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THE FRACTIONAL MODEL OF SPRING PENDULUM: NEW FEATURES WITHIN DIFFERENT KERNELS

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Abstract. In this work, new aspects of the fractional calculus (FC) are examined for a model of spring pendulum in fractional sense. First, we obtain the classical Lagrangian of the model, and as a result, we derive the classical Euler-Lagrange equations of the motion. Second, we generalize the classical Lagrangian to fractional case and derive the fractional Euler-Lagrange equations in terms of fractional derivatives with singular and nonsingular kernels, respectively. Finally, we provide the numerical solution of these equations within two fractional operators for some fractional orders and initial conditions. Numerical simulations verify that taking into account the recently features of the FC provides more realistic models demonstrating hidden aspects of the real-world phenomena.

Key words: Spring pendulum, Euler-Lagrange equation, fractional derivative, nonsingular kernel.

1. INTRODUCTION

It is well known from the literature that the equation of motion can be obtained from an energy based approach. This method (*i.e.* Lagrangian) depends on knowing how to write the kinetic energy of a system as well as its potential energy. As many interesting systems include springs, one has to know how to determine the above-mentioned energies associated to the systems with springs. For more details about such examples, one may refer to some textbooks in classical mechanics [1, 2]. The equation of motion obtained by Lagrangian method contains derivatives of generalized coordinates, and these equations are then solved by using differential equations rules. Some initial conditions are then applied to the solution to determine some constants.

The history of fractional calculus (FC) refers back to the year 1695. In recent years, considerable interest in FC has been stimulated due to its wide applications in different branches of applied sciences like physics and engineering [3–12]. Recently, the FC plays an important rule to solve fractional differential equations (FDEs). Many papers have been published on this topic; see for example [13–16]. The FDEs involve left and right fractional derivatives, so one should have a solid background on the FC.

As it is known, using the Lagrangian technique yields some differential equations, called Euler-Lagrange equations, which need to be solved by applying some initial conditions. In our case, we generalize the classical Lagrangian by using the fractional derivatives. As a result, we obtain the FDEs rather than the classical ones. These equations, namely the fractional Euler-Lagrange equations (FELEs), cannot be solved analytically so easily; hence, numerical techniques are used to solve them [17–21]. However, the numerical solution of FELEs for the spring pendulum problem will exhibit new hidden features of this system. From mathematical and practical points of view, the FELEs are new, and their solutions include more information than their integer counterparts. Hence, by modelling the classical Lagrangian of the spring pendulum system with fractional derivatives, we can prepare a realistic model describing the spring pendulum corresponding to the new FELEs. This point can be considered as one of the main utilities of the FC against the classical one, because using the fractional differential operators enables us to build a new real-world phenomenon, without breaking any existing rules of the classical calculus approach.

The outline of this paper follows here. Section 2 discusses the fractional derivatives definitions. In Sec. 3, the classical and fractional studies are carried out for the spring pendulum model. Section 4 provides the numerical solution of the derived FELEs for different values of fractional order and initial conditions. Finally, we close the paper by a conclusion in Section 5.

2. DEFINITIONS

In this Section, we give some basic definitions of the fractional derivatives in the sense of classic Caputo [5] and a new one with Mittag-Leffler nonsingular kernel (ABC) [22].

Definition 2.1 [22]. Let $f:[a,b]\rightarrow R$ be a time-dependent function and $0<\alpha\leq 1$. Then, the left and right Caputo fractional derivatives are defined as

$$ {}_{a}^{C}D_{t}^{\alpha} f = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\dot{f}(x)(t-x)^{-\alpha}\,\mathrm{d}x, \tag{1}$$

$$ {}_{t}^{C}D_{b}^{\alpha} f = -\frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}\dot{f}(x)(t-x)^{-\alpha}\,\mathrm{d}x, \tag{2}$$

respectively, where $\Gamma(\cdot)$ denotes the Euler's Gamma function.

