THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 18, Number 4/2017, pp. 337–344

NEGATIVE-ORDER FORMS FOR THE CALOGERO-BOGOYAVLENSKII-SCHIFF EQUATION AND THE MODIFIED CALOGERO-BOGOYAVLENSKII-SCHIFF EQUATION

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Abstract. In this work we develop the negative-order Calogero-Bogoyavlenskii-Schiff (CBS) equation and the negative-order modified CBS equation. By means of the recursion operators of the Korteweg-de Vries (KdV) equation and the modified KdV equation, we derive negative-order forms for the two CBS equations. We formally derive multiple soliton solutions for the newly developed equations. We compare the results of the standard CBS equation and the modified CBS equation with the results for the negative-order versions of these equations.

Key words: Calogero-Bogoyavlenskii-Schiff (CBS) equation, modified CBS equation, inverse recursion operator, multiple soliton solutions.

1. INTRODUCTION

The recursion operator plays a significant role in the field of integrable equations in (1+1) dimensions. A recursion operator is defined as an integro-differential operator that maps a generalized symmetry of a nonlinear partial differential equation to a new symmetry [1–10]. The recursion operator, as developed by Olver [4] and others, for any nonlinear evolution equation indicates that this equation has infinitely many higher-order symmetries, which is a key feature of its complete integrability. Olver [4] reported that the recursion operator maps a symmetry to a new symmetry. However, Magri [6] revealed that some systems admitted two distinct but compatible Hamiltonian structures, now known as bi-Hamiltonian system.

The hereditary symmetry $\Phi(u(x,t))$ is a recursion operator of the following hierarchy of evolution equations

$$u_t + \Phi(u)u_x = 0. \tag{1}$$

It is obvious that this equation gives rise to a variety of (1+1)-dimensional equations depending on the structure of $\Phi(u)$. The recursion operator $\Phi(v)$ for the KdV equation is given as

$$\Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}, \tag{2}$$

where ∂_x and ∂_x^{-1} denote the total derivative and its integration operator with respect to x, respectively. Using the recursion operator (2) gives the celebrated KdV equation as [11–25]

$$v_t + 6vv_x + v_{xxx} = 0, \tag{3}$$

which includes the nonlinear term vv_x and the dispersion term v_{xxx}.

However, the recursion operator for the modified KdV (mKdV) equation takes the form

$$\Phi_1(v) = \partial_x^2 + 4v^2 + 4v_x\partial_x^{-1}(\cdot v), \tag{4}$$

which gives the mKdV equation

$$v_t + 6v^2v_x + v_{xxx} = 0. \tag{5}$$

The last term in (4) is the operator that takes a polynomial $P \in R\{v\}$, multiplies it by v, then applies the D^{-1}, and finally multiplies the result by $4v_x$ [4]. The concept of the recursion operator (4) was thoroughly used in the literature, in particular in [1–10] to develop new equations in higher dimensions.

However, the Calogero-Bogoyavlenskii-Schiff (CBS) equation was first constructed by Bogoyavlenskii where the modified Lax formalism was used. It was also derived by Schiff by reducing the self-dual Yang–Mills equation [1–9].

We use the following hierarchy of evolution equations

$$v_t + \Phi(v)v_y = 0, \tag{6}$$

where Φ is the recursion operator of the KdV equation (2), and v_x in (1) is replaced by v_y. This in turn gives the CBS equation

$$v_t + v_{xxy} + 4vv_y + 2v_x\partial_x^{-1}v_y = 0, \tag{7}$$

or equivalently

$$u_{xt} + u_{xxxy} + 4u_xu_{xy} + 2u_{xx}u_y = 0, \tag{8}$$

obtained by using the potential $v = u_x$.

On the other hand, we use the following hierarchy of evolution equations

$$v_t + \Phi_1(v)v_y = 0, \tag{9}$$

where Φ_1 is the recursion operator of the modified KdV equation (4), and v_x in (1) is replaced by v_y. This in turn gives the modified Calogero-Bogoyavlenskii-Schiff equation

$$v_t + v_{xxy} + 4v^2v_y + 4v_x\partial_x^{-1}(v\cdot v_y) = 0. \tag{10}$$

Olver [4] proved a general theorem about recursion operators for symmetries of an evolution equation, where it was shown that such an operator creates a new symmetry generator when applied to a known symmetry generator. However, Verosky [8] extended the work of Olver in the negative direction to obtain a sequence of equations of increasingly negative orders. Recall that (1) indicates

$$v_t = -\Phi(v_x). \tag{11}$$

By the negative order hierarchy, we refer to

$$v_t = -\Phi^{-1}v_x, \tag{12}$$

i.e. the powers of ϕ go to the opposite direction [7–9]. In other words, the negative order equation can be denoted by

$$\Phi(v_t) = -v_x, \tag{13}$$

and similarly for the modified version we use

$$\Phi_1(v_t) = -v_x. \tag{14}$$

The determination of exact solutions, especially for integrable equations, offers a rich knowledge of the physical behavior and dynamical phenomena of the examined nonlinear equations. Examples of the methods used are the Painlevé analysis [1–9], the inverse scattering method, Lax pairs, and many others.

