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APROXIMATE ANALYTICAL SOLUTIONS FOR THIN FILM FLOW OF A FOURTH GRADE FLUID DOWN A VERTICAL CYLINDER

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Abstract: In this paper, we employ an approximate analytical method, namely optimal auxiliary functions method (OAFM), to investigate a thin film flow of a fourth grade fluid dawn a long vertical cylinder. Our solutions are compared with those obtained by numerical integration. OAFM does not depend upon large or small parameters and assures the convergence of the approximate solutions after only one iteration. OAFM is very efficient and effective.

Key words: optimal auxiliary functions method, nonlinear differential equation, optimal parameters, thin film flow.

1. INTRODUCTION

The flow of the non-Newtonian fluids is very important for physicians, applied mathematicians and engineers, because of its several applications in various fields of science and engineering. In the last few decades, these fluids have attracted considerable attention from researchers in many branches of nonlinear dynamical systems in science and technology. Examples include the pioneering work of Sakiadis [1] who investigated the flow of a viscous fluid past a moving solid surface or Chen [2] who studied mixed convection flow over a stretching surface. Zhang and Li [3] analyzed the thin film flow of the third grade fluid, Elahi and Riaz [4] investigated the non-Newtonian MHD flow with variable viscosity in a third grade fluid. Siddiqui *et al*. [5] applied homotopy perturbation method and traditional perturbation method to obtain analytic approximations to thin fluid flow of a fourth grade fluid on the outer of a long vertical cylinder. Also, Sajid *et al*. [6] discussed the steady flow of a fourth grade fluid past a porous plate and Hayat and Sajid [7] developed a series solution for the same subject. Mabood [8] analyzed the thin film flow of an incompressible third grade fluid down on an inclined plane. There are a lot of other features such as, time-dependency, history effects, other non-linear issues, yield stress, and so on [9–14].

Most physical problems are nonlinear but the linear analysis is often insufficient to describe the behavior of physical systems adequately. An exact solution for nonlinear systems is often scarce at least at the present state of knowledge. In this respect, new and innovative approaches capable to solve nonlinear dynamical systems should be known. Recently some fruitful results have been obtained for solving various nonlinear problems. There exist some analytical approaches applicable to nonlinear problems such as weighted linearization method [15], the Lindstedt-Poincare method [16], Adomian decomposition method [17], the boundary element method [18], the optimal homotopy perturbation method [13], the optimal homotopy asymptotic method [14], and so on. All of the above mentioned methods work very well for weakly nonlinear problems and some of them work well even for strongly nonlinear systems. It is very important in the case of strongly nonlinear problems to ensure the condition of convergence of the solutions.

In present work we apply OAFM to solve a boundary value problem for nonlinear differential equation of thin film flow down a vertical cylinder. The fluid used here is of fourth grade which introduces strongly nonlinearities in the study of the problem.

2. THE GOVERNING EQUATION

The fluid considered in this paper is an incompressible, of fourth grade fluid falling on the outside surface of an infinitely long vertical cylinder of radius R. The flow is in the form of a thin, uniform and axisymmetric film of thickness δ, in contact with stationary air. In cylindrical coordinates, we shall seek a velocity field of the form [5, 9, 14].

$$V=\left[0,0,u(r)\right] \tag{1}$$

If $p=p\ (z)$ is the pressure, the r-component, θ-component and z-component of momentum can be written as

$$\frac{\partial p}{\partial r}=\left(2\alpha_1+\alpha_2\right)\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right]+\frac{4}{r}\left(\gamma_1+\gamma_2+\gamma_3+\frac{\gamma_4}{2}\right)\frac{\partial}{\partial r}\left[r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4\right] \tag{2}$$

$$\frac{\partial p}{\partial\theta}=0 \tag{3}$$

$$\frac{\partial p}{\partial z}=\frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)+\frac{2}{r}\left(\beta_1+\beta_2\right)\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3\right] \tag{4}$$

respectively, where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\gamma_3$ and γ_4 are material constants. Supposing that there is no pressure gradient in z direction, Eq. (4) becomes [5].

$$r\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{2\left(\beta_1+\beta_2\right)}{\mu}\left[3r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3\right]+\frac{\rho g}{\mu}r=0. \tag{5}$$