Definition 2.2 [22]. For $f\in H^{1}(a,b)$ and $0<\alpha\leq 1$, the left and right ABC fractional derivatives are defined as

$$ {}_{a}^{ABC}D_{t}^{\alpha} f = \frac{B(\alpha)}{1-\alpha}\int_{a}^{t}\dot{f}(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)\mathrm{d}x, \tag{3}$$

$$ {}_{t}^{ABC}D_{b}^{\alpha} f = -\frac{B(\alpha)}{1-\alpha}\int_{t}^{b}\dot{f}(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)\mathrm{d}x, \tag{4}$$

respectively, where $B(\alpha)$ denotes the normalization function satisfying $B(0)=B(1)=1$ and E_{α} is the Mittag-Leffler function given by

$$ E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\alpha k+1)}. \tag{5}$$

For more details on the Caputo and ABC fractional operators, the interested reader can refer to [5] and [22], respectively.

3. CLASSICAL AND FRACTIONAL DESCRIPTIONS OF PHYSICAL MODEL

In this Section, we give a full description of spring pendulum, which is discussed in many textbooks [1, 2]. The system under consideration is illustrated in Fig. 1. As it is shown in this figure, we have a spring with constant *k*, which swings in a vertical plane. Also, a mass *m* is attached to the spring. The length of pendulum at equilibrium is *l* and *l*+*r*(*t*) is its length at any time *t*, while $\theta(t)$ is the angle of pendulum with the vertical line.

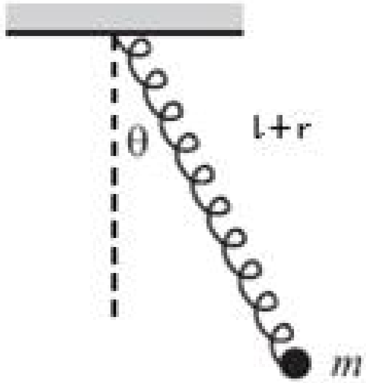

Fig. 1 – The spring pendulum.

The kinetic energy of the spring pendulum (T) as well as its potential one (U) are respectively described by

$$T=\frac{1}{2}m[\dot{r}^2+(l+r)^2\dot{\theta}^2], \tag{6}$$

$$U=\frac{1}{2}kr^2+mg(l+r)(1-\cos\theta)-mgr\,. \tag{7}$$

For the physical model under consideration, the classical Lagrangian, defined as $L_C=T-U$, reads

$$L_C=\frac{1}{2}m[\dot{r}^2+(l+r)^2\dot{\theta}^2]-\frac{1}{2}kr^2-mg(l+r)(1-\cos\theta)+mgr\,, \tag{8}$$

which results the classical Euler-Lagrange equations (CELEs) in the form

$$\ddot{r}=(l+r)\dot{\theta}^2+g\cos\theta-\omega_r^2 r\,, \tag{9}$$

$$\ddot{\theta}=\frac{-2}{(l+r)}\dot{r}\dot{\theta}-\omega_\theta^2\sin\theta\,, \tag{10}$$

where $\omega_r=\sqrt{k/m}$ is the pendulum frequency along its length and $\omega_\theta=\sqrt{g/(l+r)}$ is the pendulum frequency of oscillations. The classical system described by Eqs. (9)–(10) can be solved numerically for specified boundary conditions to investigate the trajectory/position of the spring pendulum.

Below, a fractional study is carried out, which reveals new aspects of physical system under consideration. First, we can fractionalize Eq. (8) as

$$L_F=\frac{1}{2}m\left[({}_aD_t^\alpha r)^2+(l+r)^2({}_aD_t^\alpha\theta)^2\right]-\frac{1}{2}kr^2-mg(l+r)(1-\cos\theta)+mgr,, \tag{11}$$

where ${}_aD_t^\alpha$ denotes the left Caputo or ABC fractional operator. Then, the fractional Euler-Lagrange equations (FELEs) can be obtained from

$$\frac{\partial L_F}{\partial q_i}+{}_tD_b^\alpha\frac{\partial L_F}{\partial\,{}_aD_t^\alpha q_i}+{}_aD_t^\alpha\frac{\partial L_F}{\partial\,{}_tD_b^\alpha q_i}=0\,, \tag{12}$$

in which ${}_tD_b^\alpha$ denotes the right fractional operator in the Caputo or ABC sense. Using Eq. (12) for $q_i=r,\theta$, one gets the FELEs in the form

$$ {}_tD_b^\alpha({}_aD_t^\alpha r) = -(l+r)({}_aD_t^\alpha \theta)^2 - \mathrm{g}\cos\theta + \omega_r^2 r\,, \tag{13}$$

$$ {}_tD_b^\alpha((l+r)^2\,{}_aD_t^\alpha \theta) = g(l+r)\sin\theta\,. \tag{14}$$

As $\alpha \to 1$, the FELEs (13)–(14) reduce to the CELEs previously given by Eqs. (9)–(10). Our aim now is to obtain the numerical solution of Eqs. (13)–(14) for different values of α while considering two different fractional operators namely the Caputo and ABC.