The goals of this work are two fold. In part I, we aim to establish negative-order equation for the CBS equation (8), and to compare the soliton solutions of the standard CBS equation with its negative-order form. In part II, we develop a negative-order form for the modified CBS equation, which will be derived as well. We will compare the results of the modified CBS equation with the results of the negative-order form of it. We plan to derive multiple soliton solutions for the newly developed equations and to compare the obtained results with the results of the standard CBS and the modified CBS equations.

2. PART I

In this part, we will briefly review the obtained results for the CBS equation We then will move to derive the negative-order CBS equation. We will compare the soliton results of each equation.

2.1. The standard CBS equation

As stated earlier, we will summarize the results we obtained before in [3] when we examined the CBS equation

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx}u_y = 0. \tag{15}$$

In what follows, we only list the obtained results in [3].The single soliton solutions was found to be

$$u(x,y,t) = \frac{2k_1 e^{k_1 x + r_1 y - k_1^2 r_1 t}}{1 + e^{k_1 x + r_1 y - k_1^2 r_1 t}}, \tag{16}$$

where the dispersion relation was derived as $c_1 = k_1^2 r_1$, and the solution $u(x,y,t)$ was assumed as $u(x,y,t) = 2(\ln f(x,y,t))_x$, and the auxiliary function was set as

$$f(x,y,t) = 1 + e^{\theta_1}, \tag{17}$$

where the dispersion variable θ_i is given by

$$\theta_i = k_i x + r_i y - k_i^2 r_i t, \quad i = 1,2,3. \tag{18}$$

In a like manner, we can obtain the two soliton solutions by using the auxiliary function as

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}. \tag{19}$$

where the phase shift a_{12} was found as

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{20}$$

The three soliton solutions can be obtained by using the auxiliary function as

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}. \tag{21}$$

For more information about these results, please look at Ref. [3] and some of the references therein.

2.2. The negative-order CBS equation

In this section we aim to develop the negative-order CBS equation. We first use the negative order hierarchy

$$\Phi(v_t) = -v_y, \tag{22}$$

where the recursion operator Φ for the CBS equation is the same recursion operator for the KdV equation given in (2). In other words, we use

$$\left(\partial_x^2 + 4v + 2v_x \partial_x^{-1}\right)(v_t) = -v_y, \tag{23}$$

which gives

$$v_{xxt} + 4vv_t + 2v_x \partial_x^{-1}(v_t) = -v_y, \tag{24}$$

or equivalently

$$u_{xxxt}+4u_xu_{xt}+2u_{xx}u_t+u_{xy}=0, \tag{25}$$

obtained upon using the potential $v=u_x$. Next we will study the multiple soliton solutions for the negative-order CBS equation (25).

2.2.1. Multiple soliton solutions

To determine the dispersion relation, we set

$$u(x,y,t)=R\big(\ln f(x,y,t)\big)_x, \tag{26}$$

where the auxiliary function $f(x,y,t)$ is given as

$$f(x,y,t)=e^{k_ix+r_iy-c_it},\quad i=1,2,3, \tag{27}$$

for the single soliton solution. Substituting (26) into the negative-order CBS equation (25) gives the dispersion relation by

$$c_i=\frac{r_i}{k_i^2}, i=1,2,3, \tag{28}$$

and

$$R=2. \tag{29}$$

and therefore we set the phase variable as

$$\theta_i(x,y,t)=k_ix+r_iy-\frac{r_i}{k_i^2}t,\quad i=1,2,3. \tag{30}$$

Using (26) gives the single soliton solution

$$u(x,y,t)=\frac{2k_1e^{k_1x+r_1y-\frac{r_1}{k_1^2}t}}{1+e^{k_1x+r_1y-\frac{r_1}{k_1^2}t}}, \tag{31}$$

where the solution of the negative-order CBS equation (23) is obtained by using the potential $v(x,y,t)=u_x(x,y,t)$.

For the two soliton solutions we set the auxiliary function as

$$f(x,y,t)=1+e^{\theta_1(x,y,t)}+e^{\theta_2(x,y,t)}+a_{12}e^{\theta_1(x,y,t)+\theta_2(x,y,t)}, \tag{32}$$

where a_{12} is the phase shift and $\theta_i(x,t),\ i=1,2,3$ is given in (30). Substituting (32) and (26) into the negative-order CBS equation (25) we obtain the phase shift by

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{33}$$

and hence we set the phase shifts by

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2},\quad 1\le i<j\le 3. \tag{34}$$

Notice that the negative-order CBS equation does not show resonance because $a_{12}\neq 0$ or ∞ for $|k_1|\neq|k_2|$. Combining (32) and (33) and substituting the outcome into (26), we obtain the two soliton solutions.