The boundary conditions are

$$u(R)=0,\quad \frac{\mathrm{d}u}{\mathrm{d}r}\Big|_{r=R+\delta}=0. \tag{6}$$

Defining

$$\eta=\frac{r}{\mathrm{R}},\ f=\frac{R}{\mu}u,\ d=1+\frac{\delta}{R},\ k=\frac{gR^4}{\mu^2};\ b=\frac{\mu\left(\beta_1+\beta_2\right)}{R^4\rho^2}, \tag{7}$$

the boundary value problem (5) and (6) reduces to nonlinear differential equation

$$\eta\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}+\frac{\mathrm{d}f}{\mathrm{d}\eta}+k\eta+2b\left[\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^3+3\eta\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}\right]=0 \tag{8}$$

$$f(1)=0,\ f'(d)=0. \tag{9}$$

3. BASIC IDEAS OF OAFM

Equation (8) with the boundary conditions (9) can be written in a more general form:

$$L\left[f(\eta)\right]+g(\eta)+N\left[f(\eta)\right]=0, \tag{10}$$

where L is a linear operator, g is a known function and N is a nonlinear operator, subject to the boundary condition

$$B\left(f(\eta),\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)=0. \tag{11}$$

In order to obtain an approximate solution of Eqs. (10) and (11) we assume that the approximate solution can be expressed in the form with two components:

$$\bar{f}(\eta)=f_0(\eta)+f_1(\eta,C_i),\ \ i=1,2,3,\ldots,p \tag{12}$$

where the initial and the first approximation will be determined as follows. Substituting Eq. (12) into Eq. (10), it results in

$$L\left[f_0(\eta)\right]+\mathrm{L}\left[f_1(\eta,C_i)\right]+g(\eta)+N\left[f_0(\eta)+f_1(\eta,C_i)\right]=0. \tag{13}$$

The initial approximation $f_0(\eta)$ can be obtained from the following linear equation:

$$L\left[f_0(\eta)+g(\eta)\right]=0,\ B\left[f_0(\eta),\ \frac{\mathrm{d}f_0(\eta)}{\mathrm{d}n}\right]=0 \tag{14}$$

and the first approximation from the equation

$$L\left[f_1(\eta,C_i)\right]+N\left[f_0(\eta)+f_1(\eta,C_i)\right]=0,\quad B\left[u_1,\frac{\mathrm{d}u_1}{\mathrm{d}\eta}\right]=0. \tag{15}$$

Now, the nonlinear term from Eq. (15) is expanded in the form

$$N\left[f_0(\eta)+f_1(\eta,C_i)\right]=N\left[f_0(\eta)\right]+\sum_{k=1}^{\infty}\frac{f_1^k(\eta)}{k!}N^{(k)}\left[f_0(\eta)\right]. \tag{16}$$

To avoid the difficulties that appear in solving of nonlinear differential equation (15) and to accelerate the rapid convergence of the first approximation $f_1(\eta,C_i)$ and implicit of the solution $\bar{f}(\eta)$, instead of the last term arising into Eq. (15), we propose an another expression, such that Eq. (15) can be written as

$$L\left[f_1(\eta,C_i)\right]+A_1\left[f_0(\eta),\ C_i\right]\mathrm{N}\left[f_0(\eta)\right]+A_2\left[f_0(\eta),\ C_j\right]=0,$$

$$B\left[f_1(\eta),\frac{\mathrm{d}f_1(\eta)}{\mathrm{d}\eta}\right]=0, \tag{17}$$

where A_1 and A_2 are two arbitrary auxiliary functions depending of the initial approximation $f_0(\eta)$ and a number of the unknown parameters C_i and C_j, $i = 1, 2,\ldots,s$, $j = s + 1, s + 2,\ldots,p$. The auxiliary functions A_1 and A_2 (named optimal auxiliary functions) are not unique, and are of same form like $f_0(\eta)$, or of the form of $N\left[f_0(\eta)\right]$ or combinations of the forms $f_0(\eta)$ and $N\left[f_0(\eta)\right]$. For example if $f_0(\eta)\left(\text{or } N\left[f_0(\eta)\right]\right)$ a polynomial function then $A_1\left[f_0(\eta),\ C_i\right]$ and $A_2\left[f_0(\eta),\ C_j\right]$ are a sums of polynomial functions; if $f_0(\eta)\left(\text{or } N\left[f_0(\eta)\right]\right)$ is an exponential function, then A_1 and A_2 are a sums of an exponential functions; if $f_0(\eta)\left(\text{or } N\left[f_0(\eta)\right]\right)$ is a trigonometric function, then A_1 and A_2 are a sums of the trigonometric functions, and so on. If in a special case $N\left[f_0(\eta)\right]=0$ then it is clear that $f_0(\eta)$ is an exact solution of Eq. (10). The unknown parameters C_i and C_j can be optimally identified via different methods such as the Galerkin method, the Ritz method, the least square method, the collocation method or by minimizing the square residual error:

$$J\left(C_i,C_j\right)=\int_a^b R^2\left(\eta,C_i,C_j\right)\mathrm{d}\eta, \tag{18}$$

where $R\left(\eta,C_i,C_j\right)=L\left[\overline{f}\left(\eta,C_i,C_j\right)\right]+g\left(\eta\right)+N\left[\overline{f}\left(\eta,C_i,C_j\right)\right]$, $i=1,2,\ldots,s, j=s+1,s+2,\ldots,p$, and so on. By this novel method, the approximate solution (12) is well determined. Our procedure is a powerful tool for solving nonlinear differential problems without depending on small or large parameters. It should be emphasized that our procedure contains the auxiliary functions A_1 and A_2 which provides with a simple way to adjust and control the convergence of the approximate solution after only one iteration.

4. APPROXIMATE SOLUTIONS OF THE THIN FILM FLOW WITH OAFM

In what follows we apply our procedure to obtain an approximate solution of Eqs. (8) and (9). For this purpose, we choose the linear operator, the function g and the nonlinear operator of the following forms:

$$L\left[f\left(\eta\right)\right]=\eta\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}+\frac{\mathrm{d}f}{\mathrm{d}\eta} \tag{19}$$

$$g\left(\eta\right)=k\eta \tag{20}$$

$$N\left[f\left(\eta\right)\right]=2b\left[\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^3+3\eta\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right]. \tag{21}$$

The initial approximation f_0 is obtained from Eq. (14):

$$\eta\frac{\partial^2 f_0}{\partial\eta^2}+\frac{\partial f_0}{\partial\eta}+k\eta=0,\ f_0\left(1\right)=0,\ \frac{\partial f_0}{\partial\eta}\left(d\right)=0. \tag{22}$$

The solution of Eq. (22) is

$$f_0\left(\eta\right)=\frac{k}{4}\left(1+d^2\ln\eta^2-\eta^2\right). \tag{23}$$

Substituting Eq. (23) into Eq. (21), the nonlinear operator becomes

$$N\left[f_0\left(\eta\right)\right]=-\frac{1}{2}bk^3\left(\frac{d^2}{\eta}-\eta\right)^2\left(\frac{d^2}{\eta}+2\eta\right). \tag{24}$$

The first approximation f_1 is given by Eq. (17):

$$\eta\frac{\mathrm{d}^2 f_1}{\mathrm{d}\eta^2}+\frac{\mathrm{d}f_1}{\mathrm{d}\eta}+A_1\left[f_0\left(\eta\right),\ C_i\right]N\left[f_0\left(\eta\right)\right]+A_2\left[f_0\left(\eta\right),\ C_j\right]=0,$$

$$f_1\left(1\right)=0,\ \frac{\mathrm{d}f_1}{\mathrm{d}\eta}\left(d\right)=0. \tag{25}$$

Taking into account that

$$\eta\frac{\mathrm{d}^2 f_1}{\mathrm{d}\eta^2}+\frac{\mathrm{d}f_1}{\mathrm{d}\eta}=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f_1}{\mathrm{d}\eta}\right), \tag{26}$$

$$-\left(\frac{d^2}{\eta}-\eta\right)^2\left(\frac{d^2}{\eta}+2\eta\right)=\frac{d}{d\eta}\left[\eta\left(\frac{d^2}{\eta}-\eta\right)^3\right], \tag{27}$$

then Eq. (25) may be written as

$$\frac{d}{d\eta}\left(\eta\frac{df_1}{d\eta}\right)+A_1\left[f_0(\eta),\, C_i\right]\frac{1}{2}bk^3\frac{d}{d\eta}\left[\eta\left(\frac{d^2}{\eta}-\eta\right)^3\right]+A_2\left[f_0(\eta),\, C_j\right]=0. \tag{28}$$

