ACADEMIA ROMÂNĂ
SCOSAAR

HABILITATION THESIS

Vanishing theorems on normal crossings varieties

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Domeniul fundamental: Matematică şi ştiinţe ale naturii

Domeniul de abilitare: Matematică

Teză elaborată în vederea obţinerii atestatului de abilitare în scopul conducerii lucrărilor de doctorat în domeniul Matematică

BUCUREŞTI, 2018

Abstract

Algebraic Geometry is a central field of Mathematics. It studies the common zeros of polynomials in the affine or projective space. It has intimate connections with other fields of Mathematics, such as Number Theory, Complex Geometry or Differential Geometry, or with Physics, for example String Theory.

My research interests lie in the **Classification Theory of Algebraic Varieties**, a central branch of Algebraic Geometry. The Classification Theory chooses certain minimal objects, considered to be the simplest, but from which all varieties are build up (*Minimal Model Problem*), and then describes completely the minimal objects (Embedding Problem, Classification of Singularities, Moduli Problem).

The classification of complex algebraic varieties rests on Kodaira-type vanishing theorems for Cartier divisors of the form $L \sim_{\mathbb{Q}} K_X + B$, where (X, B) is a log smooth variety. For inductive arguments with linear systems, or to study degenerations of log manifolds, it is necessary to extend the vanishing theorems to the case when (X, B) is more singular, especially X may not be normal. This Habilitation Thesis is a collection of five papers [6, 7, 8, 5, 9], leading up to the vanishing theorems for the case when (X, B) is a normal crossings log variety. We may think of normal crossings log varieties as being glueings of log smooth varieties, in the simplest possible way. The proof of the vanishing theorems is made in two steps: the log smooth case follows from Hodge Theory, and the normal crossings case is reduced to the log smooth case via simplicial methods.

An application for the results in this thesis is to improve [3, Section 3]. We proved there weaker results, under the global assumption that X is globally embedded as a normal crossings divisor in a smooth variety. We remove here this global assumption, making the hypothesis on singularities locally analytic.

The first five chapters correspond to the above mentioned papers. Chapter 6 contains plans and directions for future research. We outline now the content of the first five chapters.

Chapter 1 presents the known cyclic covering trick, with certain improvements. The cyclic covering trick is a classical tool to reduce statements about log smooth varieties (X, B) to the case when B has coefficients 0 or 1. The original part of our presentation is that we can perform this trick inside the category of quasi-smooth toroidal embeddings. We will use the results of Chapter 1 in Chapter 4.

Chapter 2 introduces toric affine varieties which may not be normal. They are defined as the spectrum of toric face rings. Toric face rings are a natural generalization of Stanley rings, studied intensively in combinatorial algebra. We present known algebraic results in a geometric way, especially the weak normality and seminormality criteria. We define weakly toroidal varieties to be those which are locally analytically isomorphic to weakly normal toric affine varieties. For weakly toroidal varieties, we construct an explicit Du Bois complex, which can be used to compute the singular cohomology of such varieties. In particular, such varieties have Du Bois singularities, a result used in Chapter 5.

Chapter 3 introduces the new class of weakly log canonical varieties, which generalize log canonical and semi-log canonical varieties. We classify those weakly toroidal varieties which have weakly log canonical singularities. Along the way, we give a criterion for a toric affine variety to satisfy Serre's S_2-property. We define codimension one residues for weakly log canonical varieties. We introduce the class of n-wlc singularities, for which we can define residues of any codimension. We will use this result in Chapter 5.

Chapter 4 presents the injectivity theorem of Esnault and Viehweg, with certain improvements. The proof is similar to that of Esnault and Viehweg: modulo the cyclic covering trick and Hironaka's desingularization, the injectivity theorem is a direct consequence of the E_1-degeneration of the Hodge to de Rham spectral sequence associated to an open manifold. Our improvement to the original result has new applications, for example an extension theorem from the non-log canonical locus of a log variety. The latter implies for example that for a log variety of Calabi-Yau type, the locus of non-log canonical singularities is connected.

Chapter 5 presents the main results, the injectivity theorems of Esnault-Viehweg and Tankeev-Kollár, the torsion freeness theorem of Kollár, the vanishing theorem of Ohsawa-Kollár. They are proved in the category of generalize normal crossings varieties, a class which contains normal crossings singularities, and is contained in the class of n-wlc singularities. The key idea is to use higher codimension residues to reduce the vanishing theorems to the log smooth case.

Rezumat

Geometria Algebrică este un domeniu fundamental al Matematicii. Studiază locul comun al zerourilor unor polinoame in spaţiul afin sau proiectiv. Este strâns legată de alte domenii ale Matematicii, de exemplu Teoria Numerelor, Geometria Complexă sau Geometria Diferenţială, sau cu Fizica, de exemplu Teoria Stringurilor.

Subiectul meu de cercetare este **Teoria de Clasificare a Varietăţilor Algebrice**, un subdomeniu central al Geometriei Algebrice. Teoria de clasificare alege anumite modele minimale, considerate a fi cele mai simple, dar din care toate varietăţile se pot construi (*Teoria Modelelor Minimale*), şi apoi descrie complet aceste obiecte minimale (Probleme de Scufundare, Clasificarea Singularităţilor, Probleme de Moduli).

Clasificarea varietăţilor algebrice complexe se bazează pe teoreme de anulare de tip Kodaira pentru divizori Cartier de tipul $L \sim_{\mathbb{Q}} K_X + B$, unde (X, B) este o varietate logaritmic netedă. Pentru argumente inductive in studiul sistemelor liniare, sau in studiul degenerărilor varietăţilor logaritmice netede, este necesar să extindem teoremele de anulare la cazul când (X, B) este mai singular, in special când X nu este normal. Această Teză de Abilitare este o colecţie de cinci lucrări [6, 7, 8, 5, 9], cu scopul final de a demonstra teoremele de anulare in cazul când (X, B) este of varietate logaritmică cu intersecţii normale. Putem considera varietăţile logaritmice cu intersecţii normale ca fiind lipiri de varietăţi logaritmice netede, in cel mai simplu mod posibil. Teoremele de anulare se obţin in doi paşi: cazul logaritmic neted rezultă din Teoria Hodge, iar cazul cu intersecţii normale se reduce la cazul logaritmic neted prin metode simpliciale.

O aplicaţie a rezultatelor acestei teze este imbunatăţirea rezultatelor din [3, Section 3]. Am demonstrat acolo rezultate mai slabe, sub ipoteza suplimentară că X este global scufundat ca divizor cu intersecţii normale într-o varietate netedă. In această teză eliminăm aceasta ipoteză globală suplimentară.

Primele cinci capitole corespund la articolele menţionate mai sus. Capitolul 6 conţine planuri si direcţii de cercetare pe viitor. Schiţăm mai jos conţinutul primelor cinci capitole.

Capitolul 1 prezintă cunoscutul truc al acoperirilor ciclice, cu anumite îmbunătăţiri. Trucul acoperirilor ciclice este un instrument clasic folosit pentru a reduce demonstraţia anumitor proprietăţi ale varietăţilor logaritmic netede (X, B) la cazul când B are coeficienţi doar 0 sau 1. Partea originală a expunerii noastre este că trucul funcţionează in categoria scufundărilor toroidale quasi-netede. Vom folosi aceste rezultate in Capitolul 4.

Capitolul 2 introduce varietăţile torice afine care nu sunt neapărat normale. Ele sunt definite ca spectrul unui inel cu faţete torice. Inelele cu faţete torice sunt o generalizare

naturală a inelelor Stanley, studiate intens in algebra combinatorică. Prezentăm rezultate algebrice cunoscute într-un limbaj geometric, în special criteriile de normalitate slabă si seminormalitate. Definim varietăţile slab toroidale ca fiind acele varietăţi care sunt local analitic izomorfe cu varietăţile torice afine slab normale. Pentru varietăţi slab toroidale, construim un complex Du Bois explicit, ce poate fi folosit in calcularea cohomologiei singulare a acestor varietăţi. In particular, aceste varietăţi au singularităţi Du Bois, un rezultat folosit in Capitolul 5.

Capitolul 3 introduce noua clasă a varietăţilor slab log canonice, care generalizează varietăţile log canonice şi semi-log canonice. Clasificăm varietăţile slab toroidale care au singularităţi slab log canonice. Pe parcurs, găsim un criteriu necesar si suficient pentru ca o varietate torică afină să satisfacă proprietatea S_2 a lui Serre. Definim reziduuri in codimensiune unu pentru varietăţi slab log canonice. Introducem clasa singularităţilor n-wlc, pentru care putem defini reziduuri in orice codimensiune. Vom folosi acest rezultat in Capitolul 5.

Capitolul 4 prezintă teorema de injectivitate Esnault-Viehweg, cu anumite îmbunătăţiri. Demonstraţia este similară cu cea dată de Esnault-Viehweg: modulo trucul acoperirilor ciclice si desingularizarea lui Hironaka, teorema de injectivitate este o consecinţă directă a E_1-degenerării sirului spectral Hodge spre de Rham asociat unei varietăţi necompacte. Imbunătăţirile noastre la rezultatul original are câteva aplicaţii noi, de exemplu o teoremă de extensie de la locul de singularităţi ne-log canonice a unei varietăţi logaritmice. In particular, rezultă că pentru o varietate logaritmică de tip Calabi-Yau, locul de singularităţi ne-log canonice este conex.

Capitolul 5 prezintă rezultatele principale, anume teoremele de injectivitate Esnault-Viehweg şi Tankeev-Kollár, teorema de lipsă a torsiunii a lui Kollár, teorema de anulare Ohsawa-Kollár. Ele sunt demonstrate in categoria varietăţilor cu intersecţii normale generalizate, o clasă care conţine singularităţile cu intersecţii normale, şi care este conţinută in clasa singularităţilor n-wlc. Ideea principală este folosirea reziduurilor de codimensiune arbitrară, pentru a reduce teoremele de anulare la cazul logaritmic neted.

Contents

Chapter 1

The cyclic covering trick

Cyclic covers are a useful tool in algebraic geometry. The simplest example is the field extension

$$K \subset K(\sqrt[n]{\varphi})$$

obtained by adjoining to a field the root of an element. For example, the equation $t^n - \varphi \prod_i z_i^{m_i}$ over K simplifies to $t^n - \prod_i z_i^{m_i}$ over $K(\sqrt[n]{\varphi})$.

In classical algebraic geometry, cyclic covers were used to construct new examples from known ones. Given a complex projective variety X, a torsion line bundle L over X induces canonically an étale cyclic Galois covering $\pi \colon X' \to X$ such that π^*L becomes trivial. If L has torsion index r and $s \in \Gamma(X, L^r)$ is nowhere zero, the covering can be constructed as the r-th root of s (as in the function field case, the pullback of s becomes an r-th power of a section of the pullback of L). We may denote it by $\pi \colon X[\sqrt[r]{s}] \to X$. The isomorphism class of π does not depend on the choice of s, since X being compact, every two nowhere zero sections differ by a non-zero constant.

Many invariants of X' can be read off those of X, but with coefficients in negative powers of L. For example,

$$\pi_* \Omega^p_{X'} = \oplus_{i=0}^{r-1} \Omega^p_X \otimes L^{-i}.$$

So one may construct manifolds with prescribed invariants by taking roots of torsion line bundles on known manifolds. The process may be reversed: known statements on the invariants of X' translate into similar statements on X, twisted by negative powers of L. For example, the Kähler differential of X' decomposes into integrable flat connections on L^{-i}, so that $\oplus_{i=0}^{r-1} \Omega^\bullet_X(L^{-i})$ is the Hodge complex $\pi_* \Omega^\bullet_{X'}$ on X. In particular, the E_1-degeneration for $(\Omega^\bullet_{X'}, F)$ translates into the E_1-degeneration for $(\Omega^\bullet_{X'}(L^{-i}), F)$, for every i. This exchange of information between the total and base space of a cyclic cover is called the *cyclic covering trick* (cf. sections 1 and 2 of [25] for example).

The range of applications of the cyclic covering trick extends dramatically if s is allowed to have zeros. In this case s is a non-zero global section of the n-th power of some line bundle L on X. The n-th root of s is defined just as above. We obtain for example the same formula

$$\pi_* \mathcal{O}_{X[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} L^{-i}.$$

The morphism π is still cyclic Galois and flat, but it ramifies over the zero locus of s. The total space $X[\sqrt[n]{s}]$ may be disconnected (even if s vanishes nowhere), it may have several irreducible components, and it may even have singularites over the zero locus of s. These singularities are partially resolved by the normalization $\bar{X}[\sqrt[n]{s}] \to X[\sqrt[n]{s}]$. The induced morphism $\bar{\pi}\colon \bar{X}[\sqrt[n]{s}] \to X$ is cyclic Galois and flat, and one computes

$$\bar{\pi}_* \mathcal{O}_{\bar{X}[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} L^{-i}(\lfloor \frac{i}{n} Z(s) \rfloor).$$

Here $Z(s)$ is the effective Cartier divisor cut out by s, and the round down of the $\mathbb{Q}$-divisor $\frac{i}{n}Z(s)$ is defined componentwise. If $\operatorname{Supp} Z(s)$ has no singularities, then $\bar{X}[\sqrt[n]{s}]$ has no singularities. Differential forms or vector fields on $\bar{X}[\sqrt[n]{s}]$ are computed in terms of X, L, and the $\mathbb{Q}$-divisor $\frac{1}{n}Z(s)$. For example

$$\bar{\pi}_* \Omega^p_{\bar{X}[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} \Omega^p_X(\log \operatorname{Supp}\{\frac{i}{n} Z(s)\}) \otimes L^{-i}(\lfloor \frac{i}{n} Z(s) \rfloor),$$

where $\{\frac{i}{n}Z(s)\}$ is the fractional part of the $\mathbb{Q}$-divisor $\frac{i}{n}Z(s)$, defined componentwise, and for a reduced divisor Σ on X, $\Omega^p_X(\log \Sigma)$ denotes the sheaf of differential p-forms ω such that both ω and $d\omega$ are regular outside Σ, and have at most logarithmic poles along the components of Σ. If the singularities of $\operatorname{Supp} Z(s)$ are at most simple normal crossing, then $\bar{X}[\sqrt[n]{s}]$ has at most quotient singularities, and if $Y \to \bar{X}[\sqrt[n]{s}]$ is a desingularization, with $\nu\colon Y \to X$ the induced generically finite morphism, then

$$\nu_* \Omega^p_Y = \oplus_{i=0}^{n-1} \Omega^p_X(\log \operatorname{Supp}\{\frac{i}{n} Z(s)\}) \otimes L^{-i}(\lfloor \frac{i}{n} Z(s) \rfloor).$$

This formula is behind the vanishing theorems used in birational classification (see [25, 42, 43]). Statements on divisors of the form $K_X + \sum_j b_j E_j + T$, with X nonsingular, $\sum_j E_j$ simple normal crossing, $b_j \in [0,1]$, and T a torsion $\mathbb{Q}$-divisor, are reduced to similar statements on Y with $b_j \in \{0,1\}$ and $T = 0$.

Cyclic covers also appear in semistable reduction [39]. In its simplest form, a complex projective family over the unit disc $f\colon \mathcal{X} \to \Delta$ has nonsingular general fibers $\mathcal{X}_t$ $(t \neq 0)$, while the special fiber $\mathcal{X}_0$ is locally cut out by monomials $\prod_{i=1}^d z_i^{m_i}$ $(m_i \in \mathbb{N})$ with respect to local coordinates $z_1, \ldots, z_d$. The family is semistable if moreover $\mathcal{X}_0$ is reduced. If we base change with $\sqrt[n]{t}$ (with n divisible by all multiplicities m_i), and normalize $\tilde{\mathcal{X}} \to \mathcal{X} \times_\Delta \tilde{\Delta}$, the new family $\tilde{\mathcal{X}} \to \tilde{\Delta}$ has reduced special fiber $\tilde{\mathcal{X}}_0$, and $\tilde{\mathcal{X}} \setminus \tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{X}}$ is a *quasi-smooth toroidal embedding*. If the irreducible components of $\mathcal{X}_0$ are nonsingular, the toroidal embedding is also strict and $\tilde{\mathcal{X}}$ admits a combinatorial desingularization. An equivalent description of $\tilde{\mathcal{X}}$ is the normalization of the n-th root of f, viewed as a holomorphic function on $\mathcal{X}$. Therefore the local computations of [39] give in fact the following statement: if X is complex manifold, and $0 \neq s \in \Gamma(X, L^n)$ is such that $\Sigma = \operatorname{Supp} Z(s)$ is a normal crossing divisor, then $\bar{X}[\sqrt[n]{s}] \setminus \bar{\pi}^{-1}(\Sigma) \subset \bar{X}[\sqrt[n]{s}]$ is a quasi-smooth toroidal embedding, and $\bar{\pi}$ is a toroidal morphism.

Cyclic covers are used to classify the singularities that appear in the birational classification of complex manifolds. Such singularities $P \in X$ are normal, and the canonical Weil divisor K_X is a torsion element of $\mathrm{Cl}(\mathcal{O}_{X,P})$. If r is the torsion index, there exists a rational function $\varphi \in \mathbb{C}(X)^*$ such that $rK_X = \mathrm{div}(\varphi)$. The normalization of X in the Kummer extension $\mathbb{C}(X) \to \mathbb{C}(X)(\sqrt[r]{\varphi})$ becomes a cyclic cover $P' \in X' \stackrel{\pi}{\to} P \in X$. It is called the *index one cover* of $P \in X$, since being étale in codimension one, $K_{X'} = \pi^* K_X \sim 0$. The known method to classify $P \in X$ is to first classify the index one cover, and then understand all possible actions of cyclic groups (see [56]).

We have discussed so far roots of rational functions, (normalized) roots of multi sections of line bundles, and index one covers of torsion $\mathbb{Q}$-divisors on normal varieties. We give a unified treatment of all these concepts, based on *normalized roots of rational functions on normal varieties*. Moreover, we show that the cyclic covering trick can be performed inside the category of quasi-smooth toroidal embeddings. In order to prove vanishing theorems, we no longer have to assume that the base is nonsingular, or to resolve the singularities of the total space of the covering.

To state the main results of this chapter, let k be an algebraically closed field.

Theorem 1.0.1. *Let X/k be an normal algebraic variety. Let φ be an invertible rational function on X, let n be a positive integer such that $\mathrm{char}\, k \nmid n$. Denote $D = \frac{1}{n}\mathrm{div}(\varphi)$, so that D is a $\mathbb{Q}$-Weil divisor on X with $nD \sim 0$. Let $\pi \colon Y \to X$ be the normalization of X with respect to the ring extension*

$$k(X) \to \frac{k(X)[T]}{(T^n - \varphi)}.$$

The right hand side is a product of fields (the function fields of the irreducible components of Y), and Y identifies with the disjoint union of the normalization of X in each field. By construction, Y/k is a normal algebraic variety (possibly disconnected).

a) The class of T becomes an invertible rational function ψ on Y such that $\psi^n = \pi^*\varphi$. We have $\pi^* D = \mathrm{div}(\psi)$ and

$$\pi_*\mathcal{O}_Y = \oplus_{i=0}^{n-1} \mathcal{O}_X(\lfloor iD \rfloor) \cdot \psi^i.$$

The morphism π is étale exactly over $X \setminus \mathrm{Supp}\{D\}$, where $\{D\}$ is the fractional part of the $\mathbb{Q}$-divisor D, defined componentwise. It is flat if and only if the Weil divisors $\lfloor iD \rfloor$ $(0 < i < n)$ are Cartier.

b) Suppose $U \subseteq X$ is a quasi-smooth toroidal embedding and $D|_U$ has integer coefficients. Then $\pi^{-1}(U) \subseteq Y$ is a quasi-smooth toroidal embedding, and π is a toroidal morphism. Denote $\Sigma_X = X \setminus U$ and $\Sigma_Y = Y \setminus \pi^{-1}(U)$. Then $\pi^*\tilde{\Omega}^p_{X/k}(\log \Sigma_X) \tilde{\to} \tilde{\Omega}^p_{Y/k}(\log \Sigma_Y)$, and by the projection formula

$$\pi_*\tilde{\Omega}^p_{Y/k}(\log \Sigma_Y) = \tilde{\Omega}^p_{X/k}(\log \Sigma_X) \otimes \pi_*\mathcal{O}_Y.$$

Theorem 1.0.2. *Suppose* $\operatorname{char} k = 0$. *Let* $U \subseteq X$ *and* $U' \subseteq X'$ *be toroidal embeddings over* k, *let* $\mu \colon X' \to X$ *be a proper morphism which induces an isomorphism* $U' \tilde{\to} U$. *Denote* $\Sigma_X = X \setminus U$ *and* $\Sigma_{X'} = X' \setminus \Sigma_{X'}$. *Then*

$$R^q \mu_* \tilde{\Omega}^p_{X'/k}(\log \Sigma_{X'}) = \begin{cases} \tilde{\Omega}^p_{X/k}(\log \Sigma_X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Theorem 1.0.1 generalizes the known cyclic covering trick (see [25, Section 3], especially 3.3-3.11), where the base space X is assumed nonsingular, to the case when X is just normal. Statement a) is elementary. The toroidal part of b) is implicit in [39] if (X, Σ_X) is log smooth, as already mentioned. The general case (Theorem 1.3.8) is proved by reduction to the following fact: the normalized root of a toric variety with respect to a torus character consists of several isomorphic copies of a toric morphism (Proposition 1.3.7). The sheaf $\tilde{\Omega}^p_{X/k}(\log \Sigma_X)$ consists of the rational p-forms ω of X such that both $\omega, d\omega$ are regular near the prime divisors of X outside Σ_X, and have at most simple poles at the prime components of Σ_X. It is called the sheaf of logarithmic p-forms of $(X/k, \Sigma_X)$, in the sense of Zariski-Steenbrink. It is constructed by ignoring closed subsets of X of codimension at least two, so in general it is singular. But if $X \setminus \Sigma_X \subset X$ is a toroidal embedding, it is locally free [63, 15]. If X is nonsingular and Σ_X is a normal crossing divisor, this sheaf coincides with the sheaf of logarithmic forms $\Omega^p_{X/k}(\log \Sigma_X)$ in the sense of Deligne (see [25] for the algebraic version, with Σ_X assumed simple normal crossing). If $\Sigma_X = 0$, then $\tilde{\Omega}^p_{X/k} = \tilde{\Omega}^p_{X/k}(\log 0)$ is the double dual of the usual sheaf of Kähler differentials $\Omega^p_{X/k}$. We note that differential forms or vector fields on Y can be computed without the toroidal assumption (Lemma 1.3.6).

Theorem 1.0.2 is the invariance of the logarithmic sheaves under different toroidal embeddings, proved by Esnault and Viehweg [23, Lemma 1.5] in the case when X is projective nonsingular and Σ_X has normal crossings. One corollary is that

$$H^q(X, \tilde{\Omega}^p_{X/k}(\log \Sigma_X)) \to H^q(X', \tilde{\Omega}^p_{X'/k}(\log \Sigma_{X'}))$$

is an isomorphism for every p, q. If X is proper and $(X, X \setminus U)$ is log smooth, the corollary follows from the E_1-degeneration of the spectral sequence induced in hypercohomology by the logarithmic De Rham complex endowed with the naive filtration (Deligne [19]). If X is projective and $X \setminus U$ is a simple normal crossing divisor, Esnault-Viehweg [23, Lemma 1.5] proved that the corollary implies Theorem 1.0.2. We use the same idea, combined with a result of Bierstone-Milman [12], in order to compactify strict log smooth toroidal embeddings (Corollary 1.1.11).

The normalization of roots of multi sections of line bundles on normal varieties, and the index one covers torsion $\mathbb{Q}$-divisors on normal varieties, are both examples of normalized roots of rational functions. In practice, index one covers are most useful. They preserve irreducibility, so one can work in the classical setting of function fields. Their drawback is that they do not commute with base change to open subsets. For this reason, at least for proofs, we need to consider normalized roots of rational functions, which commute with étale base change.

1.1 Preliminaries

We consider varieties (reduced, possibly reducible), defined over an algebraically closed field k.

1.1.1 Zariski-Steenbrink differentials

Let X/k be a normal variety. Its connected components coincide with its irreducible components. Let $k(X)$ be the ring of rational functions of X, consisting of functions regular on a dense open subset of X. It identifies with the product of the function fields of the irreducible components of X. A rational function is invertible if and only if it is not zero on each irreducible component.

Let $\omega \in \Omega^p_{k(X)/k}$ be a rational differential p-form. Let $E \subset X$ be a prime divisor. The form ω is regular at E if it can be written as a sum of forms $f_0 \cdot df_1 \wedge \cdots \wedge df_p$, with $f_i \in \mathcal{O}_{X,E}$. The form ω has at most a logarithmic pole at E if both ω and $d\omega$ have at most a simple pole at E. If t is a local parameter at the generic point of E, this is equivalent to the existence of a decomposition $\omega = \frac{dt}{t} \wedge \omega^{p-1} + \omega^p$, with ω^{p-1}, ω^p rational differentials regular at E. If $\bar{E} \to E$ is the normalization, the restriction $\omega^{p-1}|_{\bar{E}}$ is independent of the decomposition, called the residue of ω at E. The residue is zero if and only if ω is regular at E.

For an open subset $U \subseteq X$, let $\Gamma(U, \tilde{\Omega}^p_{X/k})$ consist of the rational differential p-forms which are regular at each prime divisor of X which intersects U. This defines a coherent $\mathcal{O}_X$-module $\tilde{\Omega}^p_{X/k}$, called the *sheaf of p-differential forms of X/k, in the sense of Zariski-Steenbrink* [69, 63]. If $j \colon X^0 \subset X$ is the nonsingular locus of X, $\tilde{\Omega}^p_{X/k} = j_*(\Omega^p_{X^0/k})$. If D is a Weil divisor on X, let $\Gamma(U, \tilde{\Omega}^p_{X/k}(D))$ consist of the rational differential p-forms ω such that $t_E^{\operatorname{mult}_E D}\omega$ is regular at E, for every prime divisor E, with local parameter t_E. This defines a coherent $\mathcal{O}_X$-module $\tilde{\Omega}^p_{X/k}(D)$. We have $\tilde{\Omega}^p_{X/k}(D) = j_*(\Omega^p_{X^0/k} \otimes \mathcal{O}_{X^0}(D|_{X^0}))$.

Let Σ be a reduced Weil divisor on X, or equivalently, a finite set of prime divisors on X. For an open subset $U \subseteq X$, let $\Gamma(U, \tilde{\Omega}^p_{X/k}(\log \Sigma))$ consist of the rational differential p-forms ω such that for every prime divisor E which intersects U, ω is regular (has at most a logarithmic pole) at E if $E \notin \Sigma$ ($E \in \Sigma$). This defines a coherent $\mathcal{O}_X$-module $\tilde{\Omega}^p_{X/k}(\log \Sigma)$, called the *sheaf of logarithmic p-differential forms of $(X/k, \Sigma)$, in the sense of Zariski-Steenbrink*. For a Weil divisor D on X, we can similarly define $\tilde{\Omega}^p_{X/k}(\log \Sigma)(D)$.

The tangent sheaf $\mathcal{T}_{X/k}$ is already S_2-saturated, since X is normal: a derivation θ of $k(X)/k$ is regular on an open subset $U \subseteq X$ if and only if it is regular at each prime of X which intersects U. For an open subset $U \subseteq X$, let $\Gamma(U, \tilde{\mathcal{T}}_{X/k}(-\log \Sigma))$ consist of the derivations $\theta \in \Gamma(U, \mathcal{T}_{X/k})$ such that for every prime divisor $E \in \Sigma$ which intersects U, θ preserves the maximal ideal $\mathfrak{m}_{X,E}$. This defines a coherent $\mathcal{O}_X$-module $\tilde{\mathcal{T}}_{X/k}(-\log \Sigma)$, called the *sheaf of logarithmic derivations of (X, Σ), in the sense of Zariski-Steenbrink.* For a Weil divisor D on X, we can similarly define $\tilde{\mathcal{T}}_{X/k}(D), \tilde{\mathcal{T}}_{X/k}(-\log \Sigma)(D)$.

Lemma 1.1.1. *Let $u\colon X' \to X$ be an étale morphism. In particular, X' is normal. Let Σ be a reduced Weil divisor on X, denote $\Sigma' = u^{-1}(\Sigma)$. Then we have base change isomorphisms*

$$u^*\tilde{\Omega}^p_{X/k} \tilde{\to} \tilde{\Omega}^p_{X'/k} \text{ and } u^*\tilde{\Omega}^p_{X/k}(\log \Sigma) \tilde{\to} \tilde{\Omega}^p_{X'/k}(\log \Sigma').$$

Similar statements hold for twists with Weil divisors.

