag{3.22}$$

So, the above defined k-BG-CSs $|\,z>_k$ are *certainly coherent states*.

Above we have used also the Hermitical conjugate relation of the definition of k-BG-CSs:

$${}_k\!<z\,|\,A_+\,| = z^*\ {}_k\!<z\,|\,. \tag{3.23}$$

Generally, the expectation or mean value of an operator $\mathcal{A}$ in the k-BG-CSs representation is

$${}_k\!<z\,|\,\mathcal{A}\,|\,z>_k \equiv <\mathcal{A}>_{z,\,k} = \frac{1}{{}_1F_{1,\,k}\left((k,\ k)\ ;\ (b+k,\ k)\ ;\ \dfrac{|\,z\,|^2}{k}\right)} \sum_{n=0}^{\infty} \frac{\left(|\,z\,|^2\right)^n}{(b+k)_{n,\,k}} <n\,|\,\mathcal{A}\,|\,n>. \tag{3.24}$$

On the other hand, the expectation value of the normally ordered operator's product A_+A_- is

$$<A_+A_->_{z,\,k} = \left[{}_{z,\,k}\!<z\,|\,A_+\right]\left[A_-\,|\,z>_{z,\,k}\right] = z^*z = |\,z\,|^2, \tag{3.25}$$

which means that $|\,z\,|^2$ is the eigenvalue of the normally ordered operators A_+A_-, associated with the eigenvector $|\,z>_k$. This result can be generalized using the DOOT rules, i.e. the property *a*): $\#\left(A_+A_-\right)^n\,\# = \#\left(A_+\right)^n\left(A_-\right)^n\,\#$, as well as the definition of k-BG-CSs and their Hermitic conjugate:

$$<\#\left(A_+A_-\right)^n\,\#>_{z,\,k} = \#\left[{}_{z,\,k}\!<z\,|\left(A_+\right)^n\right]\left[\left(A_-\right)^n|z\ \ >_{z,\,k}\right]\# = \left(|\,z\,|^2\right)^n. \tag{3.26}$$

The practical importance of this result is the following: whenever we have to calculate the expectation value in the k-BG-CSs representation of a function depending on the normally ordered operator product A_+A_-, due to the DOOT rules, it is sufficient to simply replace the product A_+A_- with their eigenvalue in k-BG-CSs representation, i.e. with $|\,z\,|^2$:

$$<\#f\left(A_+A_-\right)\#>_{z,\,k} = \sum_{j=0}^{\infty} c_j <\#\left(A_+A_-\right)^j\,\#>_{z,\,k} = \sum_{j=0}^{\infty} c_j\,\#\left(|\,z\,|^2\right)^j\,\# = f\left(|\,z\,|^2\right). \tag{3.27}$$

From the equality $A_+A_-\,|\,n> = \left(b+nk\right)|\,n> = \left(b+kN\right)|\,n>$ where N is the number operator (the main number of particles) , i.e. $N\,|\,n> = n|\,n>$, we obtain the following operator identity for the integer power s of the number operator N

$$N^s = \frac{1}{k^s}\#\left(A_+A_- - b\right)^s\,\# = \frac{1}{k^s}\sum_{j=0}^{s}\binom{s}{j}\left(-b\right)^{s-j}\#\left(A_+A_-\right)^j\,\#. \tag{3.28}$$

Consequently, their expectation value in the k-BG-CSs representation is

$$< N^s >_{z,\,k} = \frac{1}{k^s}\#\left(< A_+ A_- >_{z,\,k} - b\right)^s \# = \frac{1}{k^s}\left(|\,z\,|^2 - b\right)^s = \frac{1}{k^s}\sum_{j=0}^{s}\binom{s}{j}\left(-b\right)^{s-j}\left(|\,z\,|^2\right)^j . \tag{3.29}$$

Moreover, the Mandel parameter $Q_{|z|^2,\,k}$ is a measure of the departure of the occupation number distribution statistics from Poissonian statistics [12]. For k-BG-CSs it is

$$Q_{|z|^2,\,k} = \frac{< N^2 >_{z,\,k} - \left(< N >_{z,\,k}\right)^2}{< N >_{z,\,k}} - 1 = -1. \tag{3.30}$$

Because the Mandel parameter is always negative, $Q_{|z|^2;k} < 0$, the behavior of the k-BG-CSs is sub Poissonian. This assertion is confirmed also by calculating the probability that n photons (generally, n particles) will be found in the k-BG-CS $|\,z>_k$, i.e.

$$\mathrm{P}_k(n) = |< n\,|\,z >_k|^2 = \frac{1}{{}_1F_{1,\,k}\left((k,\ k);(b+k,k);\dfrac{|\,z\,|^2}{k}\right)}\frac{(|\,z\,|^2)^n}{(b+k)_{n,\,k}}. \tag{3.31}$$