We have freedom to choose

$$A_1\left[f_0(\eta),\, C_i\right]=-\frac{2}{bk^3}C_3 \tag{29}$$

$$A_2\left[f_0(\eta),\, C_i\right]=-\frac{d}{d\eta}[C_1\eta\left(\frac{d^2}{\eta}-\eta\right)+C_2\eta\left(\frac{d^2}{\eta}-\eta\right)^2+C_4\eta\left(\frac{d^2}{\eta}-\eta\right)^4+$$
$$+C_5\eta\left(\frac{d^2}{\eta}-\eta\right)^5+\ldots+C_p\eta\left(\frac{d^2}{\eta}-\eta\right)^p, \tag{30}$$

where p is an arbitrary integer positive number. Because A_1 and A_2 are not unique, we can choose and the following functions:

$$A_1\left[f_0(\eta),\, C_i\right]=-\frac{2C_1}{bk^3}\left(\frac{d^2}{\eta}-\eta\right)\frac{\frac{3d^2}{\eta}+5\eta}{\frac{d^2}{\eta}+2\eta} \tag{31}$$

$$A_2\left[f_0(\eta),\, C_i\right]=-\frac{d}{d\eta}[C_2\eta\left(\frac{d^2}{\eta}-\eta\right)^2+C_3\eta\left(\frac{d^2}{\eta}-\eta\right)^6+C_4\eta\left(\frac{d^2}{\eta}-\eta\right)^8+$$
$$+C_5\eta\left(\frac{d^2}{\eta}-\eta\right)^{10}+\ldots+C_p\eta\left(\frac{d^2}{\eta}-\eta\right)^{2p} \tag{32}$$

and so on.

Using only the expression (29) and (30) of the auxiliary functions, Eq. (28) can be written as

$$\frac{d}{d\eta}\left(\eta\frac{df_1}{d\eta}\right)-\frac{d}{d\eta}[C_1\eta\left(\frac{d^2}{\eta}-\eta\right)+C_2\eta\left(\frac{d^2}{\eta}-\eta\right)^2+C_3\eta\left(\frac{d^2}{\eta}-\eta\right)^3+\ldots+$$
$$+C_p\eta\left(\frac{d^2}{\eta}-\eta\right)^p]=0,\quad f_1(1)=\frac{df_1}{d\eta}(d)=0. \tag{33}$$

From Eq. (33) we find the following solution:

$$\frac{df_1}{d\eta}=C_1\left(\frac{d^2}{\eta}-\eta\right)^2+C_2\left(\frac{d^2}{\eta}-\eta\right)^3+\ldots+C_p\left(\frac{d^2}{\eta}-\eta\right)^p,\quad f_1(1)=\frac{df_1}{d\eta}(d)=0. \tag{34}$$

Solving Eq. (34), and then substituting this solution and Eq. (23) into Eq. (12), we obtain approximate solution of Eqs. (8) and (9) by OAFM in the form

$$\bar{f}(\eta,C_i)=\left(\frac{k}{2}+C_1\right)\left(d^2\ln\eta+\frac{1}{2}-\frac{1}{2}\eta^2\right)-C_2\left(\frac{d^4}{\eta}+2d^2\eta-\frac{1}{3}\eta^3\right)-$$
$$-C_3\left(\frac{d^6}{2\eta^2}+3d^4\ln\eta-\frac{3}{2}d^2\eta^2+\frac{1}{4}\eta^4\right)-C_4\left(\frac{d^8}{3\eta^3}-\frac{4d^6}{\eta}-6d^4\eta+\frac{4}{3}d^2\eta^3-\frac{1}{5}\eta^5\right)- \quad (35)$$
$$-C_5\left(\frac{d^{10}}{4\eta^4}-\frac{5d^8}{2\eta^2}-10d^6\ln\eta+5d^4\eta^2-\frac{5}{4}d^2\eta^4+\frac{1}{7}\eta^7\right)+\ldots\ .$$