4. NUMERICAL METHOD

In this Section, we develop an efficient numerical technique to solve the FELEs (13)–(14) within the Caputo and ABC fractional operators. Starting with the ABC fractional derivative, we first reformulate Eqs. (13)-(14) in the following way. Let us define the new variables $r_1 = r$, $r_2 = {}^{ABC}_{\ a}D_t^\alpha r$, $\theta_1 = \theta$ and $\theta_2 = {}^{ABC}_{\ a}D_t^\alpha \theta$. Then, Eqs. (13)–(14) can be rewritten as the following system of FDEs

$$\begin{cases} {}^{ABC}_{\ a}D_t^\alpha r_1 = r_2, \\ {}^{ABC}_{\ t}D_b^\alpha r_2 = -(l+r_1)\theta_2^2 - \mathrm{g}\cos\theta_1 + \omega_r^2 r_1, \\ {}^{ABC}_{\ a}D_t^\alpha \theta_1 = \theta_2, \\ {}^{ABC}_{\ t}D_b^\alpha ((l+r_1)^2\theta_2) = g(l+r_1)\sin\theta_1. \end{cases} \tag{15}$$

Using the definition of ABC fractional integral [22] and assuming that ${}^{ABC}_{\ a}D_b^\alpha r = {}^{ABC}_{\ a}D_b^\alpha \theta = 0$, Eq. (15) can be rewritten as the following system of fractional integral equations

$$\begin{cases} r_1(t) = r_1(a) + \dfrac{1-\alpha}{B(\alpha)} r_2(t) + \dfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \displaystyle\int_a^t (t-x)^{\alpha-1} r_2(x)\mathrm{d}x, \\ r_2(t) = \dfrac{1-\alpha}{B(\alpha)}\left(-(l+r_1(t))\theta_2^2(t) - \mathrm{g}\cos\theta_1(t) + \omega_r^2 r_1(t)\right) \\ \qquad + \dfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \displaystyle\int_t^b (x-t)^{\alpha-1}\left(-(l+r_1(x))\theta_2^2(x) - \mathrm{g}\cos\theta_1(x) + \omega_r^2 r_1(x)\right)\mathrm{d}x, \\ \theta_1(t) = \theta_1(a) + \dfrac{1-\alpha}{B(\alpha)}\theta_2(t) + \dfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \displaystyle\int_a^t (t-x)^{\alpha-1}\theta_2(x)\mathrm{d}x, \\ \theta_2(t) = \dfrac{1-\alpha}{B(\alpha)} \dfrac{\mathrm{g}\sin\theta_1(t)}{l+r_1(t)} + \dfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \dfrac{1}{(l+r_1(t))^2} \displaystyle\int_t^b (x-t)^{\alpha-1} g(l+r_1(x))\sin\theta_1(x)\mathrm{d}x. \end{cases} \tag{16}$$

Now, we consider a uniform mesh on $[a,b]$ and label the nodes $0,1,\ldots,N$, where N is an arbitrary positive integer and $h_N = \dfrac{b-a}{N}$ is the time step size. We denote $r_{i,j}$ and $\theta_{i,j}$ as the numerical approximations of $r_i(t_j)$ and $\theta_i(t_j)$, respectively, where $i=1,2$ and $t_j = a + jh_N$ is the time at node j for $0 \le j \le N$. Discretizing the convolution integrals in Eq. (16) by using the fractional Euler method [23], the following system of nonlinear algebraic equations is derived