Proof. Denote $U = X \setminus \operatorname{Sing} X$ and $U' = X' \setminus \operatorname{Sing} X'$. Since u is smooth, we have $U' = u^{-1}(U)$. We obtain a cartesian diagram

$$\begin{tikzcd} U' \arrow[r, "v"] \arrow[d, "i_{U'}"'] & U \arrow[d, "i_U"] \\ X' \arrow[r, "u"'] & X \end{tikzcd}$$

Since u is flat, the base change homomorphism $u^* i_{U*}\Omega^p_{U/k} \to i_{U'*}v^*\Omega^p_{U/k}$ is an isomorphism. Since v is smooth, $v^*\Omega^p_{U/k} \to \Omega^p_{U'/k}$ is an isomorphism. Therefore the base change isomorphism becomes $u^*\tilde{\Omega}^p_{X/k} \tilde{\to} \tilde{\Omega}^p_{X'/k}$.

Since u is étale, we have $\Sigma' = u^{-1}(\Sigma)$, and $\Sigma' \to \Sigma$ is étale. In particular, $\operatorname{Sing} \Sigma' = u^{-1}(\operatorname{Sing} \Sigma)$. Denote $V = X \setminus (\operatorname{Sing} X \cup \operatorname{Sing} \Sigma)$ and $V' = X' \setminus (\operatorname{Sing} X' \cup \operatorname{Sing} \Sigma')$. We obtain a cartesian diagram

$$\begin{tikzcd} V' \arrow[r, "v"] \arrow[d, "i_{V'}"'] & V \arrow[d, "i_V"] \\ X' \arrow[r, "u"'] & X \end{tikzcd}$$

Since u is flat, the base change homomorphism $u^* i_{V*}\Omega^p_{V/k}(\log \Sigma|_V) \to i_{V'*}v^*\Omega^p_{V/k}(\log \Sigma|_V)$ is an isomorphism. Since v is smooth, $v^*\Omega^p_{V/k}(\log \Sigma|_V) \to \Omega^p_{V'/k}(\log \Sigma'|_{V'})$ is an isomorphism. Therefore the base change isomorphism becomes $u^*\tilde{\Omega}^p_{X/k}(\log \Sigma) \tilde{\to} \tilde{\Omega}^p_{X'/k}(\log \Sigma')$. □

Lemma 1.1.2. *Let $u\colon X' \to X$ be an étale morphism. Let E' be a prime divisor on X', let E be the closure of $u(E')$. Let ω be a rational p-form on X. Then*

i) ω is regular at E if and only if $u^*\omega$ is regular at E'.

ii) ω has at most a log pole at E if and only if $u^*\omega$ has at most a log pole at E'.

Proof. i) There exists $l \gg 0$ such that $\omega \in \tilde{\Omega}^p_X(lE)_E$. By Lemma 1.1.1, the inclusion $\tilde{\Omega}^p_{X/k} \subset \tilde{\Omega}^p_{X/k}(lE)$ becomes after étale pullback $\tilde{\Omega}^p_{X'/k} \subset \tilde{\Omega}^p_{X'/k}(lE')$. All sheaves that appear being coherent, the section $\omega \in \tilde{\Omega}^p_{X/k}(lE)_E$ belongs to $(\tilde{\Omega}^p_{X/k})_E$ if and only the pullback section $u^*\omega \in \tilde{\Omega}^p_{X'/k}(lE')_{E'}$ belongs to $(\tilde{\Omega}^p_{X'/k})_{E'}$

ii) The proof in the logarithmic case is similar. □

Lemma 1.1.3. *Let $u\colon X' \to X$ be an étale morphism. Let D be a $\mathbb{Q}$-Weil divisor on X. Let $D' = u^*D$ be the pullback $\mathbb{Q}$-Weil divisor, defined by restricting to big open subsets. Then*

$$u^*\mathcal{O}_X(\lfloor D \rfloor) \tilde{\to} \mathcal{O}_{X'}(\lfloor D' \rfloor).$$

Proof. Restrict to the smooth loci of X' and X. Round down commutes with pullback since u is unramified in codimension one. By flat base change, the isomorphism extends to X' and X. $\square$

1.1.2 Zariski-Steenbrink differentials and derivations on toric varieties

Let M be a lattice, with dual lattice $N = M^*$. Let $T = T_N = \mathrm{Hom}(M, k^*)$ be the induced torus over k. The torus acts on the space of global regular functions, with eigenspace decomposition

$$\Gamma(T, \mathcal{O}_T) = \oplus_{m\in M} k \cdot \chi^m.$$

Recall that each element $m \in M$ induces by evaluation a torus character $\chi^m \colon T \to k^*$. Denote $\alpha_m = \frac{d(\chi^m)}{\chi^m} \in \Gamma(T, \Omega_{T/k})$. The application $m \mapsto \alpha_m$ is additive, and induces an isomorphism

$$\mathcal{O}_T \otimes_{\mathbb{Z}} M \tilde{\to} \Omega_{T/k},\ 1 \otimes m \mapsto \alpha_m.$$

This follows from computations on the affine space, since a choice of basis $m_1, \ldots, m_n$ of M, with induced characters $z_i = \chi^{m_i}$, identifies T with the principal open set $D(\prod_{i=1}^n z_i) \subset \mathbb{A}^n_k$. In particular, we obtain isomorphisms

$$\mathcal{O}_T \otimes_{\mathbb{Z}} \wedge^p M \tilde{\to} \Omega^p_{T/k},\ 1 \otimes m \mapsto \alpha_m.$$

Passing to global sections, the image of $k \otimes_{\mathbb{Z}} M$ is $V \subset \Gamma(T, \Omega_{T/k})$, the subspace of global 1-forms invariant under the torus action. We have induced eigenspace decompositions

$$\Gamma(T, \Omega^p_{T/k}) = \oplus_{m\in M} \chi^m \cdot \wedge^p V.$$

So every regular form $\omega \in \Gamma(T, \Omega^p_{T/k})$ admits a unique decomposition

$$\omega = \sum_{m\in M} \chi^m \cdot \omega(m)\ (\omega(m) \in \wedge^p V).$$

For $e \in N$, there exists a unique k-derivation θ_e of $\Gamma(T, \mathcal{O}_T)$ such that $\theta_e(\chi^m) = \langle m, e\rangle \chi^m$ for every $m \in M$. The application $\alpha \colon N \to \Gamma(T, \mathcal{T}_{T/k}), e \mapsto \theta_e$ is additive and induces an isomorphism of $\mathcal{O}_T$-modules

$$1 \otimes_{\mathbb{Z}} \alpha \colon \mathcal{O}_T \otimes_{\mathbb{Z}} N \to \mathcal{T}_{T/k}.$$

Passing to global sections, the image of $k \otimes_{\mathbb{Z}} N$ is $W \subset \Gamma(T, \mathcal{T}_{T/k})$, the space of derivations invariant under the torus action. We have an eigenspace decomposition

$$\Gamma(T, \mathcal{T}_{T/k}) = \oplus_{m\in M} \chi^m W.$$

So every k-derivation $\theta \colon k[M] \to k[M]$ has a unique decomposition

$$\theta = \sum_{m\in M} \chi^m \theta(m)\ (\theta(m) \in W).$$

As in [15, Lemma 4.3.1] or [51, Proposition 3.1], we obtain the following explict formulas:

Lemma 1.1.4. *Let $e \in N$ be a primitive vector. Then $T_N \subset T_N \operatorname{emb}(\mathbb{R}_{\geq 0} e) = X$ is an affine torus embedding such that $X \setminus T$ consists of a unique prime divisor $E = V(e)$, and both X/k and E/k are smooth. We have eigenspace decompositions:*

a) $\Gamma(X, \Omega^p_{X/k}) = \oplus_{\langle m,e\rangle=0} \chi^m \cdot \alpha(k \otimes_{\mathbb{Z}} \wedge^p(M \cap e^\perp)) \oplus \oplus_{\langle m,e\rangle>0} \chi^m \cdot \wedge^p V$.

b) $\Gamma(X, \tilde{\Omega}^p_{X/k}(\log E)) = \oplus_{\langle m,e\rangle\geq 0} \chi^m \cdot \wedge^p V$.

c) $\Gamma(X, \mathcal{T}_{X/k}) = \oplus_{\langle m,e\rangle=-1} \chi^m \cdot k\theta_e \oplus \oplus_{\langle m,e\rangle\geq 0} \chi^m \cdot W$.

d) $\Gamma(X, \tilde{\mathcal{T}}_{X/k}(-\log E)) = \oplus_{\langle m,e\rangle\geq 0} \chi^m \cdot W$.

Let $\omega \in \Gamma(T, \Omega^p_{T/k})$, and denote $\operatorname{Supp} \omega = \{m \in M; \omega(m) \neq 0\}$. Let $\theta \in \Gamma(T, \mathcal{T}_{T/k})$ and define similarly its support. Let $T = T_N \subseteq X$ be a torus embedding, let $E = V(e)$ be an invariant prime divisor on X. It corresponds to a primitive vector $e \in N$. The following properties hold:

a) ω is regular at E if and only if for every $m \in \operatorname{Supp} \omega$, either $\langle m,e\rangle > 0$, or $\langle m,e\rangle = 0$ and $\omega(m) \in \alpha(k \otimes_{\mathbb{Z}} \wedge^p(M \cap e^\perp))$.

b) ω has at most a logarithmic pole at E if and only if $\langle m,e\rangle \geq 0$ for every $m \in \operatorname{Supp} \omega$.

c) θ is regular at E if and only if for every $m \in \operatorname{Supp} \omega$, either $\langle m,e\rangle \geq 0$, or $\langle m,e\rangle = -1$ and $\theta(m) \in k\theta_e$.

d) θ is regular logarithmic at E if and only if $\langle m,e\rangle \geq 0$ for every $m \in \operatorname{Supp} \theta$.

Properties a)-d) do not depend on the toric model X, only on the valuation of $k(X)$ defined by E. So they follow from Lemma 1.1.4.

Theorem 1.1.5. *[51, Proposition 3.1] Let $T_N \subseteq X$ be a torus embedding. The complement $\Sigma = X \setminus T$ is a reduced Weil divisor on X, and we have natural isomorphisms*

$$1 \otimes_{\mathbb{Z}} \wedge^p \alpha \colon \mathcal{O}_X \otimes_{\mathbb{Z}} \wedge^p M \to \tilde{\Omega}^p_X(\log \Sigma)$$

$$1 \otimes_{\mathbb{Z}} \alpha \colon \mathcal{O}_X \otimes_{\mathbb{Z}} N \to \tilde{\mathcal{T}}_{X/k}(-\log \Sigma)$$

In particular, $\tilde{\Omega}^1_{X/k}(\log \Sigma)$ is a trivial $\mathcal{O}_X$-module of rank equal to the dimension of X, and $\wedge^p \tilde{\Omega}^1_{X/k}(\log \Sigma) \xrightarrow{\sim} \tilde{\Omega}^p_{X/k}(\log \Sigma)$. And $\tilde{\mathcal{T}}_{X/k}(-\log \Sigma)$ is a trivial $\mathcal{O}_X$-module of rank equal to the dimension of X.

1.1.3 Toroidal embeddings

A *toroidal embedding* [39] is an open subset $U \subseteq X$ in a normal variety X/k, such that for every $P \in X$, there exists an affine toric variety $Z = T_N \operatorname{emb}(\sigma)$, a point $Q \in Z$, and an isomorphism of complete local k-algebras

$$\hat{\mathcal{O}}_{X,P} \simeq \hat{\mathcal{O}}_{Z,Q}$$

such that $X \setminus U$ corresponds to $Z \setminus T_N$. It follows that U is nonsingular, and $\Sigma = X \setminus U$ has pure codimension one. If each irreducible component of Σ is normal, the toroidal embedding is called *strict*. A *toroidal morphism* $f\colon (U' \subseteq X') \to (U \subseteq X)$ of toroidal embeddings is a morphism $f\colon X' \to X$, such that for every $P' \in X'$, we can choose local formal isomorphisms as above such that $\hat{\mathcal{O}}_{X,f(P')} \to \hat{\mathcal{O}}_{X',P'}$ corresponds to the morphism $\hat{\mathcal{O}}_{Z,g(Q')} \to \hat{\mathcal{O}}_{Z',Q'}$ induced by a toric morphism $g\colon Z \to Z'$. It follows that $f(U') \subseteq U$.

Given a local formal isomorphism as above, there exists by [10, Corollary 2.6] a hut

$$\begin{tikzcd}[column sep=tiny] & U' \subset X' \ni P' \arrow[dl, "u"'] \arrow[dr, "v"] & \\ U \subset X \ni P & & T_N \subset Z \ni Q \end{tikzcd}$$

with u, v étale, $u(P') = P$, $v(P') = Q$, and $u^{-1}(U) = U' = v^{-1}(T_N)$. By Theorem 1.1.5 and Lemma 1.1.1, we obtain

Theorem 1.1.6. *Let $U \subseteq X/k$ be a toroidal embedding. Then $\tilde{\Omega}^1_{X/k}(\log \Sigma)$ is a locally trivial $\mathcal{O}_X$-module of rank equal to the dimension of X, and $\wedge^p \tilde{\Omega}^1_{X/k}(\log \Sigma) \tilde{\to} \tilde{\Omega}^p_{X/k}(\log \Sigma)$.*

Proposition 1.1.7. *Let $U \subseteq X$ and $V \subseteq Y$ be toroidal embeddings. Let $f\colon X \to Y$ be a morphism such that $f(U) \subseteq V$. The pullback homomorphism $\Omega^\bullet_{V/k} \to f_*\Omega^\bullet_{U/k}$ extends (uniquely) to a homomorphism*

$$\tilde{\Omega}^\bullet_{Y/k}(\log \Sigma_Y) \to f_*\tilde{\Omega}^\bullet_{X/k}(\log \Sigma_X).$$

Proof. We prove the claim in two steps.

Step 1: Suppose $T \subset Z$ is a torus embedding, $v\colon Y \to Z$ is a finite étale morphism, and $V = v^{-1}(T)$. Let M be the lattice of characters of the torus T. Let $m_1, \ldots, m_n$ be a basis of M, denote $t_i = \chi^{m_i}$ $(1 \le i \le n)$. Then $\tilde{\Omega}^1_{Z/k}(\log \Sigma_Z)$ is the free $\mathcal{O}_Z$-module with basis $\frac{dt_i}{t_i}$ $(1 \le i \le n)$, and $\wedge^p \Omega^1_{Z/k}(\log \Sigma_Z) \tilde{\to} \Omega^p_{Y/k}(\log \Sigma_Y)$. By Lemma 1.1.1, $v^*\tilde{\Omega}^\bullet_{Z/k}(\log \Sigma_Z) \tilde{\to} \tilde{\Omega}^\bullet_{Y/k}(\log \Sigma_Y)$. Denote $z_i = v^* t_i$ $(1 \le i \le n)$. Then $z_i \in \Gamma(V, \mathcal{O}_V^*)$, $\tilde{\Omega}^1_{Y/k}(\log \Sigma_Y)$ is the free $\mathcal{O}_Y$-module with basis

$$\omega_i = \frac{dz_i}{z_i} \in \Gamma(Y, \tilde{\Omega}^1_{Y/k}(\log \Sigma_Y)) \ (1 \le i \le n),$$

and $\wedge^p \tilde{\Omega}^1_{Y/k}(\log \Sigma_Y) \tilde{\to} \tilde{\Omega}^p_{Y/k}(\log \Sigma_Y)$. Therefore, in order to prove the claim, it suffices to show that $f^*\omega_i \in \Gamma(X, \tilde{\Omega}^1_{X/k}(\log \Sigma_X))$. But $g_i = f^* z_i \in \Gamma(U, \mathcal{O}_U^*)$, and

$$f^*\omega_i = \frac{dg_i}{g_i} \in \Gamma(X, \tilde{\Omega}^1_{X/k}(\log \Sigma_X)).$$

The claim holds in this case.

Step 2: The claim is local on Y, so we may shrink Y to an affine open neighborhood of a fixed point. By [10], there exists a hut

$$\begin{tikzcd} & V' \subset Y' \arrow[dl, "u"'] \arrow[dr, "v"] & \\ V \subset Y & & T \subset Z \end{tikzcd}$$

where $T \subseteq Z$ is a torus embedding, u, v are étale, and $u^{-1}(V) = V' = v^{-1}(T)$. Denote $X' = X \times_Y Y'$, and consider the base change diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "u'"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "u"] \end{tikzcd}$$

Denote $U' = u'^{-1}(U)$. The restriction of the cartesian diagram to open subsets is also cartesian

$$\begin{tikzcd} U \arrow[d, "f"'] & U' \arrow[l, "u'"'] \arrow[d, "f'"] \\ V & V' \arrow[l, "u"] \end{tikzcd}$$

Since u' is étale, $U' \subseteq X'$ is also a toroidal embedding. Let $\omega \in \Gamma(Y, \Omega^p_{Y/k}(\log \Sigma_Y))$. Then $\omega|_V \in \Gamma(V, \Omega^p_{V/k})$, so $\eta = f^*(\omega|_V) \in \Gamma(U, \Omega^p_{U/k})$. We have $u'^*\eta = f'^* u^*\omega$. By Lemma 1.1.1, $u^*\omega \in \Gamma(Y', \tilde{\Omega}^p_{Y'/k}(\log \Sigma_{Y'}))$. By Step 1, $f'^*u^*\omega \in \Gamma(X', \tilde{\Omega}^p_{X'/k}(\log \Sigma_{X'}))$. Therefore $u'^*\eta$ has at most logarithmic poles along the primes of $\Sigma_{X'}$. By Lemma 1.1.2, $\eta \in \Gamma(X, \tilde{\Omega}^p_{X/k}(\log \Sigma_X))$. □

1.1.4 Log smooth embeddings

A *log smooth embedding* is a toroidal embedding $U \subseteq X$ such that X/k is smooth. If we denote $\Sigma = X \setminus U$, this is equivalent to (X, Σ) being a *log smooth pair*, that is X/k is smooth and the restriction of Σ to each connected component of X is either empty, or a normal crossing divisor. A log smooth embedding is called *strict* if it is so as a toroidal embedding. This is equivalent to the property that each irreducible component of Σ is smooth, that is Σ is a simple normal crossing divisor. We obtain an equivalence between (strict) log smooth embeddings and (strict) log smooth pairs.

We assume $\operatorname{char} k = 0$. We need the special case $E = 0$ of [12, Theorem 3.4]:

Theorem 1.1.8 (Bierstone-Milman)**.** *Let X be a smooth irreducible variety, let Σ be a reduced divisor on X. Let V be an open subset of X such that $\Sigma|_V$ has at most simple normal crossing singularities. Then there exists a proper morphism $\sigma\colon X' \to X$ such that*

a) X' is smooth and $\sigma^{-1}\Sigma$ has at most simple normal crossing singularities.

b) $\sigma\colon \sigma^{-1}(V) \to V$ is an isomorphism.

Lemma 1.1.9. *Let $U \subseteq X$ be an open dense embedding, with U smooth. Then there exists a proper morphism $\sigma\colon X' \to X$ such that $\sigma\colon \sigma^{-1}(U) \to U$ is an isomorphism and $\sigma^{-1}(U) \subseteq X'$ is a strict log smooth open embedding.*

Proof. Since U is smooth, the singular locus of X is contained in $X \setminus U$. By Hironaka's strong resolution of singularities, there exists a proper morphism $\sigma\colon X' \to X$ such that X' is smooth, $\sigma^{-1}(U) \to U$ is an isomorphism, and the complement of $\sigma^{-1}(U)$ in X is a SNC divisor. The open embedding $\sigma^{-1}(U) \subseteq X'$ is therefore strict log smooth, and satisfies the desired properties. □

Lemma 1.1.10. *Let $U \subseteq X$ and $X \subseteq Y$ be open dense embeddings.*

1) If $U \subseteq Y$ is a strict log smooth embedding, so is $U \subseteq X$.

2) Suppose $U \subseteq X$ is a strict log smooth embedding. Then there exists a proper morphism $\sigma\colon Y' \to Y$ such that $\sigma\colon \sigma^{-1}(X) \to X$ is an isomorphism and the open embedding $\sigma^{-1}(U) \subseteq Y'$ is strict log smooth.

Proof. 1) Since Y is smooth, so is X. The divisor $Y \setminus U$ is SNC on Y, hence so is its restriction $(Y \setminus U)|_X = X \setminus U$. Therefore $U \subseteq X$ is strict log smooth.

2) Since $U \subseteq X$ is strict log smooth, X is smooth. By Lemma 1.1.9 for $X \subseteq Y$, we may replace Y by a modification outside X, so that $X \subseteq Y$ is also strict log smooth. Let $\Sigma = Y \setminus U = (Y \setminus X) \cup (X \setminus U)$. Then Σ is a divisor on Y. Its restriction to X is $\Sigma|_X = X \setminus U$, a SNC divisor by assumption. By Theorem 1.1.8, we may replace Y by a modification outside X so that Σ becomes a SNC divisor on Y. Therefore $U \subseteq Y$ is strict log smooth. □

Corollary 1.1.11. *Let $U \subseteq X$ be a strict log smooth embedding. Then there exists an open embedding $j\colon X \subseteq \bar{X}$ such that the induced open embedding $U \subseteq \bar{X}$ is strict log smooth, and $\bar{X}/k$ is proper.*

Proof. By Nagata, there exists an open dense embedding $X \subseteq \bar{X}$, with $\bar{X}/k$ proper. By Lemma 1.1.10.2), we may replace $\bar{X}$ by a modification outside X, so that the induced embedding $U \subseteq \bar{X}$ is strict log smooth. □

Corollary 1.1.12. *Let $U \subseteq X$ and $V \subseteq Y$ be strict log smooth open embeddings. Let $f\colon X \to Y$ be a morphism such that $f(U) \subseteq V$. Then there exists a commutative diagram*

$$\begin{array}{ccccc} U & \longrightarrow & X & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ V & \longrightarrow & Y & \longrightarrow & \bar{Y} \end{array}$$

such that

a) *the vertical arrows are open embeddings.*

b) $U \subseteq \bar{X}$ *and* $V \subseteq \bar{Y}$ *are strict log smooth embeddings.*

c) $\bar{X}$ *and* $\bar{Y}$ *are proper over* k.

Moreover, f *is proper if and only if* $X = \bar{f}^{-1}(Y)$.

Proof. By Corollary 1.1.11, there exists an open embedding $Y \subseteq \bar{Y}$ such that $\bar{Y}$ is proper and $V \subseteq \bar{Y}$ is strict log smooth. By Nagata, there exists an open embedding $X \subseteq X'$ with X' proper. Then f induces a rational map $\bar{f} \colon X' \dashrightarrow \bar{Y}$. Let Γ be the graph of $\bar{f}$, with induced morphisms to X' and $\bar{Y}$, which partially resolve $\bar{f}$. Since $\bar{f}$ is defined over X, $\Gamma \to X'$ is an isomorphism over X. We obtain a chain of open embeddings $U \subseteq X \subseteq \Gamma$. By Lemma 1.1.10.2), we may replace Γ by a modification outside X, denoted $\bar{X}$, such that $U \subseteq \bar{X}$ is strict log smooth. This ends the construction of the diagram. The last statement follows from Lemma 1.1.13. □

Lemma 1.1.13. *Let* $f \colon X \to S$ *be a proper morphism of schemes. Let* $U \subseteq X$ *and* $V \subseteq S$ *be open dense subsets such that* $f(U) \subseteq V$. *The following properties are equivalent:*

1) *The induced morphism* $g \colon U \to V$ *is proper.*

2) $U = f^{-1}(V)$.

Proof. 1) $\Longrightarrow$ 2): Consider the commutative diagram

$$\begin{tikzcd} U \arrow[rr, "\iota"] \arrow[dr, "g"'] & & f^{-1}(V) \arrow[dl, "f|_{f^{-1}(V)}"] \\ & V & \end{tikzcd}$$

Since $f|_{f^{-1}(V)}$ is proper and g is separated, it follows that the open embedding ι is proper. Since U is also dense in $f^{-1}(V)$, we obtain $U = f^{-1}(V)$.

2) $\Longrightarrow$ 1): The morphism g is obtained from f by base change with open embedding $V \subset S$. Therefore it is proper. □

1.1.5 Hypercohomology with supports

Let X be an algebraic variety. Let $U \subseteq X$ be an open subset, let $Z = X \setminus U$. Let $\alpha \colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of bounded below complexes of $\mathcal{O}_X$-modules.

Lemma 1.1.14. *Suppose the natural maps in hypercohomology induced by* α *and* $\alpha|_U$

$$\mathbb{H}^*(X, \mathcal{A}) \to \mathbb{H}^*(X, \mathcal{B}),\ \mathbb{H}^*(U, \mathcal{A}|_U) \to \mathbb{H}^*(U, \mathcal{B}|_U)$$

are isomorphisms. Then the natural map induced by α *in hypercohomology with support in* Z *is also an isomorphism:*

$$\mathbb{H}^*_Z(X, \mathcal{A}) \xrightarrow{\sim} \mathbb{H}^*_Z(X, \mathcal{B}).$$

Proof. The long exact sequences for hypercohomology with supports induce a commutative diagram with exact rows

$$\begin{tikzcd}
\mathbb{H}^{i-1}(X,\mathcal{A}) \arrow[r] \arrow[d] & \mathbb{H}^{i-1}(U,\mathcal{A}) \arrow[r] \arrow[d] & \mathbb{H}^i_Z(X,\mathcal{A}) \arrow[r] \arrow[d] & \mathbb{H}^i(X,\mathcal{A}) \arrow[r] \arrow[d] & H^i(U,\mathcal{A}) \arrow[d] \\
\mathbb{H}^{i-1}(X,\mathcal{B}) \arrow[r] & \mathbb{H}^{i-1}(U,\mathcal{B}) \arrow[r] & \mathbb{H}^i_Z(X,\mathcal{B}) \arrow[r] & \mathbb{H}^i(X,\mathcal{B}) \arrow[r] & \mathbb{H}^i(U,\mathcal{B})
\end{tikzcd}$$

All but the middle vertical arrows are isomorphisms. Then the middle vertical arrow is also an isomorphism, by the 5-lemma. □

Lemma 1.1.15. *Suppose α is a quasi-isomorphism over U. If $\dim Z = 0$ and the natural maps*

$$\mathbb{H}^*_Z(X,\mathcal{A}) \to \mathbb{H}^*_Z(X,\mathcal{B})$$

are isomorphisms, then α is a quasi-isomorphism.

Proof. This follows from the local to global spectral sequence, cf. [52, page 196]. □

Lemma 1.1.16. *Suppose Z is a finite disjoint union of closed subsets Z_i. Then the natural homomorphisms $\oplus_i \mathbb{H}^*_{Z_i}(X,\mathcal{A}) \to \mathbb{H}^*_Z(X,\mathcal{A})$ are isomorphisms.*

Proof. By induction on the cardinality of the Z_i's, and the Mayer-Vietoris sequence. □

1.1.6 Invariance of logarithmic sheaves

Suppose $\operatorname{char} k = 0$. To avoid heavy notation, we denote $\tilde{\Omega}^p_{X/k}(\log\Sigma)$ by $\Omega^p_X(\log\Sigma)$.

Theorem 1.1.17. *Let $(X',\Sigma'),(X,\Sigma)$ be strict log smooth pairs, let $f\colon X'\to X$ be a proper morphism such that $f\colon X'\setminus\Sigma'\to X\setminus\Sigma$ is an isomorphism. Then for every p, the natural homomorphism*

$$\Omega^p_X(\log\Sigma) \to Rf_*\Omega^p_{X'}(\log\Sigma')$$

is a quasi-isomorphism.

Proof. Denote $\alpha\colon \Omega^p_X(\log\Sigma)\to Rf_*\Omega^p_{X'}(\log\Sigma')$. We prove by induction on $\dim X$ that α is a quasi-isomorphism. If $\dim X = 1$, then f is an isomorphism and the claim is clear. Suppose $\dim X\ge 2$. Let Z be the complement of the largest open subset of X where α is a quasi-isomorphism. It is the union of the supports of the following $\mathcal{O}_X$-modules: the cokernel $\mathcal{C}$ of $\Omega^p_X(\log\Sigma)\to f_*\Omega^p_{X'}(\log\Sigma')$, and $R^if_*\Omega^p_{X'}(\log\Sigma')$ $(i>0)$. Suppose by contradiction that Z is nonempty.

Step 1: We claim that $\dim Z\le 0$. Indeed, the statement is local on X, so we may suppose X is affine. Let H be a general hyperplane section of X. Denote $H'=f^*H$. Then $(H,\Sigma|_H)$ and $(H',\Sigma'|_{H'})$ are strict log smooth, and $g=f|_{H'}\colon H'\to H$ maps $H\setminus(\Sigma|_H)$ isomorphically onto $H'\setminus(\Sigma'|_{H'})$. By induction, $\Omega^p_H(\log\Sigma|_H)\to Rg_*\Omega^p_{H'}(\log\Sigma'|_{H'})$ is a quasi-isomorphism.