5. NUMERICAL RESULTS BY OAFM

We illustrate the accuracy of our procedure for different values of the coefficients k and b. Also we will show that the error of the solutions decreases when the number of terms in the auxiliary functions A_1 and A_2 increases. The results obtained using OAFM are compared with numerical integration results by using the fourth-order Runge-Kutta method. Using Eq. (18), the constants C_i can be determined from conditions:

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=\frac{\partial J}{\partial C_3}=\frac{\partial J}{\partial C_4}=\ldots=\frac{\partial J}{\partial C_p}=0 \quad (36)$$

5.1. First, we consider $k = 2$, $b = 2$, $d = 1.2$, $p = 4$. From Eqs. (36) one can get

C_1= 0.02010809507891; C_2= –0.443212061022469;
C_3= –1.46971845415762; C_4= 2.05824705104109.

The approximate velocity with four constants in this case by using OAFM is

$$\frac{\partial\bar{f}(\eta)}{\partial\eta}=1.02010809507891\left(\frac{1.44}{\eta}-\eta\right)-0.443212061022469\left(\frac{1.44}{\eta}-\eta\right)^2-$$
$$-1.46971845415762\left(\frac{1.44}{\eta}-\eta\right)^3+2.05824705104109\left(\frac{1.44}{\eta}-\eta\right)^4. \quad (37)$$

5.2. In this case for same values: $k = 2$, $b = 2$, $d = 1.2$ but $p = 5$, from Eqs. (36) it holds that

C_1= 0.00493140714889; C_2= – 0.0865225298529617;
C_3= – 4.13589639137958; C_4= 9.72574923625005
C_5= – 7.43698840812736.

and velocity becomes

$$\frac{\partial\bar{f}(\eta)}{\partial\eta}=1.00493140714889\left(\frac{1.44}{\eta}-\eta\right)-0.0865225298529617\left(\frac{1.44}{\eta}-\eta\right)^2-$$
$$-4.13589639137958\left(\frac{1.44}{\eta}-\eta\right)^3+9.72574923625005\left(\frac{1.44}{\eta}-\eta\right)^4- \quad (38)$$
$$-7.43698840812736\left(\frac{1.44}{\eta}-\eta\right)^5.$$

It is easy to verify the accuracy of the obtained solutions if we compare these analytical results with numerical ones. From Tables 1 and 2 it can be seen that the analytical solutions of our problem obtained by OAFM are very accurate. The examples presented in this section lead to the very important conclusion that the accuracy of the obtained results is growing along with increasing the number of constants in the auxiliary functions. Our approach does not depend upon small parameters (in Ref. [7] is considered that $b \geq 0.3$ is a parameter corresponding to strong nonlinearity).

Table 1

Comparison between the OAFM solutions (37) and (38) with numerical solutions for $k = 2$, $b = 2$, $d = 1.2$, $p = 4$ (Eq. (37)) and $p = 5$ (Eq. (38))

η	Numerical solution	$\bar{f}(\eta)$ Eq. (37)	$\bar{f}(\eta)$ Eq. (38)	Error of Eq. (37)	Error of Eq.(38)
1	0.3149892726	0.3149902864	0.3149892725	$1.01*10^{-6}$	$3.66*10^{-7}$
1.02	0.2920872677	0.2917634189	0.2922025741	$3.25*10^{-4}$	$1.15*10^{-4}$
1.04	0.2677958645	0.2677841234	0.2677958741	$6.89*10^{-6}$	$9.48*10^{-9}$
1.06	0.2418828814	0.2422461104	0.2418448028	$3.63*10^{-4}$	$3.81*10^{-5}$
1.08	0.2140851463	0.2145652735	0.2140852072	$4.80*10^{-4}$	$6.12*10^{-8}$
1.1	0.1841229205	0.1844175114	0.1841492536	$2.94*10^{-4}$	$2.63*10^{-5}$
1.12	0.1517392282	0.1517400487	0.1517391641	$8.20*10^{-7}$	$6.4*10^{-8}$
1.14	0.1167864570	0.1166398160	0.1167536690	$1.46*10^{-4}$	$3.27*10^{-5}$
1.16	0.0793786596	0.0793792588	0.0793786281	$5.96*10^{-7}$	$3.14*10^{-8}$
1.18	0.0400814166	0.0403378899	0.0401505928	$2.56*10^{-4}$	$6.91*10^{-5}$
1.2	0	0	0	0	0