$$\begin{cases} R_1 - \dfrac{1-\alpha}{B(\alpha)} R_2 - \dfrac{\alpha}{B(\alpha)} B_N^{(\alpha)} R_2 = R_{1,0}, \\ R_2 - \dfrac{1-\alpha}{B(\alpha)} F\left(R_1, \Theta_1, \Theta_2\right) - \dfrac{\alpha}{B(\alpha)} F_N^{(\alpha)} F\left(R_1, \Theta_1, \Theta_2\right) = 0, \\ \Theta_1 - \dfrac{1-\alpha}{B(\alpha)} \Theta_2 - \dfrac{\alpha}{B(\alpha)} B_N^{(\alpha)} \Theta_2 = \Theta_{1,0}, \\ \Theta_2(t) - \dfrac{1-\alpha}{B(\alpha)} G_1\left(R_1, \Theta_1\right) - \dfrac{\alpha}{B(\alpha)} G_2\left(R_1\right) F_N^{(\alpha)} G_3\left(R_1, \Theta_1\right) = 0, \end{cases} \tag{17}$$

where

$$R_i = \begin{bmatrix} r_{i,0} \\ \vdots \\ r_{i,N} \end{bmatrix}, \ \Theta_i = \begin{bmatrix} \theta_{i,0} \\ \vdots \\ \theta_{i,N} \end{bmatrix}, \ R_{i,0} = \begin{bmatrix} r_{i,0} \\ \vdots \\ r_{i,0} \end{bmatrix}, \ \Theta_{i,0} = \begin{bmatrix} \theta_{i,0} \\ \vdots \\ \theta_{i,0} \end{bmatrix}, \quad i = 1, 2, \tag{18}$$

$$B_N^{(\alpha)} = h_N^{\alpha} \begin{bmatrix} \omega_0^{(\alpha)} & 0 & \dots & 0 \\ \omega_1^{(\alpha)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \omega_N^{(\alpha)} & \dots & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix}, \ F_N^{(\alpha)} = \left(B_N^{(\alpha)}\right)^T, \tag{19}$$

$$F\left(R_1, \Theta_1, \Theta_2\right) = \begin{bmatrix} -(l + r_{1,0})\theta_{2,0}^2 - \mathrm{g}\cos\theta_{1,0} + \omega_r^2 r_{1,0} \\ \vdots \\ -(l + r_{1,N})\theta_{2,N}^2 - \mathrm{g}\cos\theta_{1,N} + \omega_r^2 r_{1,N} \end{bmatrix}, \tag{20}$$

$$G_1\left(R_1, \Theta_1\right) = \begin{bmatrix} \dfrac{\mathrm{g}\sin\theta_{1,0}}{l + r_{1,0}} \\ \vdots \\ \dfrac{\mathrm{g}\sin\theta_{1,N}}{l + r_{1,N}} \end{bmatrix}, \ G_3\left(R_1, \Theta_1\right) = \begin{bmatrix} g(l + r_{1,0})\sin\theta_{1,0} \\ \vdots \\ g(l + r_{1,N})\sin\theta_{1,N} \end{bmatrix}, \tag{21}$$

$$G_2\left(R_1\right) = \mathrm{diag}\left(\frac{1}{(l + r_{1,0})^2}, \dots, \frac{1}{(l + r_{1,N})^2} \right), \tag{22}$$

and the binomial coefficient $\omega_j^{(\alpha)}$ can be calculated by using the recursive formula $\omega_0^{(\alpha)} = 1$ and $\omega_j^{(\alpha)} = (\frac{\alpha + j - 1}{j})\omega_{j-1}^{(\alpha)}$, $j = 1, 2, \dots$. Note that, the aforementioned results can be used in the Caputo sense by using the Caputo fractional integral [5] instead of its ABC counterpart in Eq. (16) and following the same discretizing procedure as above.

4.1. SIMULATION RESULTS

In the following simulations we take the parameters as $g = 9.81$, $k = 10\,\mathrm{N/m}$, $l = 2\,\mathrm{m}$ and $m = 1\,\mathrm{kg}$. Also, the initial values are selected to be $r(0) = 1$ and $\theta(0) = 0.1$. Figures 2–5 show the plots of $r(t)$ and

$\theta(t)$ for $\alpha = 0.85, 0.9, 0.95, 0.99$. In these figures, we consider the Caputo and ABC fractional operators for the FELEs (13)–(14), respectively. These figures indicate that the numerical solution of FELEs (13)-(14) exhibits different behaviours for different fractional operators. Thus, taking into account the new fractional derivatives provides more flexible models which help us to adjust better the dynamical behaviours of the real-world phenomena.