Since H is general, we have base change isomorphisms

$$R^i f_* \Omega^p_{X'}(\log \Sigma')|_H \xrightarrow{\sim} R^i g_* \Omega^p_{H'}(\log \Sigma'|_{H'}) \ (i \geq 0).$$

For $i > 0$, the right hand side is zero, and therefore $R^i f_* \Omega^p_{X'}(\log \Sigma')|_H$ is zero. For $i = 0$, consider the commutative diagram with exact raws

$$\begin{tikzcd}
\Omega^p_X(\log \Sigma)|_H \arrow[r] \arrow[d, "r"] & f_*\Omega^p_{X'}(\log \Sigma')|_H \arrow[r] \arrow[d, "r'"] & \mathcal{C}|_H \arrow[r] \arrow[d, "r''"] & 0 \\
\Omega^p_H(\log \Sigma|_H) \arrow[r, "\simeq"] & g_*\Omega^p_{H'}(\log \Sigma'|_{H'}) \arrow[r] & 0
\end{tikzcd}$$

where r'' is induced by r and r'. Since r' is injective and r is surjective, it follows that r'' is injective. Therefore $\mathcal{C}|_H = 0$.

We conclude that $Z \cap H = \emptyset$. Therefore $\dim Z \leq 0$.

Step 2: We claim that $\mathbb{H}^*_Z(X, \alpha)$ is an isomorphism. Indeed, denote $U' = X' \setminus \Sigma'$, $U = X \setminus \Sigma$. Then $U' \subseteq X'$ and $U \subseteq X$ are strict log smooth embeddings, and $f \colon X' \to X$ maps U' isomorphically onto U. We compactify this data as in Corollary 1.1.12:

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "f"] & \bar{X}' \arrow[d, "\bar{f}"] \\
U \arrow[r] & X \arrow[r] & \bar{X}
\end{tikzcd}$$

Since f is proper, it is obtained from $\bar{f}$ by base change with the open embedding $X \subseteq \bar{X}$. Denote $\bar{\Sigma}' = \bar{X}' \setminus U'$, $\bar{\Sigma} = \bar{X} \setminus U$. Consider the natural homomomorphism

$$\bar{\alpha} \colon \Omega^p_{\bar{X}}(\log \bar{\Sigma}) \to R\bar{f}_* \Omega^p_{\bar{X}'}(\log \bar{\Sigma}').$$

Since the second square is cartesian, $\bar{\Sigma}'|_{X'} = \Sigma'$, $\bar{\Sigma}|_X = \Sigma$, and logarithmic sheaves commute with base change by open embeddings, we obtain an identification

$$\bar{\alpha}|_X \xrightarrow{\sim} \alpha.$$

Denote $\bar{Z} = Z \cup (\bar{X} \setminus X)$ and $\bar{U} = \bar{X} \setminus \bar{Z} = X \setminus Z$. From the quasi-isomorphism $\bar{\alpha}|_{\bar{U}} \xrightarrow{\sim} \alpha|_{X \setminus Z}$, we deduce that $\bar{\alpha}$ is a quasi-isomorphism over $\bar{U}$. Therefore we obtain isomorphisms

$$\mathbb{H}^*(\bar{U}, \Omega^p_{\bar{X}}(\log \bar{\Sigma})|_{\bar{U}}) \to \mathbb{H}^*(\bar{U}, R\bar{f}_* \Omega^p_{\bar{X}'}(\log \bar{\Sigma}')|_{\bar{U}}).$$

Next, we claim that the homomorphism $\mathbb{H}^*(\bar{X}, \Omega^p_{\bar{X}}(\log \bar{\Sigma})) \to \mathbb{H}^*(\bar{X}, R\bar{f}_* \Omega^p_{\bar{X}'}(\log \bar{\Sigma}'))$ is also an isomorphism. Indeed, it identifies with the homomorphism

$$H^*(\bar{X}, \Omega^p_{\bar{X}}(\log \bar{\Sigma})) \to H^*(\bar{X}', \Omega^p_{\bar{X}'}(\log \bar{\Sigma}')).$$

This is an isomorphism by the Atiyah-Hodge lemma and Deligne's theorem on E_1 degeneration for logarithmic de Rham complexes, since $\bar{X}', \bar{X}$ are proper and $\bar{f} \colon \bar{X}' \setminus \bar{\Sigma}' \to \bar{X} \setminus \bar{\Sigma}$ is the isomorphism $f \colon U' \xrightarrow{\sim} U$.

By Lemma 1.1.14, $\mathbb{H}^*_{\bar{Z}}(\bar{X}, \bar{\alpha})$ is an isomorphism. Since $\bar{Z}$ is the disjoint union of Z with $\bar{X} \setminus X$, we deduce that $\mathbb{H}^*_Z(\bar{X}, \bar{\alpha})$ is an isomorphism. Since Z is contained in the open subset X of $\bar{X}$ and $\bar{\alpha}|_X = \alpha$, it follows by excision that $\mathbb{H}^*_Z(X, \alpha)$ is an isomorphism.

Step 3: Since $\dim Z = 0$ and $\mathbb{H}^*_Z(X, \alpha)$ is an isomorphism, Lemma 1.1.15 implies that α is a quasi-isomorphism. That is $Z = \emptyset$, a contradiction. □

Corollary 1.1.18. *Let $(X', \Sigma'), (X, \Sigma)$ be toroidal pairs, let $f \colon X' \to X$ be a proper morphism such that $f \colon X' \setminus \Sigma' \to X \setminus \Sigma$ is an isomorphism. Then for every p, the natural homomorphism*

$$\Omega^p_X(\log \Sigma) \to Rf_* \Omega^p_{X'}(\log \Sigma')$$

is a quasi-isomorphism.

Proof. We prove the claim in several steps.

Step 0: If moreover (X', Σ') is strict log smooth, if suffices to check the claim for a particular f. Indeed, suppose $g \colon (X'', \Sigma'') \to (X, \Sigma)$ is another morphism with the same properties, with (X'', Σ'') strict log smooth, and we know the claim holds for g. There exists a Hironaka hut

$$\begin{tikzcd} & (X''', \Sigma''') \arrow[dl] \arrow[dr] & \\ (X', \Sigma') \arrow[dr] & & (X'', \Sigma'') \arrow[dl] \\ & (X, \Sigma) & \end{tikzcd}$$

such that (X''', Σ''') is strict log smooth, and all arrows are isomorphisms above $X \setminus \Sigma$. The claim holds for X''/X by assumption, and for X'''/X'' by Theorem 1.1.17. Therefore it holds for X'''/X. By Theorem 1.1.17, it also holds for X'''/X'. Therefore it holds for X'/X.

Step 1: Suppose $(X', \Sigma'), (X, \Sigma)$ and f are toric. In this case, $\Omega^p_X(\log \Sigma) \simeq \mathcal{O}_X^{\oplus r}$ for some r, and $f^* \Omega^p_X(\log \Sigma) \to \Omega^p_{X'}(\log \Sigma')$ is an isomorphism. By the projection formula, our homomorphism is a quasi-isomorphism if and only if

$$\mathcal{O}_X \to Rf_* \mathcal{O}_{X'}$$

is a quasi-isomorphism. This holds, and can be proved combinatorially [15].

Step 2: Suppose (X', Σ') is strict log smooth and (X, Σ) is toric. There exists a toric log resolution $(X'', \Sigma'') \to (X, \Sigma)$, which by construction is an isomorphism over the torus $T = X \setminus \Sigma$. Moreover, (X'', Σ'') is strict log smooth. By Step 1, the claim holds for X''/X. By Step 0, it also holds for X'/X.

Step 3: (X', Σ') is strict log smooth and (X, Σ) is toroidal. The claim is local on X. After possibly shrinking X near a fixed point, there exists a hut

$$\begin{tikzcd} & (Y, \Sigma_Y) \arrow[dl] \arrow[dr] & \\ (X, \Sigma) & & (Z, \Sigma_Z) \end{tikzcd}$$

such that Y/X and Y/Z are étale, Σ_Y is the preimage of both Σ and Σ_Z, and (Z, Σ_Z) is the restriction to an open subset of a toric pair. Our sheaves commute with étale base change, so our claim on X is equivalent to the claim for the pullback of $(X', \Sigma') \to (X, \Sigma)$ to Y. On the other hand, we can construct a toric resolution, which when restricted to the open subset Z will satisfy the claim. After base change to Y, the claim still holds. We obtain two morphisms $(Y', \Sigma') \to (Y, \Sigma_Y) \leftarrow (Y'', \Sigma'')$ as in the claim, with $(Y', \Sigma'), (Y'', \Sigma'')$ strict log smooth. The claim holds for Y''/Y. By Step 0, it also holds for Y'/Y. Therefore it holds for X'/X.

Step 4: By Hironaka, there exists a diagram

$$\begin{tikzcd} (X', \Sigma') \arrow[d] & (X'', \Sigma'') \arrow[l] \arrow[dl] \\ (X, \Sigma) & \end{tikzcd}$$

such that (X'', Σ'') is strict log smooth and X''/X' is an isomorphism over $X' \setminus \Sigma'$. By Step 3, the claim holds for X''/X and X''/X'. Therefore it holds for X'/X. $\square$

1.2 Roots of sections

Let X be a scheme and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For $n \in \mathbb{Z}$, denote the tensor product $\mathcal{L}^{\otimes n}$ by $\mathcal{L}^n$.

Proposition 1.2.1. *Consider a global section $s \in \Gamma(X, \mathcal{L}^n)$, for some positive integer n. Then there exist a morphism of schemes $\pi \colon Y \to X$ and a global section $t \in \Gamma(Y, \pi^*\mathcal{L})$, such that $t^n = \pi^* s$, and the following universal property holds: if $g \colon Y' \to X$ is a morphism of schemes, and $s' \in \Gamma(Y', g^*\mathcal{L})$ is a global section such that $s'^n = g^* s$, then there exists a unique morphism $u \colon Y' \to Y$ such that $g = \pi \circ u$ and $s' = u^* t$.*

Proof. Step 1: Suppose $X = \operatorname{Spec} A$ and $\mathcal{L} = \mathcal{O}_X$. Then $s \in \Gamma(X, \mathcal{O}_X) = A$. The ring homomorphism

$$A \to \frac{A[T]}{(T^n - s)}$$

induces a finite morphism $\pi \colon Y \to X$. If we denote by $t \in \Gamma(Y, \mathcal{O}_Y)$ the class of T, we have $t^n = \pi^* s$. Let $g \colon Y' \to X$ be a morphism of schemes, and $s' \in \Gamma(Y', \mathcal{O}_{Y'})$ with $s'^n = g^* s$.

There exists a unique homomorphism of A-algebras

$$\frac{A[T]}{(T^n - s)} \to \Gamma(Y', \mathcal{O}_{Y'})$$

which maps T to s'. This translates into a morphism $u\colon Y' \to Y$ with $g = \pi \circ u$ and $s' = u^* t$.

Step 2: Consider the $\mathcal{O}_X$-algebra $\mathcal{A} = \oplus_{i=0}^{n-1} \mathcal{L}^{-i}$, with the following multiplication: if u_i, u_j are local sections of $\mathcal{A}_i$ and $\mathcal{A}_j$ respectively, their product is the local section $u_i \otimes u_j$ of $\mathcal{A}_{i+j}$ if $i + j < n$, and the local section $u_i \otimes u_j \otimes s$ of $\mathcal{A}_{i+j-n}$ if $i + j \geq n$. Let $\pi\colon Y = \operatorname{Spec}_X(\mathcal{A}) \to X$ be the induced finite morphism of schemes.

Let $u \in \Gamma(U, \mathcal{L})$ be a nowhere zero section on some open subset $U \subseteq X$. Then $s|_U = f u^n$ for some $f \in \Gamma(U, \mathcal{O}_U)$. Then $\mathcal{L}|_U = \mathcal{O}_U u$ and $\mathcal{A}|_U = \oplus_{i=0}^{n-1} \mathcal{O}_U u^{-i}$, $u^{-1} \in \Gamma(U, \mathcal{A}_1)$ satisfies $(u^{-1})^n = f$, and mapping $T \mapsto u^{-1}$ induces an isomorphism over U

$$\pi^{-1}(U) \tilde{\to} \operatorname{Spec}_U \frac{\mathcal{O}_U[T]}{(T^n - f)}.$$

Therefore the construction of $\mathcal{A}$ globalizes the local construction in Step 1.

Consider the section $u^{-1} \cdot \pi^* u \in \Gamma(\pi^{-1}(U), \pi^* \mathcal{L})$. It satisfies $(u^{-1} \cdot \pi^* u)^n = \pi^*(s|_U)$. If u' is another nowhere zero global section of $\mathcal{L}|_U$, then $u' = vu$ for some unit $v \in \Gamma(U, \mathcal{O}_U^\times)$. Since $\pi^* v = v$, we obtain $u'^{-1} \cdot \pi^* u' = u^{-1} \cdot \pi^* u$. So the section does not depend on the choice of u. Since X can be covered by affine open subsets which trivialize $\mathcal{L}$, it follows that $u^{-1} \cdot \pi^* u$ glue to a section t of $\pi^* \mathcal{L}$ whose n-th power is $\pi^* s$. The universal property can be checked on affine open subsets of X on which $\mathcal{L}$ is trivial, so it follows from Step 1. $\square$

The morphism $\pi\colon Y \to X$, endowed with the section $t \in \Gamma(Y, \pi^* \mathcal{L})$, is unique up to an isomorphism over X. It is called the *n-th root of s*. We denote Y by $X[\sqrt[n]{s}]$, and t by $\sqrt[n]{s}$.

Lemma 1.2.2. *Let $s \in \Gamma(X, \mathcal{L}^n)$ be a global section for some $n \geq 1$, let $\pi\colon X[\sqrt[n]{s}] \to X$ be the n-th root of s. The following properties hold:*

a) $\pi_* \mathcal{O}_{X[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} \mathcal{L}^{-i}$.

b) Suppose the group of units $\Gamma(X, \mathcal{O}_X^*)$ contains a primitive n-th root of 1. Then $\mathbb{Z}/n\mathbb{Z}$ acts on $X[\sqrt[n]{s}]$, the morphism π is the induced quotient map, and a) is the decomposition into eigensheaves.

c) The morphism π is finite and flat.

d) Let $f\colon X' \to X$ be a morphism. Then the n-th root of $f^* s \in \Gamma(X', f^* \mathcal{L}^n)$ is the pullback morphism $X[\sqrt[n]{s}] \times_X X' \to X'$, endowed with the pullback section.

Proof. All properties follow from the local description of the n-th root in the case when X is affine and $\mathcal{L} = \mathcal{O}_X$. Note that π is flat since $\mathcal{L}$ is locally free. In b), let $\zeta \in \Gamma(X, \mathcal{O}_X^*)$ be a primitive n-th root of 1. The action is $(\zeta, aT^i) \mapsto \zeta^i a T^i$ for the local model in Step 1, and $(\zeta, u_i) \mapsto \zeta^i u_i$ for the global model in Step 2. $\square$

Example 1.2.3. Consider $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[T]$, and $n \geq 1$. View T as a global section of $\mathcal{O}^n_{\mathbb{A}^1_{\mathbb{Z}}} = \mathcal{O}_{\mathbb{A}^1_{\mathbb{Z}}}$. The n-th root of T is the endomorphism $\pi_n \colon \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ induced by $T \mapsto T^n$, and the global section is again T. Indeed, (π_n, T) satisfies the universal property. Since roots commute with base change by Lemma 1.2.2.d), we also obtain the following description: let $f \in \Gamma(X, \mathcal{O}_X)$ and $n \geq 1$. View f as a global section of $\mathcal{O}^n_X = \mathcal{O}_X$. The n-th root of f coincides with $X \times_{\mathbb{A}^1_{\mathbb{Z}}} \mathbb{A}^1_{\mathbb{Z}}$, where $f \colon X \to \mathbb{A}^1_{\mathbb{Z}}$ is the morphism induced by f, and $\pi_n \colon \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ is the n-th root of T. We obtain a cartesian diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X[\sqrt[n]{f}] \arrow[l, "\pi"'] \arrow[d] \\ \mathbb{A}^1_{\mathbb{Z}} & \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "\pi_n"'] \end{tikzcd}$$

Remark 1.2.4. Let $m, n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{mn})$. Then the mn-th root of s is the composition of two roots: the m-th root of $s \in \Gamma(X, (\mathcal{L}^n)^m)$, followed by the n-th root of $\sqrt[m]{s}$. Indeed, this composition satisfies the universal property.

Example 1.2.5. If s vanishes nowhere and $n \in \Gamma(X, \mathcal{O}^*_X)$, then $X[\sqrt[n]{s}] \to X$ is a finite étale covering.

Remark 1.2.6. Let A be a ring, $a \in A$ and $u \in U(A)$. Then $A[\sqrt[n]{a}] \tilde{\to} A[\sqrt[n]{u^n a}]$ over A. Indeed, $T \mapsto uT$ induces an A-isomorphism $A[T]/(T^n - a) \tilde{\to} A[T]/(T^n - u^n a)$.

Remark 1.2.7. Suppose X is reduced and irreducible, and $n \in \Gamma(X, \mathcal{O}^*_X)$. Let s_1, s_2 be two nonzero global sections of $\mathcal{L}^n$ with the same zero locus, an effective Cartier divisor D. Then $s_2 = us_1$ for some unit $u \in \Gamma(X, \mathcal{O}^\times_X)$. The two cyclic covers $X[\sqrt[n]{s_i}] \to X$ $(i = 1, 2)$ become isomorphic after base change with the étale cover $\tau \colon X[\sqrt[n]{u}] \to X$. If X/k is proper, then $\Gamma(X, \mathcal{O}_X) = k$, so $u \in k^\times$. Therefore $\sqrt[n]{u} \in k^\times$ so τ is an isomorphism. It follows that the two cyclic covers are already isomorphic over X. Therefore, if X/k is integral and proper, we can speak of the *cyclic cover associated to* $\mathcal{O}_X(D) \simeq \mathcal{L}^n$.

Even if X is reduced and s is nowhere zero, the scheme $X[\sqrt[n]{s}]$ may not be reduced. For example, $\mathbf{F}_2[T]/(T^2 - 1)$ has nilpotents, as $T^2 - 1 = (T - 1)^2$.

Remark 1.2.8. Suppose $\zeta \in \Gamma(X, \mathcal{O}^\times_X)$ satisfies $\zeta^q = 1$, and $q \geq 1$ is minimal with this property. Let $s \in \Gamma(X, \mathcal{L}^n)$ and $n, q \geq 1$. Then $T^{nq} - s^q = \prod_{\zeta \in \mu_q} (T^n - \zeta s)$. Therefore $X[\sqrt[nq]{s^q}] = \cup_{\zeta \in \mu_q} X[\sqrt[n]{\zeta s}]$.

Example 1.2.9 (Singularities of semistable reduction)**.** Let $X = \mathbb{A}^d_k$ and $s = \prod_{i=1}^d z_i^{m_i} \in \Gamma(X, \mathcal{O}_X)$, for some $(m_1, \ldots, m_d) \in \mathbb{N}^d \setminus 0$. Let $n \geq 2$ with $\operatorname{char} k \nmid n$. The n-th root of s is the hypersurface

$$X[\sqrt[n]{s}] = Z(t^n - \prod_{i=1}^d z_i^{m_i}) \subset \mathbb{A}^{d+1}_k.$$

It is smooth if and only if $d = 1$ and $m_1 = 1$. Else, its singular locus is

$$\cup_{m_i \geq 2} Z(t, z_i) \cup \cup_{m_i = m_j = 1, i \neq j} Z(t, z_i, z_j).$$

The components of the former (latter) kind have codimension one (two) in $X[\sqrt[n]{s}]$. Therefore $X[\sqrt[n]{s}]$ is normal if and only if $\max_i m_i = 1$.

Denote $g = \gcd(n, m_1, \ldots, m_d)$. Let $n = gn', m_i = gm'_i$. Denote $s' = \prod_{i=1}^d z_i^{m'_i}$. Then $X[\sqrt[n]{s}]$ is a reduced k-variety, with irreducible decomposition

$$X[\sqrt[n]{s}] = \cup_{\zeta \in \mu_g} X[\sqrt[n']{\zeta s'}].$$

Each irreducible component is isomorphic over X to $X[\sqrt[n']{s'}]$. The latter is the simplicial toric variety $T_\Lambda \operatorname{emb}(\sigma)$, where $\Lambda = \{\lambda \in \mathbb{Z}^d; \sum_{i=1}^d \lambda_i m_i \in r\mathbb{Z}\}$ and $\sigma \subset \mathbb{R}^d$ is the standard positive cone (cf. [39, page 98], [63, Lemma 2.2], [42, Example 9.9, Proposition 10.10]).

If $d = 1$, the root is easier to describe. The normalization $\bar{X}[\sqrt[n']{s'}] \to X[\sqrt[n']{s'}]$ is $\mathbb{A}^1 \to \mathbb{A}^2$, $x \mapsto (x^{m_1}, x^{n'})$. So $X[\sqrt[n]{s}]$ consists of g lines through the origin in the affine plane, each line being isomorphism over X to the morphism $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^{n'}$.

1.2.1 Roots of torus characters

Let T_N be a torus defined over a field k. Let $M = N^*$ be the dual lattice. The multiplicative group of units of the torus is

$$\Gamma(T_N, \mathcal{O}^*_{T_N}) = \{c\chi^m; c \in k^\times, m \in M\}.$$

Let $v \in M$. Let $n \geq 1$. Denote $M' = M + \mathbb{Z}\frac{v}{n} \subset M_\mathbb{Q}$. The lattice dual to M' is

$$N' = \{e \in N; \langle v, e\rangle \in n\mathbb{Z}\}.$$

The set $\{i \in \mathbb{Z}; i\frac{v}{n} \in M\}$ is a subgroup of $\mathbb{Z}$, of the form $n'\mathbb{Z}$ for some divisor $1 \leq n' \mid n$. Denote $d = n/n'$ and $v' = v/d \in M$. Since $\frac{v}{n} = \frac{v'}{n'}$, we obtain $\frac{i}{n'}v \notin M$ for every $0 < i < n'$.

Proposition 1.2.10. *a) Suppose $n' = n$. Then the n-th root of the unit $\chi^v \in \Gamma(T_N, \mathcal{O}^*_{T_N})$ is the torus homomorphism $T_{N'} \to T_N$ induced by the inclusion $N' \subseteq N$, endowed with the unit $\chi^{\frac{v}{n}} \in \Gamma(T_{N'}, \mathcal{O}^*_{T_{N'}})$.*

b) Suppose k contains μ_n, the cyclic group of order n. Then $T_N[\sqrt[n]{\chi^v}]$ is reduced, with d irreducible components, and irreducible decomposition $T_N[\sqrt[n]{\chi^v}] = \sqcup_{\zeta \in \mu_d} T_N[\sqrt[n']{\zeta\chi^{v'}}]$. Each irreducible component is isomorphic to $T_{N'}$ over T_N.

Proof. a) We check that $k[M] \subseteq k[M']$ and $\chi^v = (\chi^{\frac{v}{n}})^n$ satisfy the universal property of the root. Let $g \colon k[M] \to B$ be a ring homomorphism such that $g(\chi^v) = b^n$ for some $b \in B$. The assumption $n' = n$ is equivalent to $\frac{i}{n}v \notin M$ for every $0 < i < n$. That is every element $m' \in M'$ has a unique representation

$$m' = m + \frac{i}{n}v \ (m \in M, 0 \leq i \leq n - 1).$$

Define $g' \colon k[M'] \to B$ by $g'(c\chi^{m'}) = g(c\chi^m)\cdot b^i$ ($c \in k$). This induces a ring homomorphism, the unique extension of g to $k[M']$ such that $g(\chi^{\frac{v}{n}}) = b$.

b) We have $T^n - \chi^v = \prod_{\zeta\in\mu_d}(T^{n'} - \zeta\chi^{v'})$. By a), each factor is irreducible. Therefore $T_N[\sqrt[n]{\chi^v}]$ is reduced, with d irreducible components, and irreducible decomposition

$$T_N[\sqrt[n]{\chi^v}] = \cup_{\zeta\in\mu_d} T_N[\sqrt[n']{\zeta\chi^{v'}}].$$

The union is disjoint, since T_N is normal and $T_N[\sqrt[n]{\chi^v}] \to T_N$ is étale. Since k contains μ_n, $\sqrt[n']{\zeta} \in k$ for every $\zeta \in \mu_d$. By Remark 1.2.6, each irreducible component is isomorphic over T_N with $T_{N'}$. □

Remark 1.2.11. With the same proof, we obtain: let $T_N \subset T_N \operatorname{emb}(\Delta) = X$ be a torus embedding such that χ^v is a regular function on X.

a) Suppose $n' = n$. Then the n-th root of $\chi^v \in \Gamma(X, \mathcal{O}_X)$ is the toric morphism

$$X' = T_{N'} \operatorname{emb}(\Delta) \to X = T_N \operatorname{emb}(\Delta)$$

induced by the inclusion $N' \subseteq N$, endowed with the regular function $\chi^{\frac{v}{n}} \in \Gamma(X', \mathcal{O}_{X'})$.

b) Suppose k contains μ_n. Then $X[\sqrt[n]{\chi^v}]$ is reduced, with d irreducible components, and irreducible decomposition $X[\sqrt[n]{\chi^v}] = \cup_{\zeta\in\mu_d} X[\sqrt[n']{\zeta\chi^{v'}}]$. Each irreducible component is isomorphic over X with $T_{N'} \operatorname{emb}(\Delta)$.

1.2.2 Roots of units in a field

Let K be a field. Let n be a positive integer which is not divisible by the characteristic of K. The polynomial $T^n - 1 \in K[T]$ is separable. It has n distinct roots in the algebraic closure of K, denoted

$$\mu_n(K) = \{x \in \bar{K}; x^n = 1\}.$$

The multiplicative group $\mu_n(K)$ must be cyclic, hence isomorphic to $\mathbb{Z}/n\mathbb{Z}$. An n-th root of unity $\zeta \in \mu_n(K)$ is a generator if and only if $\zeta^{n/d} \neq 1$ for every divisor $1 < d \mid n$. A generator is called a *primitive n-th root of unity of* K. We have $T^n - 1 = \prod_{\zeta\in\mu_n(K)}(T - \zeta)$ in $\bar{K}[T]$. Therefore

$$T^n - x^n = \prod_{\zeta\in\mu_n(K)} (T - \zeta x)$$

for every $x \in K^\times$. If K contains $\mu_n(K)$, the decomposition holds in $K[T]$.

Let $f \in K^\times$. Consider the integral extension $K \to K[T]/(T^n - f)$. We have

$$K[T]/(T^n - f) = \oplus_{i=0}^{n-1} K t^i.$$

Lemma 1.2.12. *Let $1 \le d \mid n$ be the maximal divisor of n such that $T^d - f$ has a root in K, say g. Suppose K contains $\mu_d(K)$ and a root of $T^4 + 4$. Let $n = dn'$. Then*

$$T^n - f = \prod_{\zeta\in\mu_d(K)} (T^{n'} - \zeta g)$$

is the decomposition into distinct irreducible factors in $K[T]$. The polynomial classes

$$P_\zeta = \frac{1}{\prod_{\zeta' \in \mu_d(K) \setminus \zeta}(\zeta g - \zeta' g)} \frac{T^n - f}{T^{n'} - \zeta g} \in \frac{K[T]}{(T^n - f)} \ (\zeta \in \mu_d(K))$$

induce an isomorphism of K-algebras

$$\prod_{\zeta \in \mu_d(K)} \frac{K[T]}{(T^{n'} - \zeta g)} \xrightarrow{\sim} \frac{K[T]}{(T^n - f)}, (\alpha_\zeta)_\zeta \mapsto \sum_\zeta \alpha_\zeta P_\zeta.$$

On the left hand side, each factor is a separable field extension of K. In particular, the ring $K[T]/(T^n - f)$ is reduced; and an integral domain if and only if $d = 1$.

The cyclic group $\mu_d(K)$ acts on $K[T]/(T^n - f)$, trivially on K and by multiplication on T. Under the isomorphism, this corresponds to the $\mu_d(K)$-action on the product given by the partial permutations $\xi\colon (\alpha_\zeta)_\zeta \mapsto (\alpha_{\xi^{n'}\zeta})_\zeta$. Moreover, if K contains $\mu_n(K)$, then $\mu_n(K)$ acts on both sides by the same formulas, and the action is transitive (hence it permutes the factor fields).