For the three soliton solutions, we set

$$f(x,y,t)=1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{23}e^{\theta_2+\theta_3}+a_{13}e^{\theta_1+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{35}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{36}$$

This shows that the three soliton solutions are obtainable. The existence of three soliton solutions often indicates the complete integrability of the equation under examination. However, other criteria, such as Lax pair should be used to confirm integrability.

We conclude from the results obtained above that the CBS equation and the negative-oder CBS equation have distinct solitons. This is due to the distinct dispersion relations. The dispersion relations for the CBS equation and the negative-order CBS equation were derived as $r_ik_i^2, i = 1,2,3$ and $r_i / k_i^2, i = 1,2,3$, respectively. However, the two equations give multiple soliton solutions where the phase shifts of the interaction of solitons are identical and both do not show resonance.

3. PART II

In a parallel manner to the analysis presented in part I, we plan to conduct a comparative study of the modified CBS equation and the negative-order modified CBS equation, which we will derive later.

3.1. The modified CBS equation

The modified CBS equation was already derived in (10) and is given as

$$v_t + v_{xxy} + 4v^2v_y + 4v_x\partial_x^{-1}(v \cdot v_y) = 0. \tag{37}$$

We first use

$$\psi_1(x,y,t) = \partial^{-1}(v \cdot v_y) = -\left(\frac{v_t + v_{xxy} + 4v^2v_y}{4v_x}\right), \tag{38}$$

which carries (37) to

$$v_t + v_{xxy} + 4v^2v_y + 4v_x\psi_1(x,y,t) = 0. \tag{39}$$

Differentiating (39) with respect to x and using (38) we obtain

$$v_{xt} + v_{xxxy} + 4v^2v_{xy} + 12vv_xv_y + 4v_{xx}\psi_1 = 0. \tag{40}$$

3.2. Multiple kink solutions

To determine the dispersion relation, we set

$$v(x,y,t) = R\left(\arctan f(x,y,t)\right)_x, \tag{41}$$

where the auxiliary function $f(x,y,t)$ is given as

$$f(x,y,t) = e^{k_ix + r_iy - c_it}, \quad i = 1,2,3, \tag{42}$$

for the single soliton solution. Substituting (41) into the modified CBS equation (40) gives the dispersion relations by

$$c_i = r_ik_i^2, \quad i = 1,2,3, \tag{43}$$

and

$$R = 2. \tag{44}$$

and therefore we set the phase variable as

$$\theta_i(x,y,t) = k_i x + r_i y - r_i k_i^2 t, \quad i = 1,2,3. \tag{45}$$

Using (41) gives the single kink solution

$$v(x,y,t) = \frac{2k_1 e^{k_1 x + r_1 y - r_1 k_1^2 t}}{1 + e^{2(k_1 x + r_1 y - r_1 k_1^2 t)}}. \tag{46}$$

For the two soliton solutions we set the auxiliary functions as

$$\begin{aligned} f(x,y,t) &= e^{\theta_1} + e^{\theta_2}, \\ g(x,y,t) &= 1 - a_{12} e^{\theta_1 + \theta_2}, \end{aligned} \tag{47}$$

where a_{12} is the phase shift and $\theta_i(x,t),\ \ i = 1,2,3$ is given in (45). Substituting (47) and (41) into the modified CBS equation (40) we obtain the phase shift by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{48}$$

and hence we set the phase shifts by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le 3. \tag{49}$$

Notice that the modified CBS equation does not show resonance because $a_{12} \neq 0$ or ∞ for $|k_1| \neq |k_2|$. Combining (47) and (48) and substituting the outcome into (40), we obtain the two soliton solutions.

For the three soliton solutions, we set

$$\begin{aligned} f(x,y,t) &= \sum_{i=1}^{3} e^{\theta_i} - b_{123} e^{\theta_1 + \theta_2 + \theta_3}, \\ g(x,y,t) &= 1 - a_{12} e^{\theta_1 + \theta_2} - a_{13} e^{\theta_1 + \theta_3} - a_{23} e^{\theta_2 + \theta_3}. \end{aligned} \tag{50}$$

Proceeding as before, we find

$$b_{123} = a_{12} a_{23} a_{13}. \tag{51}$$

This shows that the three kink solutions are obtainable. The existence of three soliton solutions often indicates the complete integrability of the equation under examination but, other criteria, such as Lax pair are needed to confirm integrability,