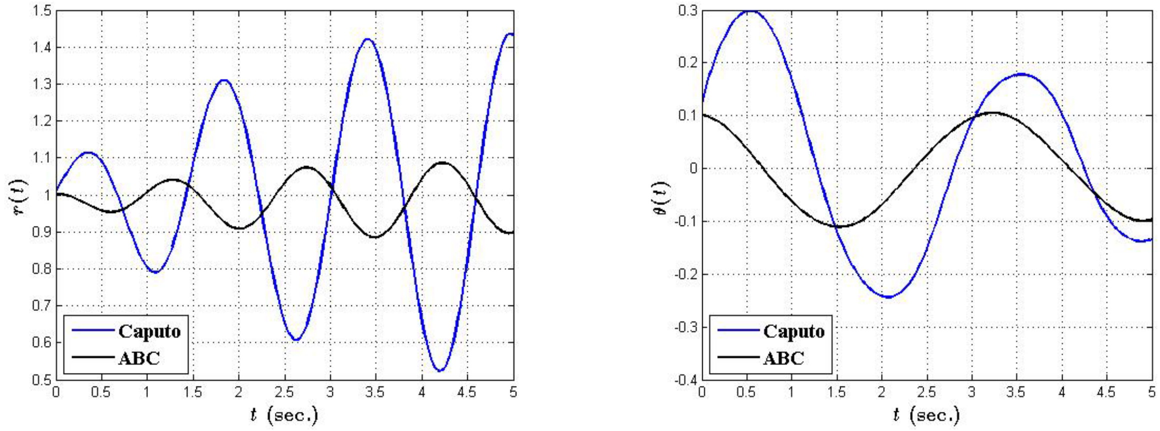

Fig. 2 – The plots of $r(t)$ and $\theta(t)$ within the Caputo and ABC fractional operators for $\alpha = 0.85$.

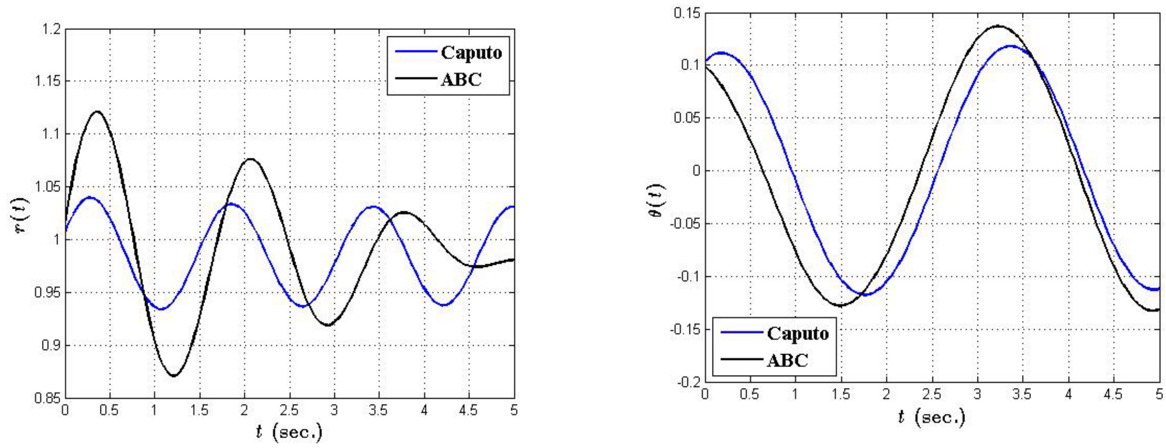

Fig. 3 – The plots of $r(t)$ and $\theta(t)$ within the Caputo and ABC fractional operators for $\alpha = 0.9$.