At the *harmonic limit* this probability distribution tends to the Poisson distribution:

$$\lim_{\substack{k\to 1\\ b\to 0}} \mathrm{P}_k(n) = e^{-\,|z|^2}\frac{|\,z\,|^2}{n\ !} = \mathrm{P}_{Poisson}(n). \tag{3.32}$$

because

$$\lim_{\substack{k\to 1\\ b\to 0}} {}_1F_{1,\,k}\left((k,k);(b+k,k);\frac{|\,z\,|^2}{k}\right) = {}_0F_0(\ ;\ ;\,|\,z\,|^2) = e^{|z|^2}. \tag{3.33}$$

4. CONCLUDING REMARKS

Starting from the new definition of Pochhammer k-symbols, introduced by Díaz and Pariguan [7], and, as a consequence, of some generalized k-hypergeometric functions, in this paper we built new kind of coherent states, the k-coherent states (k-CSs). In the paper we bordered only on the coherent states of the Barut-Girardello kind (k-BG-CSs), although the method can also be applied to build other types of coherent states (Klauder-Perelomov or Gazeau-Klauder). We demonstrated that the k-BG-CSs satisfy all the requirements imposed to a coherent state, stated by Klauder [3]: continuity in complex label, normalization, non orthogonality, unity operator resolution with unique positive weight function of the integration measure, temporal stability and action identity. For this purpose we used the rules of a previously introduced approach – the diagonal ordering operation technique (DOOT) [12] that greatly simplify the calculations. The presented approach proves that the coherent states can be built also in the frame of k-generalized functions, which give them a greater degree of generality. The value of Mandel parameter $Q_{|z|^2;k} < 0$, for pure states $|\,z>_k$, show that it is associated with sub Poissonian statistics. Also, we can point out that the above exposed approach connected with k-BG-CSs, using the DOOT, can be easy extended also on the mixed (thermal) states, as well as on the GH-CSs, so we can obtain the generalized k-hypergeometric coherent states (k-GHC-Ss), whose expansion in terms of Fock-vector basis is

$$|\,z>_k = \frac{1}{\sqrt{{}_pF_{q,k}(\{(a_i\,,\,k)\}_{i=1}^{p};\,\{(b_j\,,\,k)\}_{j=1}^{q};|\,z\,|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_{p,q;k}(n)}}|\,n> \tag{4.1}$$

and the corresponding structure constants are

$$\rho_{p,q;k}(n) = \prod_{i=1}^{p}(a_i)_{n,k}\left[\prod_{j=1}^{q}(b_j)_{n,k}\right]^{-1} = k^{p-q}\prod_{i=1}^{p}\left(\frac{a_i}{k}\right)_n\left[\prod_{j=1}^{q}\left(\frac{b_j}{k}\right)_n\right]^{-1} \tag{4.2}$$

Generally, all obtained formulae and expressions, generically denoted by $\mathcal{F}_k$ for the *k*-BG-CSs, at the limit $k \to 1$ tend to the corresponding formulae and expressions $\mathcal{F}_1$ for the usual generalized hypergeometric coherent states GH-CSs, while for the *harmonic limit* $k \to 1$ and $b \to 0$ we recover formulae and expressions $\mathcal{F}_0$ for the case of canonical CSs, characteristic for one-dimensional harmonic oscillator:

$$\lim_{k \to 1} \mathcal{F}_k = \mathcal{F}_1 , \lim_{\substack{k \to 1 \\ b \to 0}} \mathcal{F}_k = \mathcal{F}_0 . \tag{4.3}$$

Moreover, one of the consequences of the k-BG-CSs approach is the deduction of some new formulae involving the *k*-functions, namely, the Pochhammer *k*-symbols, as well as the *k*-hypergeometric functions. We insert here only one example:

$$\begin{aligned} &\int \frac{d^2 z}{2\pi} \; e^{-A|z|^2} \left(|z|^2\right)^{B-1} {}_pF_{q,k}\left(\left\{(a_i \;, \quad k)\right\}_{i=1}^{p} \; ; \; \left\{(b_j \;, \; k)\right\}_{j=1}^{q} ; \; z \; \frac{x}{k} \right) \; {}_uF_{v,k}\left(\left\{(c_l \;, \quad k)\right\}_{l=1}^{u} \; ; \; \{(\right. \\ &= k \; \frac{\Gamma(kB)}{(kA)^B} \; {}_{p+u+1}F_{q+v+1,k}\left(\left\{(a_i \;, \quad k)\right\}_{i=1}^{p} \;, \; \left\{(c_l \;, \quad k)\right\}_{l=1}^{u} , \; (kB, \; k) \; ; \; \left\{(b_j \;, \; k)\right\}_{j=1}^{q} \;, \; \{(\mathrm{d}_m \;, \; \right. \end{aligned} \tag{4.4}$$

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