Proof. The decomposition of $T^n - f$ is clear, and the factors are distinct. Suppose by contradiction that $T^{n'} - \zeta g$ is not irreducible. It follows from [46, Theorem III.9.16] that there exists $x \in K$ such that a) $\zeta g = x^p$ for some prime $p \mid n'$; or b) $\zeta g = -4x^4$ and $4 \mid n'$. In case b), $\zeta g = (yx)^4$ where y is a root of $T^4 + 4$ in K. Therefore $\zeta g = h^{d'}$ for some $h \in K$ and $1 < d' \mid n'$. Then $h^{dd'} = f$ and $d < dd' \mid n$, contradicting the maximality of d.

The standard formula for partial fractions with distinct linear factors gives $1 = \sum_{\zeta \in \mu_d(K)} P_\zeta$. We have $P_\zeta \cdot P_{\zeta'} = 0$ if $\zeta \neq \zeta'$, and $P_\zeta^2 = P_\zeta$. We obtain the desired isomorphism.

Consider the $\mu_d(K)$-action. Since P_ζ is idempotent, it follows that $(P_\zeta)^\xi = P_{\xi^{-n'}\zeta}$ for every $\xi \in \mu_d(K)$. Therefore the isomorphism transforms this action into the partial permutations $\xi\colon 1_\zeta \mapsto 1_{\xi^{-n'}\zeta}$. That is $\xi\colon (\alpha_\zeta)_\zeta \mapsto (\alpha_{\xi^{n'}\zeta})_\zeta$. Note that the action is trivial if and only if $d \mid n'$.

Suppose moreover that $\mu_n(K) \subset K$. It acts on both sides by the same formulas. Note that of $\xi \in \mu_n(K)$, then multiplication by $\xi^{-n'}$ induces a bijection of $\mu_d(K)$. The action is transitive, since for every $\zeta, \zeta' \in \mu_d(K)$ there exists $\xi \in \mu_n(K)$ such that $\xi^{n'}\zeta' = \zeta$. □

Note that any field which contains an algebraically closed field, contains a root of $T^4 + 4$. And $T^4 + 4c^4 = (T^2 - 2cT + 2c^2)(T^2 + 2cT + 2c^2)$.

Lemma 1.2.13. *Let $A \subseteq K$ be a domain, integrally closed in K. Suppose A contains $\mu_n(K)$, and the reciprocal of the Vandermonde determinant associated to some ordering of the elements of $\mu_n(K)$. Consider the ring homomorphism $A \to K[T]/(T^n - f)$. The integral closure of A is*

$$\bar{A} = \oplus_{i=0}^{n-1} \{x \in K; x^n f^i \in A\} t^i.$$

Moreover, under the product decomposition of Lemma 1.2.12, $\bar{A}$ corresponds to the product of integral closures of A in the factor fields, and each of the d-factors has the explicit description

$$\bar{A}_\zeta = \oplus_{j=0}^{n'-1} \{x \in K; x^n f^j \in A\} t_\zeta^j.$$

The integral extension $A \to \bar{A}$ is Galois, with Galois group $\mu_n(K)$. The Galois group permutes the factors of the decomposition.

Proof. Each element $\alpha \in K[T]/(T^n - f)$ has a unique representation

$$\alpha = \sum_{i=0}^{n-1} x_i t^i \ (x_i \in K).$$

Step 1: We claim that $\alpha \in \bar{A}$ if and only if $x_i t^i \in \bar{A}$, for every i. Indeed, the converse is clear. Suppose $\alpha \in \bar{A}$. The cyclic group $\mu_n(K)$ acts on $K[T]/(T^n - f)$, trivially on K, and by multiplication on T. It also acts on A, since A contains $\mu_n(K)$. Therefore $\alpha^\zeta = \sum_{i=0}^{n-1} \zeta^i x_i t^i \in \bar{A}$, for every $\zeta \in \mu_n(K)$. Choose an ordering $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ of $\mu_n(K)$. Since the Vandermonde determinant

$$\det \begin{bmatrix} \zeta_0^0 & \zeta_0^1 & \cdots & \zeta_0^{n-1} \\ \zeta_1^0 & \zeta_1^1 & \cdots & \zeta_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n-1}^0 & \zeta_{n-1}^1 & \cdots & \zeta_{n-1}^{n-1} \end{bmatrix} = \prod_{0 \le i < j \le n-1} (\zeta_j - \zeta_i)$$

is a unit in A, each $x_i t^i$ is a combination of the α^ζ's, with coefficients in A. Therefore $x_i t^i \in \bar{A}$.

Step 2: Let $x \in K$ and $0 \le i < n$. We claim that $xt^i \in \bar{A}$ if and only if $x^n f^i \in A$. Indeed, denote $y = x^n f^i \in K$. We have $y = (xt^i)^n$. If $y \in A$, then $xt^i \in \bar{A}$. Conversely, suppose there exists an equation in $K[T]/(T^n - f)$

$$(xt^i)^m + \sum_{l=1}^{m} a_l (xt^i)^{m-l} = 0 \ (a_l \in A).$$

Raising the equation to some power, we may suppose $n \mid m$. Then $m - l \equiv 0 \pmod{n}$ if and only if $n \mid l$. So the coefficient of t^0 in the equation is

$$y^{\frac{m}{n}} + \sum_{n \mid l} a_l y^{\frac{m-l}{n}} = 0.$$

Then y is algebraic over A. Therefore $y \in A$.

Step 3: Consider the product decomposition from Lemma 1.2.12. We have $(\sum_\zeta \alpha_\zeta P_\zeta)^m = \sum_\zeta \alpha_\zeta^m P_\zeta$ for every $m \ge 0$. Therefore $\sum_\zeta \alpha_\zeta P_\zeta$ is integral over A if and only if each α_ζ is integral. By Steps 1 and 2, the integral closure of A in $K[T]/(T^{n'} - \zeta g)$ is the integrally closed domain

$$\bar{A}_\zeta = \oplus_{j=0}^{n'-1} \{x \in K; x^{n'} (\zeta g)^j \in A\} t_\zeta^j.$$

Since A is integrally closed in K, $x^{n'}(\zeta g)^j \in A$ if and only if $(x^{n'}(\zeta g)^j)^d \in A$. That is $x^n f^q \in A$. Therefore

$$\bar{A}_\zeta = \oplus_{j=0}^{n'-1} \{x \in K; x^n f^j \in A\} t_\zeta^j.$$

In particular, the product decomposition of Lemma 1.2.12 induces an isomorphism of A-algebras

$$\prod_{\zeta \in \mu_d(K)} \bar{A}_\zeta \xrightarrow{\sim} \bar{A}.$$

□

Remark 1.2.14. Denote $X = \operatorname{Spec} A$. If f is a non-zero rational function on X, that is $f \in Q(A)$, then $\{x \in Q(A); x^n f^i \in A\} = \Gamma(X, \mathcal{O}_X(\lfloor \frac{i}{n} \operatorname{div}(f) \rfloor))$.

1.2.3 Irreducible components, normalization of roots

Lemma 1.2.15. *Let A be an integral domain, integrally closed in its field of fractions Q. Suppose Q contains a root of $T^4 + 4$. Let $0 \neq a \in A$ and $n \geq 1$.*

1) *Suppose $T^d - a$ has no root in A, for every divisor $1 < d \mid n$. Then $A[T]/(T^n - a)$ is an integral domain with quotient field $Q[T]/(T^n - a)$.*

2) *Suppose $\operatorname{char}(Q) \nmid n$, and A contains $\mu_n(Q)$ and the reciprocal of the Vandermonde determinant associated to some ordering of the elements of $\mu_n(Q)$ (e.g. A contains an algebraically closed field whose characteristic does not divide n). Then the ring $A[T]/(T^n - a)$ has no nilpotents, and the integral closure in its total ring of fractions $Q[T]/(T^n - a)$ is $\oplus_{i=0}^{n-1} \{q \in Q; q^n a^i \in A\} t^i$.*

Proof. 1) Since A is integrally closed, $T^d - a$ has a root in A if and only if it has a root in Q. Therefore $d = 1$ in Lemma 1.2.12. Therefore $Q[T]/(T^n - a)$ is a field. The application $A[T]/(T^n - a) \to Q[T]/(T^n - a)$ is injective. Therefore $A[T]/(T^n - a)$ is an integral domain with quotient field $Q[T]/(T^n - a)$.

2) Let $1 \leq d \mid n$ be the maximal divisor such that $a = a'^d$ for some $a' \in A$. Denote $n' = n/d$. We have

$$T^n - a = \prod_{\zeta \in \mu_d(Q)} (T^{n'} - \zeta a')$$

By the maximality of d, each $T^{n'} - \zeta a'$ is irreducible in $Q[T]$. Since $A^\times$ contains μ_d, there are no multiple factors. Therefore $A[T]/(T^n - a)$ is reduced, with irreducible components $A[T]/(T^{n'} - \zeta a')$.

Finally, since $a \in A$, the homomorphism $A \to Q[T]/(T^n - a)$ factors as $A \to A[T]/(T^n - a) \to Q[T]/(T^n - a)$. The ring extension $A \to A[T]/(T^n - a)$ is integral. Therefore the integral closure of A in $Q[T]/(T^n - a)$, computed in Lemma 1.2.13, coincides with the integral closure of $A[T]/(T^n - a) = A[\sqrt[n]{a}]$ in its total ring of fractions $Q[T]/(T^n - a)$. □

Proposition 1.2.16. *Let X be normal and irreducible scheme. Suppose the field of fractions $Q(X)$ contains a root of $T^4 + 4$. Let $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^n)$.*

a) *Suppose $T^d - s$ has no root in $\Gamma(X, \mathcal{L}^{\frac{n}{d}})$, for every divisor $1 < d \mid n$. Then $X[\sqrt[n]{s}]$ is reduced and irreducible.*

b) Let $1 \leq d \mid n$ be the maximal divisor of n with the property that $T^d - s$ has a root in $\Gamma(X, \mathcal{L}^{\frac{n}{d}})$, say s'. Suppose $\Gamma(X, \mathcal{O}_X^\times)$ contains μ_d. Then $X[\sqrt[n]{s}]$ is reduced, with irreducible decomposition

$$X[\sqrt[n]{s}] = \cup_{\zeta \in \mu_d} X[\sqrt[\frac{n}{d}]{\zeta s'}].$$

c) Suppose $s \neq 0$. Let $U \subseteq X$ be a non-empty open subset such that $\mathcal{L}|_U$ has a nowhere zero section u. Let $s|_U = fu^n$ with $f \in \Gamma(U, \mathcal{O}_X)$. The normalization of $X[\sqrt[n]{s}]$ coincides with the normalization of X in the ring extension $Q(X) \to Q(X)[T]/(T^n - f)$.

Proof. a) Let $U \subset X$ be a non-empty affine open subset such that $\mathcal{L}|_U$ has a nowhere zero section u. We claim that $T^d - s|_U$ has no root in $\Gamma(U, \mathcal{L}^{\frac{n}{d}})$, for every divisor $1 < d \mid n$. Indeed, if $s_d \in \Gamma(U, \mathcal{L}^{\frac{n}{d}})$ is a root, then since $s_d^d = s \in \Gamma(X, \mathcal{L}^n)$, it follows that s_d is the restriction to U of some $s' \in \Gamma(X, \mathcal{L}^{\frac{n}{d}})$. Then s'^d and s coincide on the dense open subset U, hence they are equal. This contradicts our assumption.

Let $s|_U = fu^n$ with $f \in A = \Gamma(U, \mathcal{O}_X)$. The claim is equivalent to the following property: $T^d - f$ has no root in A, for every divisor $1 < d \mid n$. By Lemma 1.2.15, $\pi^{-1}(U)$ is reduced and irreducible, and dominates U.

Since X can be covered by subsets U as above, it follows that $X[\sqrt[n]{s}]$ is reduced and irreducible.

b) We have $T^n - s = T^n - s'^d = \prod_{\zeta \in \mu_d}(T^{\frac{n}{d}} - \zeta s')$. The factors are distinct. Since d is maximal, $T^{d'} - \zeta s'$ has no root in $\Gamma(X, \mathcal{L}^{\frac{n}{dd'}})$, for every divisor $1 < d' \mid \frac{n}{d}$. By a), $X[\sqrt[\frac{n}{d}]{\zeta s'}]$ are the irreducible components of $X[\sqrt[n]{s}]$.

c) This follows from Lemma 1.2.15. □

From Lemma 1.2.15 and Remark 1.2.14, we deduce

Proposition 1.2.17. *Let k be an algebraically closed field. Let n be a positive integer which is not divisible by* $\mathrm{char}(k)$. *Let X/k be a normal algebraic variety, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and $0 \neq s \in \Gamma(X, \mathcal{L}^n)$. Let D be the zero locus of s, an effective Cartier divisor on X. Denote by $\bar{X}[\sqrt[n]{s}] \to X[\sqrt[n]{s}]$ the normalization, and $\bar{\pi}\colon \bar{X}[\sqrt[n]{s}] \to X$ the induced morphism.*

$$\begin{tikzcd} X[\sqrt[n]{s}] \arrow[d, "\pi"'] & \bar{X}[\sqrt[n]{s}] \arrow[l] \arrow[dl, "\bar{\pi}"] \\ X & \end{tikzcd}$$

Then $\bar{\pi}$ is a Galois ramified cover, with Galois group cyclic of order n, and eigenspace decomposition

$$\bar{\pi}_* \mathcal{O}_{\bar{X}[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} \mathcal{L}^{-i}(\lfloor \frac{i}{n} D \rfloor).$$

Call $\bar{\pi}$ the *cyclic cover obtained by taking the n-th root out of s.* Note that $\bar{\pi}$ is flat if and only if the Weil divisor $\lfloor \frac{i}{n} D \rfloor$ is Cartier, for every $0 < i < n$. If X/k is smooth, Proposition 1.2.17 was proved in [25, Corollary 3.11].

1.3 Normalized roots of rational functions

Let n be a positive integer, let k be an algebraically closed field which contains n distinct roots of unity. We consider the category of normal k-varieties and dominant morphisms. A morphism $f\colon X \to Y$ is called *dominant* if for every irreducible component X_i of X, $\overline{f(X_i)}$ is an irreducible component of Y. A composition of dominant morphisms is again dominant. But the image of a dominant morphism may not be dense (e.g. the open embedding of a connected component).

Let X, Y be normal k-varieties. A dominant morphism $f\colon X \to Y$ induces a pullback homomorphism of rings $f^*\colon k(Y) \to k(X)$. It is compatible with composition of dominant morphisms, and maps invertible rational functions to invertible rational functions. Note that f^* may not be injective.

Proposition 1.3.1. *Let X/k be a normal variety, φ an invertible rational function on X. Then there exists a normal variety Y/k, a dominant morphism $\pi\colon Y \to X$ and an invertible rational function t on Y, such that $t^n = \pi^*\varphi$, and the following universal property holds: if Y'/k is a normal variety, $g\colon Y' \to X$ is a dominant morphism, and t' is an invertible rational function on Y' such that $t'^n = g^*\varphi$, then there exists a unique dominant morphism $u\colon Y' \to Y$ such that $g = \pi \circ u$ and $t' = u^*t$.*

Proof. Step 1: Suppose $X = \operatorname{Spec} A$ and A is an integral domain. Then $\varphi \in Q(A)$. Let $\bar{A}$ be the integral closure of A in the ring extension

$$A \to \frac{Q(A)[T]}{(T^n - \varphi)}.$$

Then $A \to \bar{A}$ induces a finite morphism $\pi\colon Y \to X$. By Lemmas 1.2.12 and 1.2.13, $Q(A)[T]/(T^n - \varphi)$ is a product of fields K_ζ, and $\bar{A}$ is the product of the normalization of A in K_ζ. Each factor is a integral domain, integrally closed in its function field. Therefore Y is normal. The morphism π is dominant since it is finite. If we denote by $t \in k(Y)$ the class of T, then t is invertible and $t^n = \pi^*\varphi$.

Let Y'/k be a normal variety, $g\colon Y' \to X$ a dominant morphism, and t' an invertible rational function on Y' such that $t'^n = g^*\varphi$. There exists a unique homomorphism of $Q(A)$-algebras

$$\frac{Q(A)[T]}{(T^n - \varphi)} \to Q(Y')$$

which maps T to t'. Since A maps to $\Gamma(Y', \mathcal{O}_{Y'})$, $\bar{A}$ maps to the integral closure of $\Gamma(Y', \mathcal{O}_{Y'})$ in $Q(Y')$. Since Y' is normal, $\Gamma(Y', \mathcal{O}_{Y'})$ is integrally closed in $Q(Y')$. We obtain a morphism $\bar{A} \to \Gamma(Y', \mathcal{O}_{Y'})$. This translates into a dominant morphism $u\colon Y' \to Y$ with $g = \pi \circ u$ and $t' = u^*t$.

Step 2: Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A}(X, \varphi, n) = \oplus_{i=0}^{n-1} \mathcal{O}_X(\lfloor \frac{i}{n} \operatorname{div}(\varphi) \rfloor),$$

with the following multiplication: if u_i, u_j are local sections of $\mathcal{A}_i$ and $\mathcal{A}_j$ respectively, their product is the rational function $u_i u_j \in \mathcal{A}_{i+j}$ if $i + j < n$, and $u_i u_j \varphi \in \mathcal{A}_{i+j-n}$ if $i + j \geq n$. Let $\pi \colon Y = \operatorname{Spec}_X \mathcal{A} \to X$ be the induced finite morphism of schemes.

Let $U = \operatorname{Spec} A \subset X$ be an affine irreducible open subset. By Remark 1.2.14,

$$\Gamma(U, \mathcal{A}_i) = \{\psi \in Q(A); \psi^n \varphi^i \in A\}.$$

By Lemma 1.2.13, the integral closure of A in $Q(A)[T]/(T^n - \varphi)$ is

$$\bar{A} = \oplus_{i=0}^{n-1} \{\psi \in Q(A); \psi^n \varphi^i \in A\} t^i.$$

Therefore $\psi_i \mapsto \psi_i t^i$ induces an isomorphism of A-algebras

$$\Gamma(U, \mathcal{A}) \tilde{\to} \bar{A}.$$

Therefore the construction of $\mathcal{A}(X, \varphi, n)$ globalizes the local construction in Step 1. Since X is covered by such U, we deduce that Y is normal.

Let $V = X \setminus \operatorname{Supp}(\varphi)$ be the open dense subset of X where φ is a unit. Then $\Gamma(V, \mathcal{A}_1) = \Gamma(V, \mathcal{O}_V)$. Let $t' = 1 \in \Gamma(V, \mathcal{A}_1)$. Then $t'^n = \varphi$ in $\Gamma(V, \mathcal{A})$. So t' becomes an invertible rational function on Y such that $t'^n = \pi^* \varphi$. The universal property can be checked on irreducible affine open subsets of X, so it follows from Step 1. $\square$

The morphism $\pi \colon Y \to X$, endowed with the invertible rational function $t \in k(Y)$, is unique up to an isomorphism over X. It is called the *normalized n-th root of X with respect φ*. We denote Y by $X[\varphi, n]$, and t by $\sqrt[n]{\varphi}$. The $\mathbb{Q}$-Weil divisor $D = \frac{1}{n} \operatorname{div}(\varphi)$ satisfies $nD \sim 0$.

Lemma 1.3.2. *The following properties hold:*

a) $\pi_* \mathcal{O}_{X[\varphi,n]} = \oplus_{i=0}^{n-1} \mathcal{O}_X(\lfloor iD \rfloor)$.

b) The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts faithfully on $X[\varphi, n]$, π is the induced quotient morphism, and a) is the decomposition into eigensheaves.

c) The morphism π is finite. It is flat if and only if the Weil divisors $\lfloor D \rfloor, \ldots, \lfloor (n-1)D \rfloor$ are Cartier.

d) Let $\tau \colon X' \to X$ be an étale morphism. Then the normalized n-th root of X' with respect to $\tau^* \varphi$ is the pullback morphism $X[\varphi, n] \times_X X' \to X'$, endowed with the pullback invertible rational function.

Proof. Properties a), b), c) follow from the local description of the normalized n-th root. In b), let $\zeta \in k^*$ be a primitive n-th root of 1. The action is $(\zeta, aT^i) \mapsto \zeta^i a T^i$ for the local model in Step 1, and $(\zeta, u_i) \mapsto \zeta^i u_i$ for the global model in Step 2. For d), Lemma 1.1.3 gives $\tau^* \mathcal{A}(X, \varphi, n) \tilde{\to} \mathcal{A}(X', \tau^* \varphi, n)$. Therefore the normalized n-th root commutes with étale base change. $\square$

Lemma 1.3.3. *Let φ' be another invertible rational function on X. Then $X[\varphi, n]$ is naturally isomorphic to $X[\varphi'^n\varphi, n]$ over X.*

Proof. For an invertible $\varphi \in k(X)$ and a Weil divisor D on X, the following formula holds:

$$\mathcal{O}_X((\varphi) + D) = \varphi^{-1}\mathcal{O}_X(D).$$

Therefore the application $s_i \mapsto \varphi'^i s_i$ induces an isomorphism of $\mathcal{O}_X$-algebras

$$\mathcal{A}(X, \varphi'^n\varphi, n) \tilde{\to} \mathcal{A}(X, \varphi, n).$$

□

Suppose X is irreducible. Let $1 \le d \mid n$ be the maximal divisor such that $\varphi = \psi^d$ for some $\psi \in k(X)^*$. Then $X[\varphi, n]$ has exactly d irreducible (connected) components

$$X[\varphi, n] = \sqcup_{\zeta \in \mu_d} X[\zeta\psi, n/d].$$

Each component is isomorphic over X to $X[\psi, n/d]$.

1.3.1 Structure in codimension one

At the generic point of each prime divisor on X, the normalized root is explicitly described by the following lemma. We use the convention $\gcd(n, 0) = n$.

Lemma 1.3.4. *Suppose $\varphi = uf^m$, where $u \in \Gamma(X, \mathcal{O}_X^*)$, $f \in \Gamma(X, \mathcal{O}_X)$ is a non-zero divisor such that the divisor $\operatorname{div}(f)$ is reduced, and $m \in \mathbb{Z}$. Let $g = \gcd(n, m)$. Let $n = gn'$, and $1 \le j \le n'$ with $jm \equiv g \pmod{n}$. Then there exists an isomorphism of $\mathcal{O}_X$-algebras*

$$\pi_*\mathcal{O}_Y \simeq \mathcal{O}_X[T_1, T_2]/(T_1^g - u, T_2^{n'} - fT_1^j).$$

That is, π is isomorphic to the composition of the g-th root of the unit u, followed by the n'-th root of the regular function $\sqrt[g]{u}^j f$. The above formula simplifies to $\pi_*\mathcal{O}_Y \simeq \mathcal{O}_X[T]/(T^n - u^j f)$ if $g = 1$, and to $\pi_*\mathcal{O}_Y \simeq \mathcal{O}_X[T]/(T^n - u)$ if $g = n$.

Proof. Since $\operatorname{div}(f)$ is reduced, we have $\lfloor iD \rfloor = \lfloor \frac{im}{n} \rfloor \operatorname{div}(f)$. Therefore

$$\mathcal{A} = \oplus_{i=0}^{n-1} \mathcal{O}_X f^{-\lfloor \frac{im}{n} \rfloor} t^i \ (t^n = uf^m).$$

Let $m = gm'$. Given $0 \le i < n$, there are uniquely defined integers $0 \le \alpha < n', 0 \le \beta < g$ such that $i \equiv \alpha j \pmod{n'}$ and $\frac{i - \alpha j}{n'} \equiv \beta \pmod{g}$. In particular, $\{\frac{jm}{n}\} = \frac{1}{n'}$ and $\{\frac{jm\alpha}{n}\} = \frac{\alpha}{n'}$. Let $\gamma \in \mathbb{Z}$ with $i - \alpha j - n'\beta = n\gamma$. We obtain $m\gamma + \lfloor \frac{jm}{n} \rfloor \alpha + m'\beta = \lfloor \frac{im}{n} \rfloor$. Therefore the following holds in $\mathcal{A}$:

$$u^\gamma (f^{-\lfloor \frac{jm}{n} \rfloor} t^j)^\alpha (f^{-m'} t^{n'})^\beta = f^{-\lfloor \frac{im}{n} \rfloor} t^i.$$

It follows that the homomorphism

$$\mathcal{O}_X[T_1, T_2]/(T_1^g - u, T_2^{n'} - fT_1^j) \to \mathcal{A},\ T_1 \mapsto f^{-m'} t^{n'}, T_2 \mapsto f^{-\lfloor \frac{jm}{n} \rfloor} t^j$$

is well defined and surjective. It is injective by the uniqueness of α, β. The simplified forms of the formula are clear. □

Lemma 1.3.5. *The ramification index of π over a prime divisor E of X is*

$$\frac{n}{\gcd(n, \operatorname{ord}_E(\varphi))}.$$

In particular, π ramifies exactly over the prime divisors of $\operatorname{Supp}\{D\}$*. Moreover,*

$$\pi^*D = \sum_{E' \subset Y} \frac{\operatorname{ord}_{\pi(E')}(\varphi)}{\gcd(n, \operatorname{ord}_{\pi(E')}(\varphi))} E'.$$

Proof. Let E be a prime divisor on X. Let $m = \operatorname{ord}_E(\varphi)$. Let U be an open subset such that $U \cap E \neq \emptyset$ is nonsingular, cut out by a local parameter $f \in \Gamma(U, \mathcal{O}_X)$. By shrinking U, we may suppose $\varphi = uf^m$, for some $u \in \Gamma(U, \mathcal{O}_X^\times)$. Let $g = \gcd(n, m)$, $n = gn'$. By Lemma 1.3.4, $\pi^{-1}(U) \to U$ is the composition of the étale cover $U[\sqrt[g]{u}] \to U$, followed by the n'-th root of the regular function $\sqrt[g]{u}^j f$, which is a local parameter at each prime of $U[\sqrt[g]{u}]$ over E. Therefore the ramification index over E is n'. The pullback formula follows. Note that $n' \neq 1$ if and only if $\operatorname{mult}_E D \notin \mathbb{Z}$. □

Lemma 1.3.6. *Let Σ be a reduced Weil divisor on X which contains* $\operatorname{Supp}\{D\}$*. Let $\Sigma_Y = \pi^{-1}\Sigma$ be the preimage reduced Weil divisor. We have eigenspace decompositions:*

a) $\pi_*\tilde{\Omega}^p_{Y/k}(\log \Sigma_Y) = \oplus_{i=0}^{n-1} \tilde{\Omega}^p_{X/k}(\log \Sigma)(\lfloor iD \rfloor)$.

b) $\pi_*\tilde{\Omega}^p_{Y/k} = \oplus_{i=0}^{n-1} \tilde{\Omega}^p_{X/k}(\log \operatorname{Supp}\{iD\})(\lfloor iD \rfloor)$.

c) $\pi_*\mathcal{T}_{Y/k} = \oplus_{i=0}^{n-1} \tilde{\mathcal{T}}_{X/k}(-\log \sum_E \epsilon(d_E, i)E)(\lfloor iD \rfloor)$*, where for a rational number d we define $r(d) = \min\{r \geq 1; rd \in \mathbb{Z}\}$, and set $\epsilon(d, i)$ to be 1 if $d \notin \mathbb{Z}$ and $id + \frac{1}{r(d)} \notin \mathbb{Z}$, and zero otherwise. In particular, the invariant part of $\pi_*\mathcal{T}_{Y/k}$ is* $\tilde{\mathcal{T}}_{X/k}(-\log \operatorname{Supp}\{D\})$.

d) $\pi_*\mathcal{T}_{Y/k}(-\log \Sigma_Y) = \oplus_{i=0}^{n-1} \tilde{\mathcal{T}}_{X/k}(-\log \Sigma)(\lfloor iD \rfloor)$.

Proof. Let $V = X \setminus (\operatorname{Sing} X \cup \operatorname{Supp}\{D\})$, an open dense subset of X. Then π is étale over V. In particular, $\pi^{-1}(V)$ is also nonsingular, and $\pi^*\Omega_{V/k} \simeq \Omega_{\pi^{-1}(V)/k}$. Therefore $\pi^*\Omega^p_{V/k} \simeq \Omega^p_{\pi^{-1}(V)/k}$, and the projection formula gives

$$\pi_*\Omega^p_{\pi^{-1}(V)/k} = \oplus_{i=0}^{n-1} \Omega^p_{V/k}(iD|_V)t^i.$$

This describes the sheaves in a) and b) at the generic points of X. These sheaves are S_2, so we may determine them locally near a fixed prime divisor on X. Let E be a prime divisor on X. We may shrink X and suppose $\varphi = uf^m$, with u a unit and f a parameter for E. Let E' be a prime divisor over E. Then t_2 is a local parameter at E', and t_1 is a unit at E' (in the notations of Lemma 1.3.4). Recall that t is the n-th root of φ.