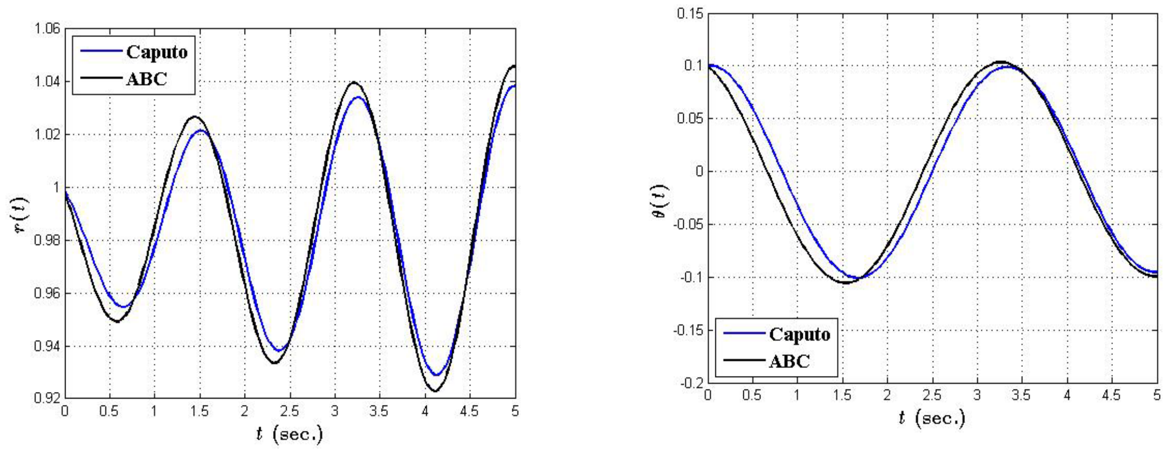

Fig. 4 – The plots of $r(t)$ and $\theta(t)$ within the Caputo and ABC fractional operators for $\alpha = 0.95$.

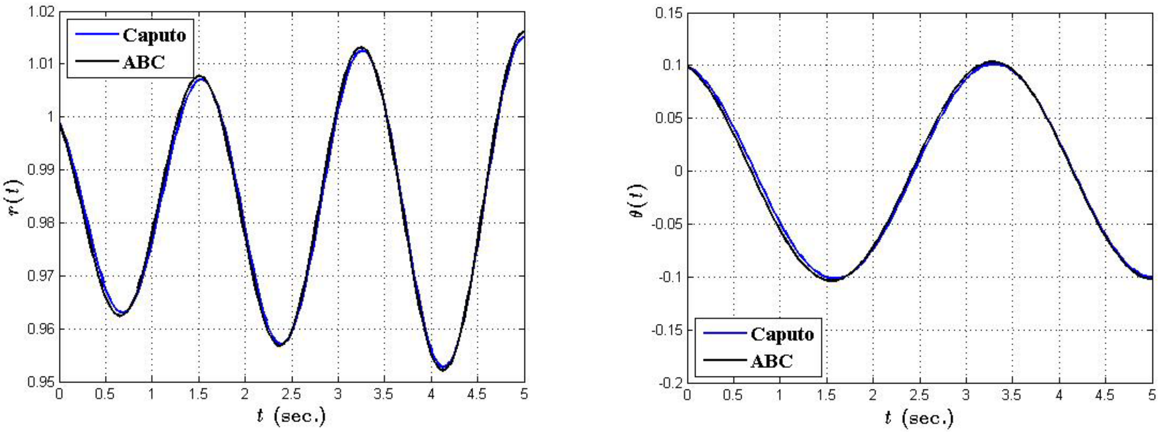

Fig. 5 – The plots of $r(t)$ and $\theta(t)$ within the Caputo and ABC fractional operators for $\alpha = 0.99$.

5. CONCLUSION

In this study, we have investigated the model of spring pendulum by using the fractional Lagrangian. For this aim, we generalized the classical Lagrangian to the fractional case and derived the FELEs in the Caputo and ABC sense. Then, we solved the proposed models within these two fractional operators by using a numerical method based on the discretization of convolution integral by the Euler convolution quadrature rule. The results reported in Figs. 2–5 indicate that the behaviours of the FELEs depend on the fractional operators. Thus, the recently features of the FC provide more realistic models, which help us to adjust better the dynamical behaviours of the real-world phenomena.

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