3.3. The negative-order modified CBS equation

We next develop the negative-order modified CBS equation. Proceeding as before, we use the negative order hierarchy

$$\Phi_1(v_t) = -v_y, \tag{52}$$

where the recursion operator Φ_1 for the modified CBS is the same recursion operator for the modified KdV equation given in (4). In other words, we use

$$\left(\partial_x^2 + 4u^2 + 4u_x \partial_x^{-1}(\cdot v)\right) u_t = -v_y, \tag{53}$$

which gives

$$v_{xxt} + 4v^2 v_t + 4v_x \partial^{-1}(v \cdot v_t) = -v_y. \tag{54}$$

Using

$$\psi(x,y,t) = \partial^{-1}(v \cdot v_t) = -\left(\frac{v_{xxt} + 4v^2 v_t + v_y}{4v_x}\right), \tag{55}$$

carries (54) to

$$v_{xxt} + 4v^2 v_t + 4v_x \psi(x, y, t) = -v_y. \tag{56}$$

Differentiating (56) with respect to x and using (55) we obtain

$$v_{xxxt} + 4v^2 v_{xt} + 12vv_x v_t + 4v_{xx}\psi(x, y, t) + v_{xy} = 0. \tag{57}$$

3.4. Multiple soliton solutions

To determine the dispersion relation, we set

$$v(x, y, t) = R\left(\arctan f(x, y, t)\right)_x, \tag{58}$$

where the auxiliary function $f(x, y, t)$ is given as

$$f(x, y, t) = e^{k_i x + r_i y - c_i t}, \quad i = 1, 2, 3, \tag{59}$$

for the single soliton solution. Substituting (58) into the negative-order modified CBS equation (57) gives the dispersion relations by

$$c_i = \frac{r_i}{k_i^2}, \quad i = 1, 2, 3, \tag{60}$$

and $R = 2.$ Therefore we set the phase variable as

$$\theta_i(x, y, t) = k_i x + r_i y - \frac{r_i}{k_i^2} t, \quad i = 1, 2, 3. \tag{61}$$

Using (58) gives the single soliton solution

$$v(x, y, t) = \frac{2k_1 e^{k_1 x + r_1 y - \frac{r_1}{k_1^2} t}}{1 + e^{2(k_1 x + r_1 y - \frac{r_1}{k_1^2} t)}}. \tag{62}$$

For the two soliton solutions we set the auxiliary functions as

$$\begin{aligned} f(x, y, t) &= e^{\theta_1} + e^{\theta_2}, \\ g(x, y, t) &= 1 - a_{12} e^{\theta_1 + \theta_2}. \end{aligned} \tag{63}$$

where a_{12} is the phase shift and $\theta_i(x, t), i = 1,2,3$ is given in (61). Substituting (63) and (58) into the negative-order CBS equation (56) we obtain the phase shift by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{64}$$

and hence we set the phase shifts by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le 3. \tag{65}$$

Notice that the negative-order CBS equation does not show resonance because $a_{12} \ne 0$ or ∞ for $| k_1 | \ne | k_2 |$. Combining (63) and (64) and substituting the outcome into (58), we obtain the two soliton solutions.

For the three soliton solutions, we set

$$\begin{aligned} f(x, y, t) &= \sum_{i=1}^{3} e^{\theta_i} - b_{123} e^{\theta_1 + \theta_2 + \theta_3}, \\ g(x, y, t) &= 1 - a_{12} e^{\theta_1 + \theta_2} - a_{13} e^{\theta_1 + \theta_3} - a_{23} e^{\theta_2 + \theta_3}. \end{aligned} \tag{66}$$

Proceeding as before, we find $b_{123} = a_{12} a_{23} a_{13}.$ This shows that the three kink solutions are obtainable.

We conclude from the results obtained above that the modified CBS equation and the negative-oder modified CBS equation have distinct kink solutions. This is due to the distinct dispersion relations. The dispersion relations for the modified CBS equation and the negative-order modified CBS equation were derived as $r_i k_i^2, i=1,2,3$ and $r_i / k_i^2, i=1,2,3$, respectively. However, the two equations have multiple kink solutions where the phase shifts of the interaction of solitons are identical and both do not show resonance.

4. DISCUSSION

We used the inverse of the recursion operators for the KdV equation and the mKdV equation, given in (2) and (4), respectively, to formally derive the negative-order CBS equation and the negative-order modified CBS equation. We have shown that the newly derived equations possess multiple soliton and multiple kink solutions. We determined the dispersion relation for each equation, and showed that the phase shifts are of the Hirota's type. Moreover, we showed that the CBS equations and their negative-order versions have distinct solutions due to the occurrence of distinct dispersion relations.

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Received May 15, 2017