Let ω be a rational p-form on X. There exists a unique integer a such that

$$f^a\omega = \frac{df}{f} \wedge \omega^{p-1} + \omega^p,$$

with ω^{p-1}, ω^p rational forms which are regular at E, and $\omega^{p-1}|_E \neq 0$. From $f = t_1^j t_2^{n'}$, we obtain

$$\frac{df}{f} = n'\frac{dt_2}{t_2} - j\frac{dt_1}{t_1}.$$

Therefore $(f^a t^{-i})(\omega t^i) - n'\frac{dt_2}{t_2} \wedge \omega^{p-1}$ is regular at E', and $\omega^{p-1}|_{E'} \neq 0$. Since $(f^a t^{-i})^n$ and $t_2^{n'(na-mi)}$ differ by a unit at E', the order of $f^a t^{-i}$ at E' if $n'(a - \frac{mi}{n})$. Therefore ωt^i has at most a logarithmic pole at E' if and only if $n'(a - \frac{mi}{n}) \leq 0$, if and only if $a \leq \lfloor \frac{im}{n} \rfloor$, if and only if $f^{\lfloor \frac{im}{n} \rfloor}\omega$ has at most a logarithmic pole at E. This proves a). Similarly, ωt^i is regular at E' if and only if $n'(a - \frac{mi}{n}) < 0$, if and only if $a < \frac{im}{n}$. That is $a \leq \lfloor \frac{im}{n} \rfloor$ if $\frac{im}{n} \notin \mathbb{Z}$, and $a < \frac{im}{n}$ if $\frac{im}{n} \in \mathbb{Z}$. That is $f^{\lfloor \frac{im}{n} \rfloor}\omega$ has at most a logarithmic pole at E if $\frac{im}{n} \notin \mathbb{Z}$, and is regular at E if $\frac{im}{n} \in \mathbb{Z}$. This proves b).

Since π is étale over V, every k-derivation θ of V lifts to a unique k-derivation $\tilde{\theta}$ of $\pi^{-1}(V)$. We have an eigenspace decomposition

$$\pi_* \mathcal{T}_{\pi^{-1}(V)/k} = \oplus_{i=0}^{n-1} \{\tilde{\theta}; \theta \in \mathcal{T}_{V/k}(iD|_V)\} t^i.$$

This determines the sheaves in c) and d) at the generic points of X. The sheaves are S_2, so we may localize near the generic point of a prime divisor E of X. We use the same notations as above. Let $A \to \bar{A}$ be the integral extension obtain by localizing π at E. We compute

$$\bar{A} = \oplus_{i=0}^{n-1} A f^{-\lfloor \frac{im}{n} \rfloor} t^i.$$

Let θ be a rational k-derivation of A. From $t^n = \varphi = uf^m$, we obtain

$$\tilde{\theta}(af^{-\lfloor \frac{im}{n} \rfloor} t^i) = (\theta(a) + a\frac{i}{n}\frac{\theta(u)}{u} + a\{\frac{im}{n}\}\frac{\theta(f)}{f}) f^{-\lfloor \frac{im}{n} \rfloor} t^i \ (a \in A).$$

c) Let $0 \leq l \leq n-1$. We claim that $\tilde{\theta} f^{-\lfloor \frac{lm}{n} \rfloor} t^l$ is a regular at E' if and only if θ is regular at E, and moreover logarithmic at E in case $\frac{m}{n} \notin \mathbb{Z}$ and $\{\frac{lm}{n}\} \neq 1 - \frac{1}{n'}$.

Indeed, the rational derivation $\tilde{\theta} f^{-\lfloor \frac{lm}{n} \rfloor} t^l$ is regular on $\bar{A}$ if and only if for every $a \in A$ and $0 \leq i \leq n-1$,

$$\tilde{\theta}(af^{-\lfloor \frac{im}{n} \rfloor} t^i) f^{-\lfloor \frac{lm}{n} \rfloor} t^l \subseteq A f^{-\lfloor \frac{(i+l \mod n)m}{n} \rfloor} t^{i+l \mod n}.$$

Since $f^{-\lfloor \frac{i(i+l \mod n)m}{n} \rfloor} t^{i+l \mod n}$ differs by $f^{-\lfloor \frac{(i+l)m}{n} \rfloor} t^{i+l}$ by a unit, the condition becomes

$$\tilde{\theta}(af^{-\lfloor \frac{im}{n} \rfloor} t^i) f^{-\lfloor \frac{lm}{n} \rfloor} t^l \subseteq A f^{-\lfloor \frac{(i+l)m}{n} \rfloor} t^{i+l}.$$

From above, this is equivalent to

$$\theta(a) + a\frac{i}{n}\frac{\theta(u)}{u} + a\{\frac{im}{n}\}\frac{\theta(f)}{f} \in A f^{\lfloor \frac{im}{n} \rfloor + \lfloor \frac{lm}{n} \rfloor - \lfloor \frac{(i+l)m}{n} \rfloor}.$$

Set $i = 0$. The condition becomes $\theta(a) \in A$. That is $\theta \in \mathrm{Der}_k(A)$. With this assumption, the condition becomes

$$\{\frac{im}{n}\}\frac{\theta(f)}{f} \in Af^{\lfloor \frac{im}{n} \rfloor + \lfloor \frac{lm}{n} \rfloor - \lfloor \frac{(i+l)m}{n} \rfloor} \ (0 \leq i \leq n-1).$$

Note $\lfloor \frac{im}{n} \rfloor + \lfloor \frac{lm}{n} \rfloor - \lfloor \frac{(i+l)m}{n} \rfloor$ is -1 or 0. The latter case happens if and only if $\{\frac{im}{n}\} + \{\frac{lm}{n}\} < 1$. If $\frac{m}{n} \in \mathbb{Z}$, this always holds. Else, let $\frac{m}{n} = \frac{m'}{n'}$ be the reduced form, with $n' > 1$. Suppose $\{\frac{lm}{n}\} = 1 - \frac{1}{n'}$. If $\{\frac{im}{n}\} + \{\frac{lm}{n}\} < 1$, then $\{\frac{im}{n}\} = 0$, so the condition holds again. Suppose $\{\frac{lm}{n}\} \neq 1 - \frac{1}{n'}$. Equivalently, $\{\frac{lm}{n}\} < 1 - \frac{1}{n'}$. Recall from Lemma 1.3.4 that $\{\frac{jm}{n}\} = \frac{1}{n'}$. The condition for $i = j$ becomes $\frac{\theta(f)}{f} \in A$.

d) We claim that $\tilde{\theta} f^{-\lfloor \frac{lm}{n} \rfloor} t^l$ is a regular and logarithmic at E' if and only if θ is regular and logarithmic at E. Indeed, a local parameter at E' is $t_2 = f^{-\lfloor \frac{jm}{n} \rfloor} t^j$ (recall $1 \leq j \leq n, jm \equiv g \mod n$). We compute

$$\begin{aligned} \frac{\tilde{\theta} f^{-\lfloor \frac{lm}{n} \rfloor} t^l(t_2)}{t_2} &= (\frac{j}{n}\frac{\theta(u)}{u} + \{\frac{jm}{n}\}\frac{\theta(f)}{f}) f^{-\lfloor \frac{lm}{n} \rfloor} t^l \\ &= (\frac{j}{n}\frac{\theta(u)}{u} + \frac{1}{n'}\frac{\theta(f)}{f}) f^{-\lfloor \frac{lm}{n} \rfloor} t^l \end{aligned}$$

The last term is regular at E' if and only if $\frac{j}{n}\frac{\theta(u)}{u} + \frac{1}{n'}\frac{\theta(f)}{f} \in A$. That is $\frac{\theta(f)}{f} \in A$. From above, the claim holds. $\square$

1.3.2 Toroidal criterion

Proposition 1.3.7. *With the notations of section 2-A, let $X = T_N \operatorname{emb}(\Delta)$ be a torus embedding. The torus character χ^v becomes a rational function φ on X.*

a) Suppose $n' = n$. Then the normalized n-th root of φ is the toric morphism

$$X' = T_{N'} \operatorname{emb}(\Delta) \to X = T_N \operatorname{emb}(\Delta)$$

induced by the inclusion $N' \subseteq N$, endowed with the rational function $\chi^{\frac{v}{n}}$ on X'.

b) Suppose k contains μ_n. Then the normalized n-th root of φ has d irreducible components, each of them isomorphic over X to the toric morphism $T_{N'} \operatorname{emb}(\Delta) \to T_N \operatorname{emb}(\Delta)$, induced by the inclusion $N' \subseteq N$.

Proof. Let $\pi \colon Y \to X$ be the normalized root of φ. Then $\pi \colon \pi^{-1}(T_N) \to T_N$ is the normalized n-th root of $\varphi|_{T_N}$. Since $\varphi|_{T_N}$ is a unit, this coincides with the n-th root of $\varphi|_{T_N}$, which is described by Proposition 1.2.10. In case a), $\pi^{-1}(T_N) \to T_N$ is isomorphic over T_N with $T_{N'} \to T_N$. Therefore Y is the normalization of X with respect to the field extension $k(T_N) \to k(T_{N'})$, which is exactly $T_{N'}(\operatorname{emb} \Delta) \to T_N \operatorname{emb}(\Delta)$. A similar argument works in case b), for each irreducible component of Y. $\square$

Theorem 1.3.8. *Let k be an algebraically closed field. Let $U \subseteq X$ be a quasi-smooth toroidal embedding defined over k. Let φ be an invertible rational function on X, let $n \geq 1$ such that $\operatorname{char} k \nmid n$. Let $D = \frac{1}{n}\operatorname{div}(\varphi)$, and suppose $D|_U$ has integer coefficients. Let $\pi \colon Y \to X$ be the normalized n-th root of φ. Then $\pi^{-1}(U) \subseteq Y$ is a quasi-smooth toroidal embedding, and π is a toroidal morphism.*

Proof. Let $P \in X$ be a point. By Lemma 1.3.2.d), we may replace X by an étale neighborhood of P. By [10, Corollary 2.6], we may suppose there exists an étale morphism

$$\tau \colon X \to Z,$$

where $Z = T_N \operatorname{emb}(\sigma)$ is an affine torus embedding defined over k, the cone σ is simplicial, and $U = \tau^{-1}(T_N)$. By [39], $U \subseteq X$ is a strict toroidal embedding.

Let D' be the part of D which is not supported by $X \setminus U$. It has integer coefficients, by assumption. By [26, Example 5.10], $\operatorname{Cl} \mathcal{O}_{X,P}$ is generated by the primes of $X \setminus U$ passing through P. Therefore there exists $\psi \in k(X)^*$ such that $\operatorname{div}(\psi) + D'$ is zero on U. Then $\operatorname{div}(\psi^n \varphi)$ is supported by $X \setminus U$. By [39], there exists $v \in N^*$ such that

$$\operatorname{div}(\psi^n \varphi) = \operatorname{div}(\pi^* \chi^v).$$

That is $u\psi^n\varphi = \pi^*\chi^v$ for some unit u. After the étale base change $X[\sqrt[n]{u}] \to X$, we may suppose $u = w^n$ for some unit w. Therefore

$$(w\psi)^n \varphi = \pi^* \chi^v.$$

By Lemma 1.3.7.b), the normalized n-th root of χ^v is a toroidal morphism. The total space is again quasi-smooth, since σ is simplicial. By étale base change, the normalized n-th root of $\pi^*\chi^v$ is also toroidal and quasi-smooth. By Lemma 1.3.3, the normalized n-th root of $(w\psi)^n\varphi$ is isomorphic to the normalized n-th root of φ. $\square$

1.3.3 Comparison with roots of sections

Let X/k be a normal variety.

Suppose $f \in \Gamma(X, \mathcal{O}_X)$ does not divide zero. Then $X[f,n]$ coincides with the normalization of $X[\sqrt[n]{f}]$ (root of regular function). If f is a unit on X, the root is already normal, and therefore $X[f,n] = X[\sqrt[n]{f}]$.

Let φ be an invertible rational function on X. Let $V = X \setminus \operatorname{Supp} D$ be the (open dense) locus where φ is a unit. The restriction of $X[\varphi, n] \to X$ to V coincides with the n-th root of the unit $\varphi|_V$. Therefore $X[\varphi, n]$ is obtained by normalizing X in the function field of each irreducible component of $V[\sqrt[n]{\varphi|_V}]$.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L}^n)$ be non-zero. Choose an open dense subset $U \subseteq X$ such that $\mathcal{L}|_U$ has a nowhere zero section u. Let $s|_U = \varphi u^n$ with $\varphi \in \Gamma(U, \mathcal{O}_U)$. Then the normalization of $X[\sqrt[n]{s}]$ coincides with $X[\varphi, n]$.

1.4 Index one covers of torsion divisors

Let k be an algebraically closed field. Let X/k be an irreducible, normal algebraic variety. Let D be a $\mathbb{Q}$-Weil divisor on X which is *torsion*, that is $iD \sim 0$ for some $i \geq 1$. The *index* of D is

$$r = \min\{i \geq 1; iD \sim 0\}.$$

We suppose $\operatorname{char} k \nmid r$. Choose a non-zero rational function $\varphi \in k(X)^\times$ such that $\operatorname{div}(\varphi) = rD$.

Lemma 1.4.1. *The polynomial $T^r - \varphi \in k(X)[T]$ is irreducible.*

Proof. We may apply Lemma 1.2.12. Suppose $1 \leq d \mid n$ and $T^d - \varphi$ has a root $\psi \in K$. Then $\operatorname{div}(\psi) = \frac{r}{d}D$, hence $\frac{r}{d}D \sim 0$. The minimality of r implies $d = 1$. □

We deduce that $k(X)[T]/(T^r - \varphi)$ is a field, denoted $k(X)(\sqrt[r]{\varphi})$. The Kummer field extension

$$k(X) \subset k(X)(\sqrt[r]{\varphi})$$

has Galois group μ_r. Let ψ be a root of $T^r - \varphi$ in this extension. The Galois group action induces an eigenspace decomposition

$$k(X)(\sqrt[r]{\varphi}) = \oplus_{i=0}^{r-1} k(X) \cdot \psi^i.$$

Let $\pi \colon Y \to X$ be the normalization of X in the Kummer extension. By construction, Y/k is an irreducible, normal algebraic variety with quotient field $k(X)(\sqrt[r]{\varphi})$. The root ψ identifies with a rational function on Y such that $\psi^r = \pi^*\varphi$. In particular,

$$\operatorname{div}(\psi) = \pi^* D.$$

So π^*D is linearly trivial on Y. The morphism π is finite, determined as follows:

Lemma 1.4.2. *The Galois group μ_r acts on Y relative to X. The eigenspace decomposition is*

$$\pi_*\mathcal{O}_Y = \oplus_{i=0}^{r-1} \mathcal{O}_X(\lfloor iD \rfloor) \cdot \psi^i.$$

Proof. We have $\pi_*\mathcal{O}_Y = \oplus_{i=0}^{r-1} \mathcal{F}_i \cdot \psi^i$ for some subspaces $\mathcal{F}_i \subset k(X)$. Locally on X, a non-zero rational function $a \in k(X)^\times$ belongs to $\mathcal{F}_i$ if and only if $\pi^*a \cdot \psi^i \in \mathcal{O}_Y$. Since Y is normal, this is equivalent to $\operatorname{div}(\pi^*a) + i\pi^*D \geq 0$, that is $\operatorname{div}(a) + iD \geq 0$, or $a \in \mathcal{O}_X(\lfloor iD \rfloor)$. Therefore $\mathcal{F}_i = \mathcal{O}_X(\lfloor iD \rfloor)$. □

We deduce that $(Y/X, \psi)$ is the normalized r-th root of X with respect to φ.

Call $\pi \colon Y \to X$ the *index one cover associated to the torsion $\mathbb{Q}$-divisor D*. It depends on the choice of φ. If φ_1, φ_2 are two choices, they differ by a unit $u \in \Gamma(X, \mathcal{O}_X^\times)$, and the two associated morphisms $Y_i \to X$ $(i = 1, 2)$ become isomorphic after base change with the étale covering $X[\sqrt[r]{u}] \to X$. If X/k is proper, it follows that $Y_i \to X$ $(i = 1, 2)$ are isomorphic, and therefore π does not depend on the choice of φ.

Let $D' = D + (f)$. Then D' is again torsion, of the same index. We have $rD' = (\varphi f^r)$ and $(\psi f)^r = \varphi f^r$. Therefore the Kummer field is the same, so $Y \to X$ is also an index one cover of D'. In conclusion, for two linearly equivalent torsion $\mathbb{Q}$-divisors, one may choose isomorphic index one covers. In general, any two become isomorphic after an étale base change of X (taking the r-th root of some global unit).

Index one covers do not commute with restriction to open subsets, since the index may drop after restricting to an open subset.

Chapter 2

Du Bois complex for weakly toroidal varieties

Our motivation to study toric face rings is to construct toric examples of semi-log canonical singularities (cf. [44]). It is known that for the class of log canonical singularities, (normal) toric examples form a reasonably large subclass, which is useful for testing open problems. These models can be defined either geometrically as normal affine equivariant torus embeddings $T \subset X$, or algebraically as $X = \operatorname{Spec} \mathbb{C}[M \cap \sigma]$, where M is lattice and $\sigma \subset M_{\mathbb{R}}$ is a rational polyhedral cone. Here $\mathbb{C}[M \cap \sigma] = \oplus_{m \in M \cap \sigma} \mathbb{C} \cdot \chi^m$ is a semigroup ring with multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

Semi-log canonical singularities appear at the boundary of the moduli space of canonically polarized varieties with log canonical singularities. These singularities are weakly normal, but not necessarily normal or even irreducible. Here are two examples:

- The *pinch point* is the surface singularity with local analytic model
$$0 \in X : (xy^2 - z^2 = 0) \subset \mathbb{C}^3.$$
We have $X = \operatorname{Spec} \mathbb{C}[S]$, where $\mathbb{C}[S] = \oplus_{s \in S} \mathbb{C} \cdot \chi^s$ is the semigroup algebra associated to the semigroup $S = \mathbb{N}^2_{x_2 > 0} \sqcup 2\mathbb{N} \times 0$. The multiplication is given by $\chi^s \cdot \chi^{s'} = \chi^{s+s'}$. The torus $T = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^2]$ acts naturally on X, and $T \subset X$ becomes an affine equivariant torus embeddings, which is irreducible but not normal.

- The *normal crossings singularity* has the local analytic model
$$0 \in X : (\prod_{i=1}^{q} z_i = 0) \subset \mathbb{C}^{d+1} \ (1 \leq q \leq d+1).$$
The torus $T = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^{d+1}]$ acts naturally on $\mathbb{C}^{d+1}$ and leaves X invariant. In fact, T acts on each irreducible component of X, inducing a structure of equivariant embedding of a torus which is a quotient of T. Corresponding to this action, we can write $X = \operatorname{Spec} \mathbb{C}[\cup_{i=1}^q S_i]$, where $S_i = \{s \in \mathbb{N}^{d+1}; s_i = 0\}$ and $\mathbb{C}[\cup_{i=1}^q S_i]$ is the

Stanley-Reisner ring with $\mathbb{C}$-vector space structure $\oplus_{s\in\cup_{i=1}^q S_i}\mathbb{C}\cdot\chi^s$, and multiplication defined as follows: $\chi^s\cdot\chi^{s'}$ is $\chi^{s+s'}$ if there exists i such that $s,s'\in S_i$, and zero otherwise.

Toric face rings are a natural generalization of both semigroup rings and Stanley-Reisner rings. We will use the definition of Ichim and Römer [33], which is based on previous work of Stanley, Reisner, Bruns and others (see the introduction of [33]). A *toric face ring* $\mathbb{C}[\mathcal{M}]$ is associated to a *monoidal complex* $\mathcal{M}=(M,\Delta,(S_\sigma)_{\sigma\in\Delta})$, the data consisting of a lattice M, a fan Δ of rational polyhedral cones in M, and a collection of semigroups $S_\sigma\subseteq M\cap\sigma$, such that S_σ generates the cone σ and $S_\tau=S_\sigma\cap\tau$ if τ is a face of σ. The $\mathbb{C}$-vector space structure is

$$\mathbb{C}[\mathcal{M}]=\oplus_{s\in\cup_{\sigma\in\Delta}S_\sigma}\mathbb{C}\cdot\chi^s,$$

with the following multiplication: $\chi^s\cdot\chi^{s'}$ is $\chi^{s+s'}$ if there exists $\sigma\in\Delta$ such that $s,s'\in S_\sigma$, and zero otherwise.

Toric face rings are glueings of semigroup rings, as $\mathbb{C}[\mathcal{M}]\simeq\varprojlim_{\sigma\in\Delta}\mathbb{C}[S_\sigma]$. The affine variety $X=\operatorname{Spec}\mathbb{C}[\mathcal{M}]$ has a natural action by the torus $T=\operatorname{Spec}\mathbb{C}[M]$, and the cones of the fan are in one to one correspondence with the orbits of the action. The T-invariant closed subvarieties of X are also induced by a toric face ring, obtained by restricting the fan Δ to a subfan.

We say that X has *weakly toroidal singularities* if X is weakly normal, and locally analytically isomorphic to $\operatorname{Spec}\mathbb{C}[\mathcal{M}]$ for some monoidal complex $\mathcal{M}$. In the next chapter, we determine when X has semi-log canonical singularities. In this chapter, we aim to understand the topology of X.

Let $X/\mathbb{C}$ be a proper variety. If X is smooth, the cohomology of X^{an} is determined by differential forms on X [19]: the filtered complex (Ω^*_X,F), where Ω^*_X is the de Rham complex of regular differential forms on X, and F is the naive filtration, induces in hypercohomology a spectral sequence

$$E_1^{pq}=H^q(X,\Omega^p_X)\Longrightarrow\operatorname{Gr}^p_F H^{p+q}(X^{an},\mathbb{C})$$

which degenerates at E_1, and converges to the Hodge filtration on the cohomology groups of X^{an}. If X has singularities, its topology is determined by rational forms on a smooth simplicial resolution [20, 22]: if $\epsilon\colon X_\bullet\to X$ is a smooth simplicial resolution, the Deligne-Du Bois filtered complex

$$(\underline{\Omega}^*_X,F):=R\epsilon_*(\Omega^*_{X_\bullet},F)$$

induces in hypercohomology a spectral sequence

$$E_1^{pq}=\mathbb{H}^q(X,\operatorname{Gr}^p_F\underline{\Omega}^p_X[p])\Longrightarrow\operatorname{Gr}^p_F H^{p+q}(X^{an},\mathbb{C})$$

which degenerates at E_1, and converges to the Hodge filtration on the cohomology groups of X^{an}. The filtered complex $(\underline{\Omega}^*_X,F)$ does not depend on the choice of ϵ, when viewed in the derived category of filtered complexes on X. It is a rather complicated object in general: for example F may not be a naive filtration, so each $\operatorname{Gr}^p_F\underline{\Omega}^*_X[p]$ is a complex.

Steenbrink, Danilov, Du Bois and Ishida have observed that if the singularities of X are simple enough, one may still compute the cohomology of X using differential forms on X:

- Suppose X has only quotient singularities, or toroidal singularities (i.e. locally analytically isomorphic to a normal affine toric variety). Let $w\colon U \subset X$ be the smooth locus of X. The complement has codimension at least 2, since X is normal. In particular,
$$\tilde{\Omega}^p_X := w_*(\Omega^p_U)$$
is a coherent $\mathcal{O}_X$-module. Then $(\tilde{\Omega}^*_X, F_{naive})$ is a canonical (functorial) choice for the Deligne-Du Bois complex. In particular, the cohomology of X is determined by rational differential forms on X which are regular on the smooth locus of X [61, 15, 16].

- Suppose X has normal crossings singularities. Let $\epsilon_0\colon X_0 \to X$ be the normalization of X, let $X_1 = X_0 \times_X X_0$. Both X_0 and X_1 are smooth, and if we define
$$\tilde{\Omega}^p_X := \mathrm{Ker}(\epsilon_0 \Omega^p_{X_0} \rightrightarrows \epsilon_1 \Omega^p_{X_1}),$$
then $(\tilde{\Omega}^*_X, F_{naive})$ is a canonical (functorial) choice for the Deligne-Du Bois complex [22].

- Let $Y = \operatorname{Spec} \mathbb{C}[M \cap \sigma]$ be a normal affine toric variety. Let $X \subset Y$ be a T-invariant closed subvariety. One can define combinatorially a coherent $\mathcal{O}_X$-module $\tilde{\Omega}^p_X$ (a glueing of certain regular forms on the orbits of X), such that $(\tilde{\Omega}^*_X, F_{naive})$ is a canonical (functorial) choice for the Deligne-Du Bois complex. The same holds for a semi-toroidal variety with a good filtration [35].

The aim of this chapter is to unify all these results, and extend them to varieties with weakly toroidal singularities. What all these examples have in common is the *vanishing property*
$$R^i \epsilon_* \Omega^p_{X_\bullet} = 0 \ (i > 0),$$
where $\epsilon\colon X_\bullet \to X$ is a smooth simplicial resolution. This means that in the filtered derived category, the Deligne-Du Bois complex is equivalent to $(\tilde{\Omega}^*_X, F_{naive})$, where
$$\tilde{\Omega}^p_X := h^0(\underline{\Omega}^p_X) = \epsilon_*(\Omega^p_{X_\bullet}) = \mathrm{Ker}(\epsilon_0 \Omega^p_{X_0} \rightrightarrows \epsilon_1 \Omega^p_{X_1})$$
is the cohomology in degree zero of the complex $\underline{\Omega}^p_X$. As defined, $\tilde{\Omega}^p_X$ is uniquely defined only up to an isomorphism. But if we require that $\epsilon_0\colon X_0 \to X$ is a desingularization, then $\tilde{\Omega}^p_X$ is uniquely defined, and has a description in terms of rational differential forms on X. More precisely, let $\epsilon_0\colon X_0 \to X$ be a desingularization, let $X_1 \to X_0 \times_X X_0$ be a desingularization. Then $\tilde{\Omega}^p_X$ consists of rational differential p-forms ω on X such that ω is regular on the smooth locus of X, the rational differential $\epsilon_0^*\omega$ extends to a regular p-form

everywhere on X_0, and $p_1^*\epsilon_0^*\omega = p_2^*\epsilon_0^*\omega$ on X_1. The $\mathcal{O}_X$-module $\tilde{\Omega}_X^p$ coincides with the *sheaf of h-differential forms* $(\Omega_h^p)|_X$ introduced by Huber and Jörder [32]. It is functorial in X.

The main result of this chapter is that weakly toroidal singularities satisfy the same vanishing property, hence the sheaf of h-differentials, endowed with the naive filtration, computes the cohomology of X^{an}:

Theorem 2.0.3. *Let $X/\mathbb{C}$ be a variety with weakly toroidal singularities.*

a) The filtered complex $(\tilde{\Omega}_X^*, F_{naive})$, consisting of the sheaf of h-differential forms on X and its naive filtration, is a canonical (and functorial) choice for the Deligne-Du Bois complex of X.

b) X has Du Bois singularities (i.e. $\mathcal{O}_X = \tilde{\Omega}_X^0$).

c) Moreover, suppose $X/\mathbb{C}$ is proper. Then $(\tilde{\Omega}_X^*, F_{naive})$ induces in hypercohomology a spectral sequence

$$E_1^{pq} = H^q(X, \tilde{\Omega}_X^p) \Longrightarrow \operatorname{Gr}_F^p H^{p+q}(X^{an}, \mathbb{C})$$

which degenerates at E_1, and converges to the Hodge filtration on the cohomology groups of X^{an}.

This is proved in Theorems 2.4.3, 2.4.6 and Corollary 2.4.4. A similar result holds for pairs (X, Y) with weakly toroidal singularities. In Theorem 2.0.3.c), we can say nothing about the weight filtration on $H^*(X^{an}, \mathbb{C})$.

We outline the structure of this chapter. We recall in Section 1 the main result of Du Bois [22], defining from this point of view the sheaf of h-differentials of Huber and Jörder [32], and recall the combinatorial description of differential forms on smooth toric varieties (used in Section 3). Section 2 brings together mostly known results on affine equivariant embeddings of the torus, and toric face rings. Especially, we see the combinatorial construction of weak (semi-) normalization of a toric face ring. In Section 3 we give a combinatorial description for the sheaf of h-differentials on the spectrum of a toric face ring, and prove the main vanishing result (Theorem 2.3.3). The proof is by induction on dimension; it is simpler but inspired from the proof of similar results of Danilov and Ishida. We also extend Theorem 2.3.3 to toric pairs. In Section 4 we generalize the results of Section 3 to varieties with weakly toroidal singularities (pairs as well).