5.3. For $k = 1$, $b = 2$, $d = 1.2$, $p = 4$ we obtain

$$C_1 = 0.00022854018451;\quad C_2 = -0.00216198649445;$$
$$C_3 = -0.53701336395192;\quad C_4 = 0.47615841198521.$$

$$\frac{\partial \bar{f}(\eta)}{\partial \eta} = 0.50022854018451\left(\frac{1.44}{\eta}-\eta\right) - 0.00216198649445\left(\frac{1.44}{\eta}-\eta\right)^2 - -0.537013363951926\left(\frac{1.44}{\eta}-\eta\right)^3 + 0.476158411985218\left(\frac{1.44}{\eta}-\eta\right)^4. \tag{39}$$

5.4. In the last case, for $k = 1$, $b = 2$, $d = 1.2$ but $p = 5$ we have

$$C_1 = -0.00056524385115;\quad C_2 = 0.01651354436441;$$
$$C_3 = -0.676547804124;\quad C_4 = 0.87717256349806;$$
$$C_5 = -0.38869164165841.$$

$$\frac{\partial \bar{f}(\eta)}{\partial \eta} = 0.49943475061489\left(\frac{1.44}{\eta}-\eta\right) + 0.01651354436441\left(\frac{1.44}{\eta}-\eta\right)^2 - -0.676547804124\left(\frac{1.44}{\eta}-\eta\right)^3 + 0.87717256379806\left(\frac{1.44}{\eta}-\eta\right)^4 \tag{40}$$
$$-0.38869164165841\left(\frac{1.44}{\eta}-\eta\right)^5.$$

Table 2

Comparison between the OAFM solution (39) and (40) with numerical solutions for $k = 1$, $b = 2$, $d = 1.2$, $p = 4$(Eq. (39)) and $p = 5$(Eq. (40))

η	Numerical solution	$\bar{f}(\eta)$ Eq. (39)	$\bar{f}(\eta)$ Eq. (40)	Error of Eq. (39)	Error of Eq. (40)
1	0.1917839251	0.1917878034	0.1917843781	$9.41*10^{-5}$	$4.53*10^{-7}$
1.02	0.1745937891	0.1745669533	0.1745912673	$2.68*10^{-5}$	$2.52*10^{-6}$
1.04	0.1568673294	0.1568678463	0.156867684	$5.64*10^{-6}$	$3.55*10^{-7}$
1.06	0.1385961651	0.138557774	0.1385972549	$3.83*10^{-5}$	$1.11*10^{-6}$
1.08	0.1197907602	0.1198159933	0.1197910245	$2.52*10^{-5}$	$2.64*10^{-7}$
1.1	0.10048675874	0.1004998341	0.1004859761	$1.30*10^{-5}$	$7.88*10^{-7}$

(continued)

1.12	0.0807509285	0.0807519952	0.0807510956	$1.07*10^{-6}$	$7.67*10^{-7}$
1.14	0.0606850155	0.0606806934	0.0606866534	$4.30*10^{-6}$	$1.66*10^{-6}$
1.16	0.0404254003	0.0404257002	0.0404254824	$3*10^{-7}$	$8.20*10^{-8}$
1.18	0.0201368303	0.0201799364	0.0201313916	$4.31*10^{-5}$	$5.43*10^{-6}$
1.2	0	0	0	0	0

6. CONCLUSIONS

In this work, we introduce a new method (OAFM) to propose analytic approximate solutions to the thin film flow of a fourth grade fluid down a long vertical cylinder. Our procedure is valid even if the nonlinear equation does not contain small or large parameters.

The proposed construction of the OAFM is different from any other approaches especially referring to the auxiliary functions A_1 and A_2, and the presence of some parameters C_1, C_2... which ensure a very rapid convergence of the solutions. The results obtained by OAFM are growing along with increasing of the number of the parameters C_i. The OAFM provides us with simple but rigorous way to control and adjust the convergence of the solutions through several convergence – control parameters C_i which are optimally determined. It should be emphasized that very good approximations are obtained after only one iteration and in only a few terms. Optimal auxiliary functions method is effective, efficient and easy to use.

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