2.1 Preliminary

2.1.1 Simplicial resolutions

See [20] for the definition of simplicial schemes. Let X/k be a scheme of finite type over a field of characteristic zero. A *resolution of X* is an augmented simplicial k-scheme $\epsilon\colon X_\bullet \to X$ such that

- the transition maps of $X_\bullet$ and the ϵ_n's are all proper, and

- $(\mathbb{Q}_l)_X \to R\epsilon_*((\mathbb{Q}_l)_{X_\bullet})$ is an isomorphism (étale topology).

The resolution is called *smooth* if the components of $X_\bullet$ are smooth.

Lemma 2.1.1. *[22, 2.1.4, 2.4] Let $X'_\bullet \to X$ and $X''_\bullet \to X$ be two (resp. smooth) resolutions. Then there exists a commutative diagram*

$$\begin{tikzcd} & X_\bullet \arrow[dl] \arrow[dr] & \\ X'_\bullet \arrow[dr] & & X''_\bullet \arrow[dl] \\ & X & \end{tikzcd}$$

such that the composition $X_\bullet \to X$ is a (resp. smooth) resolution.

Theorem 2.1.2. *[22, 3.11, 3.17,4.2] Consider a commutative diagram*

$$\begin{tikzcd} X'_\bullet \arrow[rr, "\alpha"] \arrow[dr, "\epsilon'"'] & & X_\bullet \arrow[dl, "\epsilon"] \\ & X & \end{tikzcd}$$

where ϵ, ϵ' are smooth resolutions. Then $R\epsilon_*(\Omega^p_{X_\bullet} \to \alpha_*\Omega^p_{X'_\bullet} \to R\alpha_*\Omega^p_{X'_\bullet})$ induces a quasi-isomorphism

$$R\epsilon_*(\Omega^p_{X_\bullet}) \to R\epsilon'_*(\Omega^p_{X'_\bullet}).$$

Taking cohomology in degree zero, we obtain that $\epsilon_*(\Omega^p_{X_\bullet}) \to \epsilon'_*(\Omega^p_{X'_\bullet})$ is an isomorphism.

Corollary 2.1.3. *Let $\epsilon\colon X_\bullet \to X$ be a smooth resolution. If X is smooth, the natural homomorphism*

$$\epsilon^*\colon \Omega^p_X \to R\epsilon_*(\Omega^p_{X_\bullet})$$

is a quasi-isomorphism. That is $\Omega^p_X \tilde{\to} \epsilon_*(\Omega^p_{X_\bullet})$ and $R^i\epsilon_*(\Omega^p_{X_\bullet}) = 0$ $(i > 0)$.

Proof. Factor ϵ through the constant resolution of X. □

2.1.2 h-Differentials

Let X/k be a scheme of finite type, defined over a field of characteristic zero. Let $w\colon X^o \subseteq X$ be the smooth locus of X/k. Let $\epsilon\colon X_\bullet \to X$ be a smooth simplicial resolution. By Corollary 2.1.3, $\Omega^p_{X/k} \to \epsilon_*(\Omega^p_{X_\bullet})$ is an isomorphism over X^o. Define a coherent $\mathcal{O}_X$-module $\tilde{\Omega}^p_{X/k}$ as follows: if $U \subseteq X$ is an open subset, $\Gamma(U, \tilde{\Omega}^p_{X/k})$ consists of those differential forms $\omega \in \Gamma(U \cap X^o, \Omega^p_{U\cap X^0})$ such that $\epsilon^*\omega \in \Gamma(\epsilon^{-1}(U \cap X^o), \Omega^p_{X_\bullet})$ extends to a section of $\Gamma(\epsilon^{-1}(U), \Omega^p_{X_\bullet})$.

By Lemma 2.1.1 and Theorem 2.1.2, the definition of $\Omega^p_{X/k}$ does not depend on the choice of ϵ. Moreover, for every smooth simplicial resolution $\epsilon\colon X_\bullet \to X$, we have an isomorphism

$$\epsilon_0^*\colon \tilde{\Omega}^p_X \tilde{\to} \epsilon_*(\Omega^p_{X_\bullet}).$$

We have an induced k-linear differential $d\colon \tilde{\Omega}^p_X \to \tilde{\Omega}^{p+1}_X$, which defines a differential complex $\tilde{\Omega}^*_X$.

The correspondence $X \mapsto \tilde{\Omega}^*_X$ is functorial. Indeed, let $f\colon X' \to X$ be a morphism. There exists a commutative diagram

$$\begin{tikzcd} X'_\bullet \arrow[r, "f_\bullet"] \arrow[d, "\epsilon'"'] & X_\bullet \arrow[d, "\epsilon"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

where ϵ and ϵ' are smooth simplicial resolutions. The natural homomorphism $f_\bullet^*\colon \Omega^p_{X_\bullet/k} \to f_{\bullet *}\Omega^p_{X'_\bullet/k}$ pushes forward to $f^*\colon \tilde{\Omega}^p_{X/k} \to f_*\tilde{\Omega}^p_{X'/k}$. The latter does not depend on the choice of ϵ, ϵ' and $f_\bullet$. Transitivity follows from this.

The natural homomorphism $\Omega^p_{X/k} \to \tilde{\Omega}^p_{X/k}$ is an isomorphism over the smooth locus of X.

The sheaf $\tilde{\Omega}^p_{X/k}$ coincides with the *sheaf of h-differential forms* $(\Omega^p_h)|_X$ introduced in [32].

2.1.3 Differential forms on smooth toric varieties

- Let M be a lattice, let $\sigma \subset M_\mathbb{R}$ be a cone generated by finitely many elements of M. Then $M \cap \sigma - M \cap \sigma = M \cap (\sigma - \sigma)$.

- Let $T = \operatorname{Spec} k[M]$ be a torus. Then $\Gamma(T, \Omega^p_T) = \oplus_{m \in M} \chi^m \cdot \wedge^p V$, where V is the k-vector space of T-invariant global 1-forms on T. We have an isomorphism

$$\alpha\colon k \otimes_\mathbb{Z} M \tilde{\to} V, 1 \otimes m \mapsto \frac{d(\chi^m)}{\chi^m}.$$

Moreover, $d\omega = 0$ for every $\omega \in V$.

- Let $X = T_N \operatorname{emb}(\Delta)$ be a smooth torus embedding. Suppose $\operatorname{Supp} \Delta = \sigma^\vee$, where $\sigma \subset M_\mathbb{R}$ is a rationally polyhedral cone. Then $\Gamma(X, \Omega^p_X) = \oplus_{\tau \prec \sigma} \oplus_{m \in M \cap \operatorname{relint} \tau} \chi^m \cdot \wedge^p \alpha(M \cap \tau - M \cap \tau)$ [15].

Proof. The restriction $\Gamma(X, \Omega^p_X) \to \Gamma(T, \Omega^p_T)$ is injective. Every element of the right hand side has a unique decomposition $\omega = \sum_{m \in M} \chi^m \omega_m$, with $\omega_m \in \wedge^p V$. It remains to identify which ω lift to X. Each ω extends as a form on X with at most logarithmic poles along $X \setminus T$. Then ω lifts to a regular form on X if and only if ω is regular at the generic point of $V(e)$, for every invariant prime $V(e) \subset X$, if and only if $e \in \Delta(1)$ and $\omega_m \neq 0$ implies $\langle m, e \rangle \geq 0$, and $\langle m, e \rangle = 0$ implies $\omega_m \in \wedge^p \alpha(M \cap e^\perp)$. This gives the claim. $\square$

For each $m \in M \cap \sigma$, denote by σ_m the unique face of σ which contains m in its relative interior. Denote by V_m the invariant regular 1-forms on the torus $\operatorname{Spec} k[M \cap \sigma_m - M \cap \sigma_m]$. Then we can rewrite

$$\Gamma(X, \Omega^p_X) = \oplus_{m \in M \cap \sigma} \chi^m \cdot \wedge^p V_m.$$

2.2 Toric face rings

All varieties considered are reduced schemes of finite type, defined over an algebraically closed field k, of characteristic $p \geq 0$.

Let M be a lattice. It induces a k-algebra $k[M] = \oplus_{m \in M} k \cdot \chi^m$, with multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. The variety $T = \operatorname{Spec} k[M]$ is called a *torus over* k. It is endowed a natural multiplication $T \times T \to T$, given by translation on M.

2.2.1 Equivariant affine embeddings of torus

Let $S \subseteq M$ be a finitely generated semigroup such that $S - S = M$. It induces a k-algebra $k[S] = \oplus_{m \in S} k \cdot \chi^m$, with the multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. The affine variety $X = \operatorname{Spec} k[S]$ is an *equivariant embedding of* T [39], i.e. it is equipped with a torus action $T \times X \to X$, and X admits an open dense orbit isomorphic to T, such that the restriction of the action to this orbit corresponds to the torus multiplication.

The correspondence $S \mapsto \operatorname{Spec} k[S]$ is a bijection between finitely generated semigroups $S \subseteq M$ such that $S - S = M$, and isomorphism classes of affine equivariant embeddings of T [39, Proposition 1]. The semigroup is recovered as the set of exponents of the torus action.

For the rest of this section, let $X = \operatorname{Spec} k[S]$. It is affine, reduced and irreducible. The torus orbits are in one to one correspondence with the faces of the cone $\sigma_S \subseteq M_{\mathbb{R}}$ generated by S. If σ is a face of σ_S, the ideal $k[S \setminus \sigma]$ defines a T-invariant closed irreducible subvariety $X_\sigma \subseteq X$. We have $\tau \prec \sigma$ if and only if $X_\tau \subseteq X_\sigma$. The orbit corresponding to the face $\sigma \prec \sigma_S$ is $O_\sigma = X_\sigma \setminus \cup_{\tau \prec \sigma, \tau \neq \sigma} X_\tau$, and is isomorphic to the torus $\operatorname{Spec} k[S \cap \sigma - S \cap \sigma]$.

The normalization of X is $\bar{X} = \operatorname{Spec} k[\bar{S}] \to \operatorname{Spec} k[S] = X$, where $\bar{S} = \cup_{n \geq 1} \{m \in M; nm \in S\} = M \cap \sigma_S$ (see [39, Chapter 1] for proofs of the above statements).

Recall [65] that the *seminormalization of* X, denoted $X^{sn} \to X$, is defined as a universal homeomorphism $f \colon Y \to X$ such that $k(f(y)) \to k(y)$ is an isomorphism for all Grothendieck points $y \in Y$, and f is maximal with this property. It follows that $X^{sn} \to X$ is birational, and topologically a homeomorphism. We call X *seminormal* if its seminormalization is an isomorphism.

Proposition 2.2.1. *[31, Proposition 5.32] The seminormalization of* X *is* $\operatorname{Spec} k[S^{sn}] \to \operatorname{Spec} k[S]$*, where*

$$S^{sn} = \sqcup_{\sigma \prec \sigma_S} (S \cap \sigma - S \cap \sigma) \cap \operatorname{relint} \sigma.$$

Proof. The seminormalization is the spectrum of the ring

$$R = \cap_{x \in X} \{f \in k[\bar{S}]; f_x \in \mathcal{O}_{X,x} + \operatorname{Rad}(\pi_* \mathcal{O}_{\bar{X}})_x\}.$$

The normalization is a toric morphism, hence R is T-invariant. Therefore $R = k[S^{sn}]$ for a certain semigroup $S \subseteq S^{sn} \subseteq \bar{S}$ which we identify.

Let $m \in S^{sn}$. Let x be the generic point of X_σ, for a face $\sigma \prec \sigma_S$. There is a unique (invariant) point x' lying over x, which is $\bar{X}_\sigma$. The map $x' \to x$ corresponds to the morphism of tori

$$O_{x'} = \mathrm{Spec}(k[\bar{S} \cap \sigma - \bar{S} \cap \sigma]) \to \mathrm{Spec}(k[S \cap \sigma - S \cap \sigma]) = O_x.$$

Now $\chi^m|_{O_{x'}}$ is χ^m if $m \in \sigma$, and 0 otherwise. So the condition over x is that if the face $\sigma \prec \sigma_S$ contains m, then $m \in S \cap \sigma - S \cap \sigma$. The condition for χ^m over all torus invariant points of X is thus equivalent to: if σ is the unique face of σ_S which contains m in its relative interior, then $m \in S \cap \sigma - S \cap \sigma$. That is m belongs to

$$S' = \sqcup_{\sigma \prec \sigma_S} (S \cap \sigma - S \cap \sigma) \cap \mathrm{relint}\, \sigma.$$

To check that χ^m satisfies the gluing condition over all points of X, it suffices now to show that $\pi' \colon \mathrm{Spec}\, k[S'] \to \mathrm{Spec}\, k[S]$ is a homeomorphism, which induces isomorphism between residue fields. This map respects the orbit decompositions $\mathrm{Spec}\, k[S'] = \sqcup_\sigma O'_\sigma \to \mathrm{Spec}\, k[S] = \sqcup_\sigma O_\sigma$, and $O'_\sigma \to O_\sigma$ is isomorphic to $\mathrm{Spec}\, k[S' \cap \sigma - S' \cap \sigma] \to \mathrm{Spec}\, k[S \cap \sigma - S \cap \sigma]$. The latter is an isomorphism since $S \cap \sigma - S \cap \sigma = S' \cap \sigma - S' \cap \sigma$. We deduce that π' is bijective. It is also proper, hence open. Therefore π' is a homeomorphism. Since the maps between orbits are isomorphisms, and the orbits are locally closed, it follows that π' induces isomorphisms between residue fields.

We conclude that $S^{sn} = S'$. $\square$

Lemma 2.2.2. $S^{sn} = \{m \in M; nm \in S \ \forall n \gg 0\}$.

Proof. $\supseteq$: let $m \in M$ with $nm, (n+1)m \in S$ for some $n > 0$. Then $m \in \sigma_S$. Let $\sigma \prec \sigma_S$ such that $m \in \mathrm{relint}\, \sigma$. Then $m = (n+1)m - nm \in S_\sigma - S_\sigma$.

$\subseteq$: let $m \in S^{sn}$. Let $\sigma \prec \sigma_S$ such that $m \in \mathrm{relint}\, \sigma$. Let $(s_i)_i$ be a finite system of generators of $S \cap \sigma$. Then $m = \sum_i z_i s_i$ for some $z_i \in \mathbb{Z}$. Since $m \in \mathrm{relint}\, \sigma$, we can write $qm = \sum_i q_i s_i$, with $q, q_i \in \mathbb{Z}_{>0}$. There exists $l \geq 0$ such that $z_i + lq_i \geq 0$ for all i. Then $lqm, (1 + lq)m \in S$. Therefore $nm \in S$ for every $n \geq (lq - 1)lq$. $\square$

Corollary 2.2.3. *X is seminormal if and only if $S \cap \mathrm{relint}\, \sigma = (S \cap \sigma - S \cap \sigma) \cap \mathrm{relint}\, \sigma$, for every face $\sigma \prec \sigma_S$.*

Recall [2] that the *weak normalization of* X, denoted $X^{wn} \to X$, is defined as a birational universal homeomorphism $f \colon Y \to X$, maximal with this property. It follows that $X^{wn} \to X$ is birational and a topological homeomorphism. We call X *weakly normal* if its weak normalization is an isomorphism.

The normalization of X factors as $\bar{X} \to X^{wn} \xrightarrow{u} X^{sn} \to X$, where u is a topological homeomorphism. If $\mathrm{char}\, k = 0$, then u is an isomorphism.

Proposition 2.2.4. *The weak normalization of X is* $\mathrm{Spec}\, k[S^{wn}] \to \mathrm{Spec}\, k[S]$, *where*

$$S^{wn} = \sqcup_{\sigma \prec \sigma_S} \cup_{e \geq 0} \{m \in \bar{S}; p^e m \in (S \cap \sigma - S \cap \sigma) \cap \mathrm{relint}\, \sigma\}.$$

If $p = 0$, we set $p^e = 1$.

Proof. The weak normalization is the spectrum of the ring

$$R = \cap_{x \in X}\{f \in k[\bar{S}]; f_x^{p^e} \in \mathcal{O}_{X,x} + \mathrm{Rad}(\pi_*\mathcal{O}_{\bar{X}})_x, \exists e \geq 0\}.$$

The proof is similar to that for seminormalization. We only need to use that if $\Lambda \subseteq \Lambda' \subseteq p^{-e}\Lambda$ are lattices, then $\mathrm{Spec}\, k[\Lambda'] \to \mathrm{Spec}\, k[\Lambda]$ is a universal homeomorphism (use relative Frobenius). □

Note that $S^{wn} = \cup_{e \geq 0}\{m \in \bar{S}; p^e m \in S^{sn}\}$.

Remark 2.2.5. Let $S \subseteq S'$ be an inclusion of finitely generated semigroups, such that $X' = \mathrm{Spec}\, k[S'] \to \mathrm{Spec}\, k[S] = X$ is a finite morphism (i.e. for every $s' \in S'$, there exists $n \geq 1$ such that $ns' \in S$). Then the seminormalization of X in X' is associated to

$$\sqcup_{\sigma \prec \sigma_S} S' \cap (S \cap \sigma - S \cap \sigma) \cap \mathrm{relint}\, \sigma$$

and the weak normalization of X in X' is associated to

$$\sqcup_{\sigma \prec \sigma_S} \cup_{e \geq 0} \{m \in S'; p^e m \in (S \cap \sigma - S \cap \sigma) \cap \mathrm{relint}\, \sigma\}.$$

Example 2.2.6. Let d be a positive integer. The extension $k[T] \subset k[T^d]$ is seminormal. It is weakly normal if and only if $p \nmid d$. Its weak normalization is $k[T] \subset k[T^{\frac{d}{d_p}}] \subset k[T^d]$, where d_p is the largest divisor of d which is not divisible by p.

Example 2.2.7. The semigroup $S = \{(x_1, x_2) \in \mathbb{N}^2; x_2 > 0\} \sqcup 2\mathbb{N} \times 0$ induces the k-algebra $k[S] \simeq k[X, Y, Z]/(ZX^2 - Y^2)$. If $\mathrm{char}\, k = 2$, then $\mathrm{Spec}\, k[S]$ is seminormal, but not weakly normal.

Example 2.2.8. Let $\dim S = 1$. Then $\mathrm{Spec}\, k[S]$ is seminormal if and only if it is smooth, if and only if S is isomorphic to $\mathbb{N}$ or $\mathbb{Z}$.

Lemma 2.2.9. *Let $\sigma \subset M_\mathbb{R}$ be a convex cone with non-empty interior. Then $M \cap \mathrm{int}\, \sigma - M \cap \mathrm{int}\, \sigma = M$.*

Proof. Let $m \in M$. Choose $m' \in M \cap \mathrm{int}\, \sigma$. Then $m' + \epsilon m \in \mathrm{int}\, \sigma$ for $0 \leq \epsilon \ll 1$. Therefore $nm' + m \in \mathrm{int}\, \sigma$ for $n \gg 0$. Then $m = (nm' + m) - (nm') \in M \cap \mathrm{int}\, \sigma - M \cap \mathrm{int}\, \sigma$. □

Proposition 2.2.10 (Classification of seminormal and weakly normal semigroups)**.** *Let $\sigma \subset M_\mathbb{R}$ be a rational polyhedral cone, which generates $M_\mathbb{R}$. There is a one to one correspondence between semigroups S such that $S - S = M$, $\sigma_S = \sigma$ and $\mathrm{Spec}\, k[S]$ is seminormal, and collections $(\Lambda_\tau)_{\tau \prec \sigma}$ of sublattices of finite index $\Lambda_\tau \subseteq M \cap \tau - M \cap \tau$ such that $\Lambda_\sigma = M$ and $\Lambda_{\tau'} \subset \Lambda_\tau$ if $\tau' \prec \tau$. The correspondence, and its inverse, is*

$$S \mapsto (S_\tau - S_\tau)_{\tau \prec \sigma} \text{ and } (\Lambda_\tau)_{\tau \prec \sigma} \mapsto \sqcup_{\tau \prec \sigma} \Lambda_\tau \cap \mathrm{relint}\, \tau.$$

Moreover, $\mathrm{Spec}\, k[S]$ is weakly normal if and only if p does not divide the index of the sublattice $\Lambda_\tau = S_\tau - S_\tau \subseteq \bar{S} \cap \tau - \bar{S} \cap \tau$, for every face $\tau \prec \sigma$.

Proof. Use Lemma 2.2.9 to show that the two are inverse. Moreover, if $\operatorname{Spec} k[S]$ is seminormal, it is weakly normal if and only if for all $\tau \prec \sigma$, if $m \in (\bar{S}_\tau - \bar{S}_\tau) \cap \operatorname{relint} \tau$ and $pm \in \Lambda_\tau \cap \operatorname{relint} \tau$, then $m \in \Lambda_\tau \cap \operatorname{relint} \tau$. By Lemma 2.2.9, this is equivalent to the index of the sublattice $S_\tau - S_\tau \subseteq \bar{S} \cap \tau - \bar{S} \cap \tau$ not being divisible by p. □

Thus, a seminormal variety $X = \operatorname{Spec} k[S]$ is obtained from its normalization $\bar{X} = \operatorname{Spec} k[\bar{S}]$ by self-glueing some invariant subvarieties $\bar{X}_\sigma$ ($\sigma \prec \sigma_S$), according to the finite index sublattices $\Lambda_\sigma \subseteq M \cap \sigma - M \cap \sigma$.

2.2.2 Spectrum of a toric face ring

A *monoidal complex* $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$ consists of a lattice M, a rational fan Δ with respect to M (i.e. a finite collection of rational polyhedral cones in $M_\mathbb{R}$, such that every face of a cone of Δ is also in Δ, and any two cones of Δ intersect along a common face), and a collection of finitely generated semigroups $S_\sigma \subseteq M \cap \sigma$, such that S_σ generates σ and $S_\tau = S_\sigma \cap \tau$ if $\tau \prec \sigma$.

If $0 \in \Delta$, that is each cone of Δ admits the origin as a face, this is the definition introduced by Ichim and Römer [33].

The *support* of $\mathcal{M}$ is the set $|\mathcal{M}| = \cup_{\sigma \in \Delta} S_\sigma \subseteq M$. The *toric face ring* of $\mathcal{M}$ is the k-algebra

$$k[\mathcal{M}] = \oplus_{m \in |\mathcal{M}|} k \cdot \chi^m,$$

with the following multiplication: $\chi^m \cdot \chi^{m'}$ is $\chi^{m+m'}$ if m, m' are contained in some S_σ, and 0 otherwise. Note that $k[\mathcal{M}] \simeq \varprojlim_{\sigma \in \Delta} k[S_\sigma]$.

For the rest of this section, let $X = \operatorname{Spec} k[\mathcal{M}]$. We call X/k the *toric variety* associated to the monoidal complex $\mathcal{M}$. The torus $T = \operatorname{Spec} k[M]$ acts on X, with $g^*(\chi^m) = g(m)\chi^m$. So the support of $\mathcal{M}$ is recovered as the set of weights of the torus action.

Example 2.2.11. Let M be a lattice, and $S \subseteq M$ a finitely generated semigroup such that $S - S = M$. Let Δ be a subfan of the fan of faces of σ_S. Then $(M, \Delta, (S \cap \sigma)_{\sigma \in \Delta})$ is a monoidal complex, and $\operatorname{Spec} k[\mathcal{M}]$ is a closed subvariety of $\operatorname{Spec} k[S]$ which is torus invariant.

Example 2.2.12. [60] Let Δ be a rational fan with respect to M. For $\sigma \in \Delta$, define $S_\sigma = M \cap \sigma$. This defines a monoidal complex with toric face ring $k[M, \Delta] = \oplus_{m \in M \cap \operatorname{Supp} \Delta} k \cdot \chi^m$.

A T-invariant ideal $I \subseteq k[\mathcal{M}]$ is radical if and only if $I = k[\mathcal{M} \setminus A]$, where $A = \cup_{\sigma \in \Delta'} S_\sigma$, where Δ' is a subfan of Δ. The quotient $k[\mathcal{M}]/I$ is $k[\mathcal{M}']$, where $\mathcal{M}' = (M, \Delta', (S_\sigma)_{\sigma \in \Delta'})$. In particular, $I \subseteq k[\mathcal{M}]$ is a T-invariant prime ideal if and only if $I = k[\mathcal{M} \setminus A]$ with $A = S_\sigma$ for some $\sigma \in \Delta$. The quotient $k[\mathcal{M}]/I$ is $k[S_\sigma]$.

We obtain a one to one correspondences between: i) T-invariant closed reduced subvarieties of X and subfans of Δ; ii) T-orbits of X and the cones of Δ. If $\sigma \in \Delta$, then the ideal $k[\mathcal{M} \setminus \sigma]$ defines a T-invariant closed reduced subvariety $X_\sigma \subseteq X$. We have $\tau \prec \sigma$ if and only if $X_\tau \subseteq X_\sigma$. The orbit corresponding to the cone $\sigma \in \Delta$ is $O_\sigma = X_\sigma \setminus \cup_{\tau \prec \sigma, \tau \neq \sigma} X_\tau$, and is isomorphic to the torus $\operatorname{Spec} k[S \cap \sigma - S \cap \sigma]$. We obtain

$$X = \cup_{\sigma \in \Delta} X_\sigma = \sqcup_{\sigma \in \Delta} O_\sigma.$$

The smallest cone of Δ is $\tau = \cap_{\sigma \in \Delta} \sigma$. The orbit O_τ is the unique orbit which is closed. In particular, $0 \in \Delta$ if and only if $\tau = 0$. That is the torus action on X has a (unique) fixed point, and the cones of Δ are pointed as in [33].

Each X_σ is an affine equivariant embedding of the torus T_σ, where $T_\sigma = \operatorname{Spec} k[S_\sigma - S_\sigma]$ is a quotient of T. The action of T on X_σ factors through the action of T_σ.

The irreducible components of X are X_F, where F are the *facets* of Δ (cones of Δ which are maximal with respect to inclusion). The torus T acts on each irreducible component of X.

The toric variety X is irreducible if and only if Δ has a unique maximal cone, if and only if $X = \operatorname{Spec} k[S]$ is an equivariant torus embedding (see [33] for proofs of the above statements).

Remark 2.2.13. A geometric characterization of $X = \operatorname{Spec} k[\mathcal{M}]$ is as follows: X/k is a reduced affine algebraic variety, endowed with an action by a torus T/k, subject to the following axioms:

a) T acts on each irreducible component X_i of X, and the action factors through a torus quotient $T \to T_i$ such that $T_i \subseteq X_i$ becomes an equivariant affine torus embedding.

b) The scheme intersection $X_i \cap X_j$ is reduced, and the induced action of T on $X_i \cap X_j$ factors through a torus quotient $T \to T_{ij}$ such that $T_{ij} \subseteq X_i \cap X_j$ becomes an equivariant affine torus embedding.

We have seen above that $X = \operatorname{Spec} k[\mathcal{M}]$ satisfies properties a) and b). Conversely, we recover the monomial complex as follows: $T = \operatorname{Spec} k[M]$ for some lattice M. Each irreducible component of X is of the form $X_i = \operatorname{Spec} k[S_i]$ for some finitely generated semigroup $S_i \subseteq M$. Let $F_i \subseteq M_\mathbb{R}$ be the cone generated by S_i. Define Δ to be the collection of F_i and their faces. Each $\sigma \in \Delta$ is a face of some F_i, and we set $S_\sigma = S_i \cap \sigma$. To verify that Δ is actually a fan, it suffices to show that two maximal cones F_i, F_j intersect along a common face. By b), $X_i \cap X_j = \operatorname{Spec} k[S_{ij}]$ for some finitely generated semigroup $S_{ij} \subseteq M$. Since $X_i \cap X_j$ is a T-invariant closed reduced subvariety of X_i, there exists a face τ_{ij} of F_i such that $S_{ij} = S_i \cap \tau_{ij}$. By a similar argument, there exists a face τ_{ji} of F_j such that $S_{ij} = S_j \cap \tau_{ji}$. Then τ_{ij}, τ_{ji} coincide, equal to the cone generated by S_{ij}, also equal to $F_i \cap F_j$. Therefore $F_i \cap F_j$ is a face in both F_i and F_j.

Proposition 2.2.14 (Nguyen [50])**.** *For $\sigma \in \Delta$, let $S_\sigma^{sn} = \sqcup_{\tau \prec \sigma} (S_\tau - S_\tau) \cap \operatorname{relint} \tau$ be the seminormalization of S_σ. Then $\mathcal{M}^{sn} = (M, \Delta, (S_\sigma^{sn})_{\sigma \in \Delta})$ is a monoidal complex, and the seminormalization of X is $\operatorname{Spec} k[\mathcal{M}^{sn}] \to \operatorname{Spec} k[\mathcal{M}]$.*

Proof. For cones $\tau, \sigma \in \Delta$, we have $\sigma \cap \operatorname{relint} \tau \neq \emptyset$ if and only if $\tau \prec \sigma$. Therefore $S_\tau^{sn} = \tau \cap S_\sigma^{sn}$ if $\tau \prec \sigma$. We conclude that $\mathcal{M}^{sn} = (M, \Delta, (S_\sigma^{sn})_{\sigma \in \Delta})$ is a monoidal complex.

Let $\{F\}$ be the facets of Δ. The normalization of X is

$$\bar{X} = \sqcup_F \operatorname{Spec} k[(S_F - S_F) \cap F].$$

The torus T acts on $\bar{X}$ too, and is compatible with $\pi \colon \bar{X} \to X$. The seminormalization is the spectrum of the ring

$$R = \cap_{x \in X} \{f \in \mathcal{O}(\bar{X}); f_x \in \mathcal{O}_{X,x} + \operatorname{Rad}(\mathcal{O}_{\bar{X}})_x\}.$$

The torus T acts on R, and therefore $R = \prod_F k[S'_F]$ for certain semigroups $S'_F \subseteq (S_F - S_F) \cap F$, which remains to be identified.

Let $\sigma \in \Delta$. It defines a T-invariant subvariety $X_\sigma \subset X$. Its preimage $\pi^{-1}(X_\sigma)$ is $\sqcup_F (\bar{X}_F)_{\sigma \cap F}$. So if x is the generic point of X_σ, $\pi^{-1}(x)$ consists of the generic points of $X_\sigma \subset \operatorname{Spec} k[F \cap (S_F - S_F)]$, after all facets F which contain σ.

We deduce that $f = (\chi^{m_F})_F$ satisfies the glueing condition over the generic point of X_σ if and only if either $M_F \notin \sigma$ for all $F \supseteq \sigma$, or there exists $m \in (S_\sigma - S_\sigma) \cap \sigma$ such that $m_F = m$ for all $F \supseteq \sigma$. Choose a component F_1, and let $m_{F_1} \in \operatorname{relint} \tau$. For $\sigma = \tau$, we obtain $m_F = m \in (S_\tau - S_\tau) \cap \tau$ for all $F \supseteq \tau$. But $m \in F$ if and only if $\tau \subseteq F$. Therefore $f = \pi^* \chi^m$, with

$$m \in \sqcup_{\sigma \in \Delta} (S_\sigma - S_\sigma) \cap \operatorname{relint} \sigma.$$

One checks that it's enough to glue only over invariant points. Therefore $X^{sn} = \operatorname{Spec} k[\mathcal{M}^{sn}]$. $\square$

Similarly, we obtain

Proposition 2.2.15. *For each $\sigma \in \Delta$, let S_σ^{wn} be the weak-normalization of S_σ:*

$$S_\sigma^{wn} = \sqcup_{\tau \prec \sigma} \cup_{e \geq 0} \{m \in \overline{S_\sigma}; p^e m \in (S_\tau - S_\tau) \cap \operatorname{relint} \tau\}.$$

Then $\mathcal{M}^{wn} = (M, \Delta, (S_\sigma^{wn})_{\sigma \in \Delta})$ is a monoidal complex, and the weak-normalization of X is

$$\operatorname{Spec} k[\mathcal{M}^{wn}] \to \operatorname{Spec} k[\mathcal{M}].$$

Corollary 2.2.16. *X is seminormal (resp. weakly normal) if and only if X_F is seminormal (resp. weakly normal) for every facet F of Δ, if and only if X_σ is seminormal (resp. weakly normal) for every face $\sigma \in \Delta$.*

In particular, if X is seminormal (weakly normal), so is any union of torus invariant closed subvarieties of X.

Proposition 2.2.17 (Classification of seminormal and weakly normal toric face rings)**.** *Let M be a lattice and Δ a finite rational fan in $M_\mathbb{R}$. There is a one to one correspondence between collections of semigroups $(S_\sigma)_\sigma \in \Delta$ such that $(M, \Delta, (S_\sigma)_{\sigma \in \Delta})$ is a monoidal* *complex with* $\operatorname{Spec} k[\mathcal{M}]$ *seminormal, and collections* $(\Lambda_\sigma)_{\sigma\in\Delta}$ *of sublattices of finite index* $\Lambda_\sigma \subseteq M\cap\sigma - M\cap\sigma$ *such that* $\Lambda_\tau \subset \Lambda_\sigma$ *if* $\tau \prec \sigma$*. The correspondence, and its inverse, is*

$$(S_\sigma)_\sigma \mapsto (S_\sigma - S_\sigma)_\sigma \text{ and } (\Lambda_\sigma)_\sigma \mapsto (\sqcup_{\tau\prec\sigma}\Lambda_\tau \cap \operatorname{relint}\tau)_\sigma.$$

Moreover, $\operatorname{Spec} k[\mathcal{M}]$ *is weakly normal if and only if* p *does not divide the index of the sublattice* $S_\sigma - S_\sigma \subset (S_F - S_F)\cap\sigma - (S_F - S_F)\cap\sigma$*, for every* $\sigma \prec F$ *in* Δ*, with* F *a facet of* Δ*.*

Remark 2.2.18. Let x be a point which belongs to the closed orbit of X. Then X is seminormal (resp. weakly normal) if and only if $\mathcal{O}_{X,x}$ is seminormal (resp. weakly normal). Indeed, the direct implication is clear. For the converse, note that the proofs of Propositions 2.2.14 and 2.2.15 show that X is seminormal (resp. weakly normal) if and only if so are $\mathcal{O}_{X,X_\sigma}$ for all $\sigma\in\Delta$. Since $x\in X_\sigma$ for all $\sigma\in\Delta$, the converse holds as well.

Remark 2.2.19. Consider the germ of X near a closed point x. There exists a unique cone $\tau\in\Delta$ such that $x\in O_\tau$. The smallest T-invariant open subset of X which contains x is $U=\sqcup_{\tau\prec\sigma\in\Delta}O_\sigma$. If we choose $s\in S_\tau\cap\operatorname{relint}\tau$, then U coincides with the principal open set $D(\chi^s)$. We deduce that $U=\operatorname{Spec} k[\mathcal{M}_x]$, where $\mathcal{M}_x$ is the monoidal complex $(M, \{\sigma-\tau\}_{\tau\prec\sigma\in\Delta}, (S_\sigma - S_\tau)_{\tau\prec\sigma\in\Delta})$. We have an isomorphism of germs $(X,x)=(U,x)$, and x is contained in the orbit associated to $\tau-\tau$, the smallest cone of $\Delta(\mathcal{M}_x)$.

Consider the quotient $\pi\colon M\to M' = M/(M\cap\tau - M\cap\tau)$. For $\tau\prec\sigma\in\Delta$, denote $\sigma'=\pi(\sigma)\subseteq M'_{\mathbb{R}}$ and $S_{\sigma'}=\pi(S_\sigma)$. Then $\pi^{-1}(\sigma')=\sigma-\tau$, the cone generated by $S_\sigma - S_\tau$. Note that $\pi^{-1}(S_{\sigma'}) = S_\sigma + M\cap\tau - M\cap\tau$ is usually larger than $S_\sigma - S_\tau$.

Suppose $S_\sigma = M\cap\sigma$ for every $\tau\prec\sigma\in\Delta$. Then $S_\sigma - S_\tau = \pi^{-1}(S_{\sigma'})$. The choice of a splitting of π induces an isomorphism $X\simeq T''\times\operatorname{Spec} k[\mathcal{M}']$, where T'' is the torus $\operatorname{Spec} k[M\cap\tau - M\cap\tau]$ and $\mathcal{M}'$ is the monoidal complex $(M', \{\sigma'\}, \{S_{\sigma'}\})$. We have $0\in\Delta(\mathcal{M}')$, so $\operatorname{Spec} k[\mathcal{M}']$ has a fixed point x'. The isomorphism maps x onto (x'', x'), where $x''\in T''$ is a closed point. In particular, $(X,x)\simeq(T'',x'')\times(\operatorname{Spec} k[\mathcal{M}'], x')$.

2.3 Du Bois complex for the spectrum of a toric face ring

Let $X=\operatorname{Spec} k[\mathcal{M}]$ be the affine variety associated to a monoidal complex $\mathcal{M}=(M,\Delta,(S_\sigma)_{\sigma\in\Delta})$. Suppose X is weakly normal.

For $m\in\cup_{\sigma\in\Delta}S_\sigma$, denote by σ_m the unique cone of Δ which contains m in its relative interior. Denote by V_m the invariant regular 1-forms on the torus $\operatorname{Spec} k[S_{\sigma_m} - S_{\sigma_m}]$. For each p, denote

$$A^p(X) = \oplus_{m\in\cup_{\sigma\in\Delta}S_\sigma}\chi^m\cdot\wedge^p V_m.$$

If $m,m'\in S_\sigma$ for some $\sigma\in\Delta$, then σ_m is a face of $\sigma_{m+m'}$, hence $V_m\subseteq V_{m+m'}$. Therefore $A^p(X)$ becomes a $\Gamma(X,\mathcal{O}_X)$-module in a natural way: $\chi^{m'}\cdot(\chi^m\omega_m) = (\chi^{m'}\cdot\chi^m)\omega_m$. It induces a coherent $\mathcal{O}_X$-module, denoted $\tilde{\Omega}^p_X$, with $\Gamma(X,\tilde{\Omega}^p_X) = A^p(X)$.

Lemma 2.3.1. *For every morphism $f\colon X' \to X$ from a smooth variety X', we can naturally define a pullback homomorphism $f^*\colon A^p(X) \to \Gamma(X', \Omega^p_{X'})$. Moreover, each commutative diagram*

$$\begin{array}{ccc} X' & \xleftarrow{v} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \hookleftarrow & X_\tau \end{array}$$

with X', Y' smooth and $\tau \in \Delta$, induces a commutative diagram

$$\begin{array}{ccc} \Gamma(X', \Omega^p_{X'}) & \xrightarrow{v^*} & \Gamma(Y', \Omega^p_{Y'}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f'^*} \\ A^p(X) & \xrightarrow{|_{X_\tau}} & A^p(X_\tau) \end{array}$$

where $\chi^m\omega_m|_{X_\tau}$ is $\chi^m\omega_m$ if $m \in \tau$, and 0 otherwise.

Proof. Let $f\colon X' \to X$ be a morphism from a smooth variety X'. To define f^*, we may suppose X' is irreducible. Let τ be the smallest cone of Δ such that $f(X') \subseteq X_\tau$. In particular, $f(X')$ intersects the orbit O_τ. Let $f'\colon X' \to X_\tau$ be the induced morphism.

Let $\omega \in A^p(X)$. Let $\omega_\tau = \omega|_{X_\tau} \in A^p(X_\tau)$ be its combinatorial restriction, defined above. It is a regular differential p-form on the orbit O_τ, which is smooth, being isomorphic to a torus. Therefore $f'^*(\omega_\tau)$ is a well defined rational differential p-form on X' (regular on $f'^{-1}(O_\tau)$). Define

$$f^*\omega = f'^*(\omega_\tau).$$

We claim that $f'^*(\omega_\tau)$ is regular everywhere on X'. Indeed, choose a toric desingularization $\mu_\tau\colon Y_\tau \to X_\tau$. By Hironaka's resolution of the indeterminacy locus of the rational map $X' \dashrightarrow Y_\tau$, we obtain a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y_\tau \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu_\tau} \\ X' & \xrightarrow{f'} & X_\tau \end{array}$$

From the combinatorial description of differential forms on the smooth toric variety Y_τ, the rational form $\mu_\tau^*(\omega_\tau)$ is in fact regular everywhere on Y_τ. Then $h^*\mu_\tau^*(\omega_\tau)$ is regular on Y'. Therefore $\mu^*(f'^*\omega_\tau)$ is regular on Y'. But μ is a proper birational contraction and X', Y' are smooth, so $\Omega^p_{X'} = \mu_*(\Omega^p_{Y'})$. Therefore the rational form $f'^*\omega_\tau$ is in fact regular on X'.

It remains to verify the commutativity of the square diagram. We may suppose X' and Y' are irreducible. Let $f(X') \subseteq X_\sigma$ and $f(Y') \subseteq X_\tau$, with σ and τ minimal with this property. We obtain a commutative diagram

$$\begin{array}{ccc} X' & \xleftarrow{v} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X_\sigma & \hookleftarrow & X_\tau \end{array}$$

We may replace f by a toric desingularization of X_σ. Then $v(Y')$ is contained in $f^{-1}(X_\tau)$, which is a union of closed invariant subvarieties of X'. Each of these closed invariant subvarieties is smooth, since X' is smooth. We may replace Y' by an invariant closed subvariety of X' which contains $v(Y')$. It remains to check the claim for the special type of diagrams

$$\begin{tikzcd} X' \arrow[d, "f"'] & Y' \arrow[l, hook'] \arrow[d, "f'"] \\ X_\sigma & X_\tau \arrow[l, hook'] \end{tikzcd}$$

where f is a toric desingularization, $Y' \subset X'$ is a closed invariant subvariety, and $f'(Y') \cap O_\tau \neq \emptyset$. By the explicit combinatorial formula for differential forms on smooth toric varieties, the diagram

$$\begin{tikzcd} \Gamma(X', \Omega^p_{X'}) \arrow[r, "|_{Y'}"] & \Gamma(Y', \Omega^p_{Y'}) \\ A^p(X_\sigma) \arrow[u, "f^*"] \arrow[r, "|_{X_\tau}"] & A^p(X_\tau) \arrow[u, "f'^*"'] \end{tikzcd}$$

is commutative. □

Lemma 2.3.2. *Consider a commutative diagram*

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X'' \arrow[l, "v"'] \arrow[dl, "f''"] \\ X & \end{tikzcd}$$

with X', X'' smooth. Then the induced diagram of pullbacks

$$\begin{tikzcd} \Gamma(X', \Omega^p_{X'}) \arrow[r, "v^*"] & \Gamma(X'', \Omega^p_{X''}) \\ A^p(X) \arrow[u, "f'^*"] \arrow[ur, "f''^*"'] & \end{tikzcd}$$

is commutative.

Proof. We may suppose X'' is irreducible. Then there exists $\tau \in \Delta$ such that $f''(X'') \subset X_\tau$. We obtain a diagram

$$\begin{tikzcd} X' \arrow[d] & X'' \arrow[l] \arrow[dl] \arrow[d] \\ X & X_\tau \arrow[l, hook'] \end{tikzcd}$$

and the claim follows by applying Lemma 2.3.1 twice. □

Theorem 2.3.3. *Let $\epsilon\colon X_\bullet \to X$ be a smooth simplicial resolution. Then the natural homomorphism $\tilde{\Omega}^p_X \to R\epsilon_*(\Omega^p_{X_\bullet})$ is a quasi-isomorphism (i.e. $\tilde{\Omega}^p_X \xrightarrow{\sim} \epsilon_*(\Omega^p_{X_\bullet})$ and $R^i\epsilon_*(\Omega^p_{X_\bullet}) = 0$ for $i > 0$).*

Proof. Let $\delta_0, \delta_1 \colon X_1 \to X_0$ be the two face morphisms. Then $\delta_0^*\epsilon_0^* = \epsilon_1^* = \delta_1^*\epsilon_0^*$, by Lemma 2.3.2. Therefore the pullback homomorphism ϵ_0^* maps $\tilde{\Omega}^p_X$ into $\mathrm{Ker}(\epsilon_{0*}\Omega^p_{X_0} \rightrightarrows \epsilon_{1*}\Omega^p_{X_1}) = \epsilon_*(\Omega^p_{X_\bullet})$. This defines a natural homomorphism $\tilde{\Omega}^p_X \to R\epsilon_*(\Omega^p_{X_\bullet})$. We show that this is a quasi-isomorphism, by induction on $\dim X$.

Let Σ be the (toric) boundary of X. The restriction $\tilde{\Omega}^p_X \to \tilde{\Omega}^p_\Sigma$ is surjective. Denote its kernel by $\tilde{\Omega}^p_{(X,\Sigma)}$. Denote by $\underline{\Omega}^p_X$ the complex on the right hand side. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Omega}^p_{(X,\Sigma)} \arrow[r] & \underline{\Omega}^p_X \arrow[r] & \underline{\Omega}^p_\Sigma \arrow[r] & 0 \\
0 \arrow[r] & \tilde{\Omega}^p_{(X,\Sigma)} \arrow[r] \arrow[u, "\alpha"] & \tilde{\Omega}^p_X \arrow[r] \arrow[u, "\beta"] & \tilde{\Omega}^p_\Sigma \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where the bottom row is exact, and the top row is an exact triangle in the derived category. Since Σ is again weakly normal, γ is a quasi-isomorphism by induction on dimension. If α is a quasi-isomorphism, then β is a quasi-isomorphism.

We claim that α is a quasi-isomorphism. Indeed, let $\pi \colon \bar{X} \to X$ be the normalization. Each component of $\bar{X}$ is an affine, normal toric variety. We construct a desingularization $\bar{f} \colon Y \to \bar{X}$ by choosing a toric desingularization for each component of $\bar{X}$. Let $f = \pi \circ \bar{f} \colon Y \to X$ be the induced desingularization. Let Σ' and $\bar{\Sigma}$ be the (toric) boundaries of Y and $\bar{X}$ respectively. Since $f \colon Y \setminus \Sigma' \to X \setminus \Sigma$ is an isomorphism, we obtain a quasi-isomorphism $\underline{\Omega}^p_{(X,\Sigma)} \to Rf_*\underline{\Omega}^p_{(Y,\Sigma')}$ (see the proof of [22, Proposition 3.9]). Since Y is smooth and Σ' is a normal crossings divisor in Y, $\Omega^p_{(Y,\Sigma')} \to \underline{\Omega}^p_{(Y,\Sigma')}$ is a quasi-isomorphism. By [16, Proposition 1.8], $\tilde{\Omega}^p_{(\bar{X},\bar{\Sigma})} \to R\bar{f}_*\Omega^p_{(Y,\Sigma')}$ is a quasi-isomorphism. Since π is finite, $\pi_*\tilde{\Omega}^p_{(\bar{X},\bar{\Sigma})} \to Rf_*\Omega^p_{(Y,\Sigma')}$ is a quasi-isomorphism. As X is weakly normal, we see combinatorially that $\tilde{\Omega}^p_{(X,\Sigma)} = \pi_*\tilde{\Omega}^p_{(\bar{X},\bar{\Sigma})}$ (since if $\mathrm{Spec}\, k[S]$ is weakly normal, then $S \cap \mathrm{relint}\, \sigma_S = (S - S) \cap \mathrm{relint}\, \sigma_S$). From the commutative diagram

$$\begin{tikzcd}
\pi_*\tilde{\Omega}^p_{(\bar{X},\bar{\Sigma})} \arrow[r] & Rf_*\underline{\Omega}^p_{(Y,\Sigma')} \\
\tilde{\Omega}^p_{(X,\Sigma)} \arrow[r] \arrow[u] & \underline{\Omega}^p_{(X,\Sigma)} \arrow[u]
\end{tikzcd}$$

we conclude that $\tilde{\Omega}^p_{(X,\Sigma)} \to \underline{\Omega}^p_{(X,\Sigma)}$ is a quasi-isomorphism. □

Let $d \colon A^p(X) \to A^{p+1}(X)$ be the k-linear map such that $d(\chi^m\omega_m) = \chi^m \cdot (\frac{d\chi^m}{\chi^m} \wedge \omega_m)$. It induces a structure of complex with k-linear differential $\tilde{\Omega}^*_X$. Let F be its naive filtration.

Corollary 2.3.4. *Let $\epsilon \colon X_\bullet \to X$ be a smooth simplicial resolution. Then the natural homomorphism $(\tilde{\Omega}^*_X, F) \to R\epsilon_*(\Omega^*_{X_\bullet}, F)$ is a filtered quasi-isomorphism.*

Note that $\mathcal{O}_X = \tilde{\Omega}^0_X$.

Lemma 2.3.5. *Let $f\colon X' \to X$ be a desingularization, let $X'' \to X' \times_X X'$ be a desingularization. We obtain a commutative diagram*

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X'' \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_2"] \arrow[dl, "f''"] \\ X & \end{tikzcd}$$

Then $\tilde{\Omega}^p_X \xrightarrow{\sim} \operatorname{Ker}(f'_*\Omega^p_{X'} \rightrightarrows f''_*\Omega^p_{X''}) = \{\omega' \in f'_*\Omega^p_{X'}; p_1^*\omega' = p_2^*\omega'\}$.

Proof. Let $X_0 = X'$, $\epsilon_0 = f'$. Let $X_1 = X'' \sqcup X'$, let $\delta_0, \delta_1\colon X_1 \to X_0$ be the identity on X', and p_1, p_2 respectively on X''. Let $s_0\colon X_0 \to X_1$ be the extension of the identity of X'. Let $\epsilon_1 = f''$. Both desingularizations are proper and surjective, so we obtain a 1-truncated smooth proper hypercovering

$$\begin{tikzcd} X_0 \arrow[d, "\epsilon_0"'] \arrow[r] & X_1 \arrow[l, shift right=2, "\delta_0"'] \arrow[l, shift left=2, "\delta_1"] \arrow[dl, "\epsilon_1"] \\ X & \end{tikzcd}$$

We can extend the 1-truncated augmented simplicial object to a smooth proper hypercovering $\epsilon\colon X_\bullet \to X$ (see [20, 6.7.4]). By Theorem 2.3.3, $\tilde{\Omega}^p_X \xrightarrow{\sim} \epsilon_*(\Omega^p_{X_\bullet})$. But

$$\epsilon_*(\Omega^p_{X_\bullet}) = \operatorname{Ker}(\epsilon_{0*}\Omega^p_{X_0} \rightrightarrows \epsilon_{1*}\Omega^p_{X_1}) = \operatorname{Ker}(f'_*\Omega^p_{X'} \rightrightarrows f''_*\Omega^p_{X''}).$$

□

In particular, $\tilde{\Omega}^p_X$ coincides with the sheaf of h-differential forms [32].

2.3.1 Toric pairs

Let $X = \operatorname{Spec} k[\mathcal{M}]$ be a weakly normal affine variety associated to a monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$. The torus $\operatorname{Spec} k[M]$ acts on X. Let $Y \subset X$ be an invariant closed subscheme, with reduced structure. Then $Y = \operatorname{Spec} k[\mathcal{M}']$, where $\mathcal{M}' = (M, \Delta', (S_\sigma)_{\sigma \in \Delta'})$ and Δ' is a subfan of Δ, is also weakly normal. The restriction $A^p(X) \to A^p(Y)$ is surjective. Denote the kernel by $A^p(X, Y)$. We have

$$A^p(X, Y) = \oplus_{m \in \cup_{\sigma \in \Delta} S_\sigma \setminus \cup_{\tau \in \Delta'} S_\tau} \chi^m \cdot \wedge^p V_m.$$

Denote by $\tilde{\Omega}^p_{(X,Y)}$ the coherent $\mathcal{O}_X$-module induced by $A^p(X, Y)$. We obtain a short exact sequence

$$0 \to \tilde{\Omega}^p_{(X,Y)} \to \tilde{\Omega}^p_X \xrightarrow{|_Y} \tilde{\Omega}^p_Y \to 0.$$

We constructed a differential complex $\tilde{\Omega}^*_{(X,Y)}$. If we denote by F the naive filtration, Corollary 2.3.4 gives a filtered quasi-isomorphism

$$(\tilde{\Omega}^*_{(X,Y)}, F) \to (\underline{\Omega}^*_{(X,Y)}, F).$$

Note that $\mathcal{I}_{Y \subset X} = \tilde{\Omega}^0_{(X,Y)}$.

Remark 2.3.6. $\Gamma(X, \tilde{\Omega}^p_{(X,Y)}) = \oplus_{\sigma \in \Delta, X_\sigma \not\subset Y} \Gamma(X_\sigma, \tilde{\Omega}^p_{(X_\sigma, \partial X_\sigma)})$, where $\partial X_\sigma = X_\sigma \setminus O_\sigma$ is the toric boundary of the irreducible toric variety X_σ.

Remark 2.3.7. The sheaf of h-differentials can be computed without the weakly normal assumption. Let $X = \operatorname{Spec} k[\mathcal{M}]$ be the variety associated to a monoidal complex. Let $f\colon X^{wn} \to X$ be the weak normalization, described in Proposition 2.2.15. Then $\tilde{\Omega}^p_X = f_*\tilde{\Omega}^p_{X^{wn}}$. If $Y \subset X$ is a union of closed torus invariant subvarieties, then $f^{-1}(Y)$ is weakly normal, hence $f^{-1}(Y) = Y^{wn}$. We obtain $\tilde{\Omega}^p_{(X,Y)} = f_*\tilde{\Omega}^p_{(X^{wn},Y^{wn})}$.

2.4 Weakly toroidal varieties

Let k be an algebraically closed field of characteristic zero. An algebraic variety X/k has *weakly toroidal singularities* if for every closed point $x \in X$, there exists an isomorphism of complete local k-algebras $\mathcal{O}^\wedge_{X,x} \simeq \mathcal{O}^\wedge_{X',x'}$, where $X' = \operatorname{Spec} k[\mathcal{M}]$ is weakly normal, associated to a monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma\in\Delta})$, and x' is a closed point contained in the closed orbit of X'. We say that (X', x') is a local model for (X, x).

Example 2.4.1. Let $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma\in\Delta})$ be a monoidal complex. Then $X = \operatorname{Spec} k[\mathcal{M}]$ has weakly toroidal singularities if and only if X is weakly normal (by Remarks 2.2.18 and 2.2.19).

Remark 2.4.2. Suppose a local model of (X, x) satisfies $S_\sigma = M \cap \sigma$ for all $\sigma \in \Delta$. Then we can find another local model such that x' is a fixed point of the torus action (by Remark 2.2.19). In particular, for Danilov's toroidal singularities [16] and Ishida's polyhedral singularities [35] we can always find local models near a fixed point.

Let X/k have weakly toroidal singularities. Then X is normal if and only if X is toroidal in the sense of Danilov, that is the local models are (X', x') with X' an affine toric normal variety, and x' a torus invariant closed point of X'.

Theorem 2.4.3. *Let X have weakly toroidal singularities. Let $\epsilon\colon X_\bullet \to X$ be a smooth simplicial resolution. Then $\tilde{\Omega}^p_X \to R\epsilon_*(\Omega^p_{X_\bullet})$ is a quasi-isomorphism (i.e. $\tilde{\Omega}^p_X \tilde{\to} \epsilon_*(\Omega^p_{X_\bullet})$ and $R^i\epsilon_*(\Omega^p_{X_\bullet}) = 0$ for $i > 0$).*

Proof. The statement is local, and invariant under étale base change. By [10], we may suppose $X = \operatorname{Spec} k[\mathcal{M}]$ is a weakly normal local model. Then we may apply Theorem 2.3.3. □

Thus, the filtered complex $(\tilde{\Omega}^*_X, F)$, with F the naive filtration, is a canonical choice for the Du Bois complex of X.

Corollary 2.4.4. *Let X have weakly toroidal singularities. Then X has Du Bois singularities.*

Proof. We claim that $\mathcal{O}_X = \tilde{\Omega}^0_X$. The statement is local, and invariant under étale base change. By [10], we may suppose $X = \operatorname{Spec} k[\mathcal{M}]$ is a weakly normal local model. By definition, $A^0(X) = \Gamma(X, \mathcal{O}_X)$. Therefore the claim holds. □

Lemma 2.4.5 (Poincaré lemma)**.** *Let $X/\mathbb{C}$ have weakly toroidal singularities. Then $\mathbb{C}_{X^{an}} \to \tilde{\Omega}^*_{X^{an}}$ is a quasi-isomorphism.*

Proof. Let $\epsilon\colon X_\bullet \to X$ be a smooth simplicial resolution. Consider the commutative diagram

$$\begin{array}{ccc} R\epsilon_*\mathbb{C}_{X_\bullet^{an}} & \longrightarrow & R\epsilon_*\Omega^*_{X_\bullet^{an}} \\ \uparrow & & \uparrow \\ \mathbb{C}_{X^{an}} & \longrightarrow & \tilde{\Omega}^*_{X^{an}} \end{array}$$

The left vertical arrow is a quasi-isomorphism from the definition of ϵ. The right vertical arrow is a quasi-isomorphism by Theorem 2.4.3. The top horizontal arrow is a quasi-isomorphism, by the Poincaré lemma on each component X_n^{an} $(n \geq 0)$. Therefore the bottom horizontal arrow is a quasi-isomorphism. □

Theorem 2.4.6. *Let $X/\mathbb{C}$ be proper, with weakly toroidal singularities. Then the spectral sequence*

$$E_1^{pq} = H^q(X, \tilde{\Omega}^p_X) \Longrightarrow \mathrm{Gr}^p_F H^{p+q}(X^{an}; \mathbb{C})$$

degenerates at E_1, and converges to the Hodge filtration on the cohomology groups of X^{an}.

Proof. This follows from Theorem 2.4.3 and [22]. More precisely, let $\epsilon\colon X_\bullet \to X$ be a smooth simplicial resolution. Then $(\Omega^*_{X_\bullet}, W, F)$, with W the trivial filtration, is the analytical part of a cohomological moved Hodge $\mathbb{Z}$-complex on $X_\bullet$ (see [20, Example 8.1.12] with $Y_\bullet = \emptyset$). Since $X_\bullet$ is proper, we obtain by [20, Theorem 8.1.15.(i), Scolie 8.1.9.(v)] a spectral sequence

$$E_1^{pq} = H^q(X_\bullet, \tilde{\Omega}^p_{X_\bullet}) \Longrightarrow \mathrm{Gr}^p_F H^{p+q}(X_\bullet^{an}; \mathbb{C})$$

which degenerates at E_1, and converges to the Hodge filtration on the cohomology groups of $X_\bullet^{an}$. By Theorem 2.4.3 and Lemma 2.4.5, this pushes down on X to our claim. □

Finally, we check that $\tilde{\Omega}^p_X$ coincides with the sheaves defined by Danilov [16] and Ishida [35]:

- Suppose X is toroidal. If $f\colon X' \to X$ is a desingularization and $w\colon U \subseteq X$ is the inclusion of the smooth locus, then $\tilde{\Omega}^p_X = f_*(\Omega^p_{X'}) = w_*(\Omega^p_U)$.

  Indeed, by [10] and étale base change, we may suppose X is an affine toric normal variety. We may replace f by a toric desingularization. Danilov shows in [15, Lemma 1.5] that $A^p(X) = \Gamma(X', \Omega^p_{X'}) = \Gamma(U, \Omega^p_U)$.

- Suppose X is a torus invariant closed reduced subvariety of an affine toric normal variety. Then $A^p(X)$ coincides with Ishida's module $\tilde{\Omega}^p_{B(\Phi)}$ defined in [35, page 119].

2.4.1 Weakly toroidal pairs

A *weakly toroidal pair* (X,Y) consists of a weakly normal algebraic variety X/k and a closed reduced subvariety $Y \subseteq X$, such that for every closed point $x \in X$ there exists an isomorphism of complete local k-algebras $\mathcal{O}^\wedge_{X,x} \simeq \mathcal{O}^\wedge_{X',x'}$, mapping $\mathcal{I}^\wedge_{Y,x}$ onto $\mathcal{I}^\wedge_{Y',x'}$, where $X' = \operatorname{Spec} k[\mathcal{M}]$ is the affine variety associated to some monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$, $Y' = \cup_{\sigma \in \Delta'} X'_\sigma \subseteq X'$ is a closed reduced subvariety which is invariant under the action of the torus $\operatorname{Spec} k[M]$, and x' is a closed point contained in the closed orbit of X'.

Example 2.4.7. Let $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma\in\Delta})$ and Δ' a subfan of Δ. Consider $X = \operatorname{Spec} k[\mathcal{M}]$ and $Y = \cup_{\sigma\in\Delta'} X_\sigma$. Then (X,Y) is a weakly toroidal pair if and only if X is weakly normal.

Example 2.4.8. Suppose X is weakly toroidal. Let $\operatorname{Sing} X$ and C be the singular and non-normal locus of X, respectively. Then $(X, \operatorname{Sing} X)$ and (X, C) are weakly toroidal pairs.

If (X,Y) is a weakly toroidal pair, then X and Y are weakly toroidal.
Let (X,Y) be a weakly toroidal pair. Define $\tilde{\Omega}^p_{(X,Y)} = \operatorname{Ker}(\tilde{\Omega}^p_X \to \tilde{\Omega}^p_Y)$.

Lemma 2.4.9. *Let (X,Y) be a weakly toroidal pair. We have a short exact sequence*

$$0 \to \tilde{\Omega}^p_{(X,Y)} \to \tilde{\Omega}^p_X \to \tilde{\Omega}^p_Y \to 0.$$

Proof. By étale base change and [10], we may suppose (X,Y) is a local model. Then $A^p(X) \to A^p(Y)$ is surjective, by the combinatorial formulas for the two modules. $\square$

Theorem 2.4.10. *Let (X,Y) be a weakly toroidal pair. Let $\epsilon \colon X_\bullet \to X$ be a smooth simplicial resolution, such that $\epsilon^{-1}(Y) = Y_\bullet$ is locally on $X_\bullet$ either empty, or a normal crossing divisor. Then $\tilde{\Omega}^p_{(X,Y)} \to R\epsilon_*(\Omega^p_{(X_\bullet,Y_\bullet)})$ is a quasi-isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R\epsilon_*\Omega^p_{(X_\bullet,Y_\bullet)} \arrow[r] & R\epsilon_*\Omega^p_{X_\bullet} \arrow[r] & R\epsilon_*\Omega^p_{Y_\bullet} \arrow[r] & 0 \\
0 \arrow[r] & \tilde{\Omega}^p_{(X_\bullet,Y_\bullet)} \arrow[r] \arrow[u, "\alpha"] & \tilde{\Omega}^p_X \arrow[r] \arrow[u, "\beta"] & \tilde{\Omega}^p_Y \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where the top row is an exact triangle, and the bottom row is a short exact sequence. Since β, γ are quasi-isomorphisms, so is α. $\square$

Thus, the filtered complex $(\tilde{\Omega}^*_{(X,Y)}, F)$, with F the naive filtration, is a canonical choice for the Du Bois complex of the pair (X,Y) (see [45]).

Corollary 2.4.11. *Let (X,Y) be a weakly toroidal pair. Then (X,Y) has Du Bois singularities.*

Proof. We have to show that $\mathcal{I}_{Y\subset X} = \tilde{\Omega}^0_{(X,Y)}$. The statement is local, and invariant under étale base change. By [10], we may suppose $X = \operatorname{Spec} k[\mathcal{M}]$ is a weakly normal local model, and $Y \subset X$ is a torus invariant closed subvariety. By definition, $A^0(X,Y) = \Gamma(X, \mathcal{I}_{Y\subset X})$. Therefore the claim holds. □

As above, we obtain the Poincaré lemma for pairs: $\mathbb{C}_{(X^{an},Y^{an})} \xrightarrow{\sim} \tilde{\Omega}^*_{(X^{an},Y^{an})}$. Similarly, we obtain

Theorem 2.4.12. *Let (X,Y) be a weakly toroidal pair, with $X/\mathbb{C}$ proper. Then the spectral sequence*

$$E_1^{pq} = H^q(X, \tilde{\Omega}^p_{(X,Y)}) \Longrightarrow \operatorname{Gr}^p_F H^{p+q}(X^{an}, Y^{an}; \mathbb{C})$$

degenerates at E_1, and converges to the Hodge filtration on the relative cohomology groups of (X^{an}, Y^{an}).

We can also generalize [15, Propositions 1.8, 2.8] as follows:

Proposition 2.4.13. *Let (X',Y') and (X,Y) be weakly toroidal pairs. Let $f\colon X' \to X$ be a proper surjective morphism such that $Y' = f^{-1}(Y)$ and $f\colon X' \setminus Y' \to X \setminus Y$ is an isomorphism. Then*

$$\tilde{\Omega}^p_{(X,Y)} \to Rf_*\tilde{\Omega}^p_{(X',Y')}$$

is a quasi-isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\underline{\Omega}^p_{(X,Y)} & \longrightarrow & Rf_*\underline{\Omega}^p_{(X',Y')} \\
\uparrow & & \uparrow \\
\tilde{\Omega}^p_{(X,Y)} & \longrightarrow & Rf_*\tilde{\Omega}^p_{(X',Y')}
\end{array}$$

The vertical arrows are quasi-isomorphisms, by Theorem 2.4.10. The top horizontal arrow is a quasi-isomorphism by the proof of [22, Proposition 4.11]. Therefore the bottom horizontal arrow is a quasi-isomorphism as well. □

Chapter 3

Weakly log canonical varieties

Our motivation is to better understand semi-log canonical singularities (cf. [44]) by constructing toric examples. Semi-log canonical singularities are possibly not normal, and even reducible. So by a toric variety we mean $\operatorname{Spec} k[\mathcal{M}]$, the spectrum of a toric face ring $k[\mathcal{M}]$ associated to a monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$. From the algebraic point of view, toric face rings were introduced as a generalization of Stanley-Reisner rings, studied by Stanley, Reisner, Bruns, Ichim, Römer and others (see the introductions of [33, 7] for example). From the geometric point of view, Alexeev [1] introduced another generalization of Stanley-Reisner rings, the so called stable toric varieties, obtained by glueing toric varieties (possibly not affine) along orbits.

In order to understand residues for varieties with normal crossings singularities, we were forced to enlarge the category of semi-log canonical singularities to the class of weakly log canonical singularities. To see this, let us consider the normal crossings model $\Sigma = \cup_{i=1}^n H_i \subset \mathbb{A}^n_{\mathbb{C}}$, where $H_i : (z_i = 0)$ is the i-th standard hyperplane. It is Cohen Macaulay and Gorenstein, and codimension one residues onto components of Σ glue to a residue isomorphism $\operatorname{Res}\colon \omega_{\mathbb{A}^n}(\log \Sigma)|_\Sigma \xrightarrow{\sim} \omega_\Sigma$, where ω_Σ is a dualizing sheaf. It follows that Σ has semi-log canonical singularities and $\omega_\Sigma \simeq \mathcal{O}_\Sigma$. The complement $T = \mathbb{A}^n \setminus \Sigma$ is the n-dimensional torus, which acts naturally on $\mathbb{A}^n$. The invariant closed irreducible subvarieties of codimension p are $H_{i_1} \cap \cdots \cap H_{i_p}$ for $i_1 < \cdots < i_p$. A natural way to realize Σ as a glueing of smooth varieties (cf. [19]) is to consider the decreasing filtration of algebraic varieties

$$X_1 \supset X_2 \supset \cdots$$

where $X_1 = \Sigma$ and $X_{p+1} = \operatorname{Sing}(X_p)$ for $p \geq 1$. It turns out that X_p is the union of T-invariant closed irreducible subvarieties of $\mathbb{A}^n$ of codimension p, that is $X_p = \cup_{i_1 < \cdots < i_p} H_{i_1} \cap \cdots \cap H_{i_p}$ (the reader may check that X_p is the affine toric variety associated to the following monoidal complex: lattice $\mathbb{Z}^n$, fan consisting of all faces $\sigma \prec \mathbb{R}^n_{\geq 0}$ of codimension at least p, and semigroups $S_\sigma = \mathbb{Z}^n \cap \sigma$). After extending the filtration with $X_0 = \mathbb{A}^n$, we would like to realize it as a chain of semi-log canonical structures and (glueing of) codimension one residues

$$(\mathbb{A}^n, \Sigma) \rightsquigarrow (\Sigma, 0) \rightsquigarrow (X_2, 0) \rightsquigarrow \cdots .$$

The varieties X_p are weakly normal and Cohen Macaulay, but not nodal in codimension one if $p > 1$. The dualizing sheaf of X_2 is not invertible in codimension one, so we cannot define the sheaves $\omega^{[n]}_{X_2}$ ($n \in \mathbb{Z}$), and $(X_2, 0)$ is not semi-log canonical. We observe in this chapter that the filtration may still be viewed as a chain of log structures, provided we enlarge the category of semi-log canonical singularities to a certain class called *weakly log canonical singularities*. We show that for $p > 0$, $(X_p, 0)$ has weakly log canonical singularities, $\omega^{[2]}_{(X_p,0)} \simeq \mathcal{O}_{X_p}$, and codimension one residues onto components of X_{p+1} glue to a residue isomorphism $\operatorname{Res}^{[2]} \colon \omega^{[2]}_{(X_p,0)}|_{X_{p+1}} \tilde{\to} \omega^{[2]}_{(X_{p+1},0)}$ (see Proposition 3.5.6).

A semi-log canonical singularity X is defined as a singularity such that a) X is S_2 and nodal in codimension one, b) certain pluricanonical sheaves $\omega^{[r]}_X$ are invertible, and c) the induced log structure on the normalization has log canonical singularities. We define weakly log canonical singularities by replacing axiom a) with a'): X is S_2 and weakly normal. The known pluricanonical sheaves $\omega^{[r]}_X$ are replaced by certain pluricanonical sheaves $\omega^{[r]}_{(X,0)}$, consisting of rational differential r-forms on X which have constant residues over each codimension one non-normal point of X. Semi-log canonical singularities are a subclass of weakly log canonical singularities, as it turns out that $\omega^{[r]}_X = \omega^{[r]}_{(X,0)}$ ($r \in 2\mathbb{Z}$) if X has semi-log canonical singularities. Among weakly log canonical singularities, semi-log canonical singularities are those which have multiplicity 1 or 2 in codimension one.

We classify toric varieties $X = \operatorname{Spec} k[\mathcal{M}]$ which are weakly (semi-) log canonical. The classification is combinatorial, expressed in terms of the log structure on the normalization, and certain incidence numbers of the irreducible components in their invariant codimension one subvarieties. The irreducible case is much simpler than the reducible case. Along the way, we find a criterion for X to satisfy Serre's property S_2, which extends Terai's criterion [64].

A key feature of weakly log canonical singularities is the definition of residues onto lc centers of codimension one. We make this explicit in the toric case. We also construct residues to higher codimension lc centers, under the assumption that the irreducible components of the toric variety are normal. In particular, we obtain higher codimension residues for normal crossings pairs.

We outline the structure of this chapter. In Section 1 we collect known results on log pairs and codimension one residues, and exemplify them in the (normal) toric case. In Section 2, we find a criterion (Theorem 3.2.10) for $\operatorname{Spec} k[\mathcal{M}]$ to satisfy Serre's property S_2. The irreducible case was known [13], and our criterion generalizes that of Terai [64]. The weak normality criterion for $\operatorname{Spec} k[\mathcal{M}]$ was also known (see [7] for a survey and references). In Section 3 we define weakly normal log pairs, and the class of weakly log canonical singularities. Compared to semi-log canonical pairs, weakly normal log pairs are allowed boundaries with negative coefficients, and a certain locus where it is not weakly log canonical. Hopefully, this will be useful in future applications. In Section 4, we find a criterion for $\operatorname{Spec} k[\mathcal{M}]$, endowed with a torus invariant boundary B, to be a weakly normal log pair (Proposition 3.4.2 for the irreducible case, Proposition 3.4.10 for the reducible case). We also investigate the LCS-locus, or non-klt locus of a toric weakly normal pair, which is useful for inductive arguments. In Section 5 we construct residues of toric weakly log canonical pairs onto lc centers of arbitrary codimension, under the assumption that the irreducible components of the toric variety are normal. We extend these results to weakly log canonical pairs which are locally analytically isomorphic to such toric models (Theorem 3.5.8). In particular, we obtain higher codimension residues for normal crossings pairs (Corollary 3.5.10).

3.1 Preliminary on log pairs, codimension one residues

3.1.1 Rational pluri-differential forms on normal varieties

Let X/k be a normal algebraic variety, irreducible, of dimension d. A prime divisor on X is a codimension one subvariety P in X.

A non-zero rational function $f \in k(X)^\times$ induces the principal Weil divisor on X

$$(f) = \operatorname{div}_X(f) = \sum_P v_P(f) \cdot P,$$

where the sum runs after all prime divisors of X. Note that $v_P(f)$ is the maximal $m \in \mathbb{Z}$ such that $t_P^{-m} f$ is regular at P, where t_P is a local parameter at P.

A non-zero rational differential d-form $\omega \in \wedge^d \Omega^1_{k(X)/k} \setminus 0$ induces a Weil divisor on X

$$(\omega) = \sum_P v_P(\omega) \cdot P,$$

where $v_P(\omega)$ is the maximal $m \in \mathbb{Z}$ such that $t_P^{-m}\omega$ is regular at P, where t_P is a local parameter at P. If $\omega' \in \wedge^d \Omega^1_{k(X)/k} \setminus 0$, then $\omega' = f\omega$ for some $f \in k(X)^\times$, and $(\omega') = (f) + (\omega)$. Therefore the linear equivalence class of (ω) is an invariant of X, called the *canonical divisor* of X, denoted K_X. Sometimes we also denote by K_X any divisor in this class, but this may cause confusion.

Let $r \in \mathbb{Z}$. A non-zero rational r-pluri-differential form $\omega \in (\wedge^d \Omega^1_{k(X)/k})^{\otimes r} \setminus 0$ induces a Weil divisor on X

$$(\omega) = \sum_P v_P(\omega) \cdot P,$$

where if we write $\omega = f\omega_0^r$ with $f \in k(X)^\times$ and $\omega_0 \in \Omega^1_{k(X)/k} \setminus 0$, we define $(\omega) = (f) + r(\omega_0)$. This is well defined, and $(\omega) \sim rK_X$.

The following properties hold: $(f\omega) = (f) + (\omega)$, $(\omega_1\omega_2) = (\omega_1) + (\omega_2)$. Note that rational functions identify with rational differential 0-forms.

Let $P \subset X$ be a prime divisor. A rational differential p-form $\omega \in \wedge^p \Omega^1_{k(X)/k}$ has *at most a logarithmic pole* at P if both ω and $d\omega$ have at most a simple pole at P. Equivalently, there exists a decomposition $\omega = (dt/t) \wedge \omega^{p-1} + \omega^p$, with t a local parameter at P, and ω^{p-1}, ω^p regular at P. Define the *Poincaré residue of* ω *at* P to be the rational differential form

$$\operatorname{Res}_P \omega = \omega^{p-1}|_P \in \wedge^{p-1} \Omega^1_{k(P)/k}.$$

The definition is independent of the decomposition. It is additive in ω, and if $f \in k(X)$ is regular at P, then $f|_P \in k(P)$ and $\mathrm{Res}_P(f\omega) = f|_P \cdot \mathrm{Res}_P(\omega)$.

Note that $\omega \in \wedge^d \Omega^1_{k(X)/k}$ automatically satisfies $d\omega = 0$. Therefore ω has at most a logarithmic pole at P if and only if $(\omega) + P \geq 0$ near P.

3.1.2 Log pairs and varieties

Let X/k be a normal algebraic variety. Let B be a $\mathbb{Q}$-Weil divisor on X: a formal sum of prime divisors on X, with rational coefficients, or equivalently, the formal closure of a $\mathbb{Q}$-Cartier divisor defined on the smooth locus of X. For $n \in \mathbb{Z}$, define a coherent $\mathcal{O}_X$-module $\omega^{[n]}_{(X/k,B)}$ by setting for each open subset $U \subseteq X$

$$\Gamma(U, \omega^{[n]}_{(X/k,B)}) = \{0\} \cup \{\omega \in (\wedge^d \Omega^1_{k(X)/k})^{\otimes n}; (\omega) + nB \geq 0 \text{ on } U\}.$$

On $V = X \setminus (\operatorname{Sing} X \cup \operatorname{Supp} B)$, $\omega^{[n]}_{(X/k,B)}|_V$ coincides with the invertible $\mathcal{O}_V$-module $(\wedge^d \Omega^1_{V/k})^{\otimes n}$.

Lemma 3.1.1. *Let $U \subseteq X$ be an open subset. Let $\omega \in (\wedge^d \Omega^1_{k(X)/k})^{\otimes n} \setminus 0$ be a non-zero rational pluri-differential form. Then $1 \mapsto \omega$ induces an isomorphism $\mathcal{O}_U \xrightarrow{\sim} \omega^{[n]}_{(X/k,B)}|_U$ if and only if $(\omega) + \lfloor nB \rfloor = 0$ on U.*

Proof. Indeed, the homomorphism is well defined only if $D = (\omega) + \lfloor nB \rfloor|_U \geq 0$. The homomorphism is an isomorphism if and only if $\mathcal{O}_U = \mathcal{O}_U(D)$, that is $D = 0$, since U is normal. □

The choice of a non-zero rational top differential form on X induces an isomorphism between the sheaf of rational pluri-differentials $\omega^{[n]}_{(X/k,B)}$ and the sheaf of rational functions $\mathcal{O}_X(nK_X + \lfloor nB \rfloor)$.

We have a natural multiplication map $\omega^{[m]}_{(X/k,B)} \otimes_{\mathcal{O}_X} \omega^{[n]}_{(X/k,B)} \to \omega^{[m+n]}_{(X/k,B)}$, which is an isomorphism if mB has integer coefficients and $\omega^{[m]}_{(X/k,B)}$ is invertible. In particular, if rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible, then $(\omega^{[r]}_{(X/k,B)})^{\otimes n} \xrightarrow{\sim} \omega^{[rn]}_{(X/k,B)}$ for all $n \in \mathbb{Z}$, and the graded $\mathcal{O}_X$-algebra $\oplus_{n \in \mathbb{N}} \omega^{[n]}_{(X/k,B)}$ is finitely generated.

Definition 3.1.2. A *log pair* $(X/k, B)$ consists of a normal algebraic variety X/k and the (formal) closure B of a $\mathbb{Q}$-Weil divisor on the smooth locus of X/k, subject to the following property: there exists an integer $r \geq 1$ such that rB has integer coefficients and the $\mathcal{O}_X$-module $\omega^{[r]}_{(X/k,B)}$ is locally free (i.e. invertible).

If B is effective, we call $(X/k, B)$ a *log variety*.

3.1.3 Log canonical singularities, lc centers

We assume log resolutions are known to exist (e.g. if $\mathrm{char}(k) = 0$, by Hironaka, or in the category of toric log pairs). Let $(X/k, B)$ be a log pair. There exists a *log resolution* $\mu\colon X' \to (X, B_X)$, that is a desingularization $\mu\colon X' \to X$ such that $\mathrm{Exc}\,\mu \cup \mu^{-1}(\mathrm{Supp}\,B)$ is a normal crossings divisor. Let $r \geq 1$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. If ω is a local generator, then $\mu^*\omega$ is a local generator of $\omega^{[r]}_{(X'/k,B_{X'})}$, where $B_{X'}$ is a $\mathbb{Q}$-divisor on X' such that $rB_{X'}$ has integer coefficients (locally, $B_{X'} = -\frac{1}{r}(\mu^*\omega)$). The $\mathbb{Q}$-divisor $B_{X'}$ may not be effective even if B is effective, and this is the reason why we consider log pairs, although we are mainly interested with log varieties.

We obtain a log crepant desingularization $\mu\colon (X', B_{X'}) \to (X, B)$, with X' smooth and $\mathrm{Supp}(B_{X'})$ a normal crossings divisor, and an isomorphism $\mu^*\omega^{[r]}_{(X/k,B)} \stackrel{\sim}{\to} \omega^{[r]}_{(X',B_{X'})}$.

If the coefficients of $B_{X'}$ are at most 1, we say that (X, B) has *log canonical singularities*. This definition is independent of the choice of μ. If $B^{>1}_{X'}$ denotes the part of $B_{X'}$ which has coefficients strictly larger than 1, then $\mu(\mathrm{Supp}(B^{>1}_{X'}))$ is a closed subset of X, called the *non-lc locus* of (X, B), denoted $(X, B)_{-\infty}$. It is the complement in X of the largest open subset where (X, B) has log canonical singularities. An *lc center* of (X, B) is either X, or $\mu(E)$ for some prime divisor E on some log resolution $X' \to X$, with $\mathrm{mult}_E(B_{X'}) = 1$ and $\mu(E) \not\subseteq (X, B)_{-\infty}$. If $\mu\colon (X', B_{X'}) \to (X, B)$ is a log resolution such that $B^{=1}_{X'}$ has simple normal crossings, the lc centers of (X, B) different from X are exactly the images, not contained in $(X, B)_{-\infty}$, of the intersections of the components of $B^{=1}_{X'}$. In particular, (X, B) has only finitely many lc centers.

3.1.4 Residues in codimension one lc centers, different

Let $(X/k, B)$ be a log pair, let $E \subset X$ be a prime divisor with $\mathrm{mult}_E(B) = 1$. Let $\omega \in \Gamma(X, \omega^{[l]}_{(X/k,B)})$. Near the generic point of E, $\omega^{[1]}_{(X/k,B)}$ is invertible, say with generator ω_0. We can write $\omega = f\omega_0^{\otimes l}$, with $f \in k(X)^\times$ regular at the generic point of E. Define the *residue of* ω *at* E to be the rational pluri-differential form

$$\mathrm{Res}^{[l]}_E\,\omega = f|_E \cdot (\mathrm{Res}_E\,\omega_0)^{\otimes l} \in (\wedge^{d-1}\Omega^1_{k(E)/k})^{\otimes l}.$$

The definition is independent of the choice of f and ω_0. It is additive in ω, and if $g \in k(X)$ is regular at the generic point of E, then $\mathrm{Res}^{[l]}_E(g \cdot \omega) = g|_E \cdot \mathrm{Res}^{[l]}_E\,\omega$. The residue operation induces a natural map

$$\mathrm{Res}^{[l]}_E\colon \omega^{[l]}_{(X/k,B)} \to \omega^{[l]}_{k(E)/k},$$

which is compatible with multiplication of pluri-differential rational forms.

Let $r \geq 1$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Let $E^n \to E$ be the normalization and $j\colon E^n \to X$ the induced morphism. Choose an open subset $U \subseteq X$ which intersects E, and a nowhere zero section ω of $\omega^{[r]}_{(X/k,B)}|_U$. Then $\mathrm{Res}^{[r]}_E\,\omega$ is a non-zero rational pluri-differential form on E^n. The identity

$$(\mathrm{Res}^{[r]}_E\,\omega)|_{j^{-1}(U)} + D|_{j^{-1}(U)} = 0$$

defines a Weil divisor D on $j^{-1}(U)$. It does not depend on the choice of ω, and it glues to a Weil divisor D on E^n. The $\mathbb{Q}$-Weil divisor $B_{E^n} = \frac{1}{r}D$ is called the *different of* (X, B) *on* E^n. It follows that rB_{E^n} has integer coefficients, $\omega^{[r]}_{(E^n/k,B_{E^n})}$ is invertible, and the residue at E induces an isomorphism

$$\mathrm{Res}^{[r]}_E : j^*\omega^{[r]}_{(X/k,B)} \xrightarrow{\sim} \omega^{[r]}_{(E^n/k,B_{E^n})}.$$

If $l \geq 1$ is an integer, then $\omega^{[rl]}_{(X/k,B)}$ is again invertible. It defines the same different, and the isomorphism $\mathrm{Res}^{[rl]}_E$ identifies with $(\mathrm{Res}^{[r]}_E)^{\otimes l}$. We deduce that the different B_{E^n} is independent of the choice of r, and $(E^n/k, B_{E^n})$ is again a log pair. The following properties hold:

- If $B \geq 0$, then $B_{E^n} \geq 0$.
- Let B' such that $\mathrm{mult}_E B' = 1$ and $B' - B$ is $\mathbb{Q}$-Cartier. Then $B'_{E^n} = B_{E^n} + j^*(B' - B)$.

3.1.5 Volume forms on the torus

Let T/k be a torus, of dimension d. Then $T = \mathrm{Spec}\, k[M]$ for some lattice M. Let $\mathcal{B} = (m_1, \ldots, m_d)$ be an ordered basis of the lattice M. Then

$$\omega_{\mathcal{B}} = \frac{d\chi^{m_1}}{\chi^{m_1}} \wedge \cdots \wedge \frac{d\chi^{m_d}}{\chi^{m_d}}$$

is a T-invariant global section of $\wedge^d \Omega^1_{T/k}$, which is nowhere zero. It induces an isomorphism

$$\mathcal{O}_T \xrightarrow{\sim} \wedge^d \Omega^1_{T/k}.$$

Let $\mathcal{B}' = (m'_1, \ldots, m'_d)$ be another ordered basis of M. Then $\omega_{\mathcal{B}} = \epsilon \cdot \omega_{\mathcal{B}'}$, where the sign $\epsilon = \pm 1$ is computed either by the identity $\wedge^d_{i=1} m_i = \epsilon \cdot \wedge^d_{i=1} m'_i$ in $\wedge^d M$, or as the determinant of the matrix (a_{ij}) given by $m_i = \sum_j a_{ij} m'_j$. Therefore $\omega_{\mathcal{B}}$ depends on the choice of the ordered basis of M only up to a sign. If the sign does not matter, we denote $\omega_{\mathcal{B}}$ by ω_T or ω_M. For example, if n is an *even* integer, we denote $\omega^{\otimes n}_{\mathcal{B}}$ by $\omega^{\otimes n}_T$.

The above trivialization of $\wedge^d \Omega^1_{T/k}$ depends on the choice of the ordered basis up to a sign. Its invariant form is $\mathcal{O}_T \otimes_{\mathbb{Z}} \wedge^d M \xrightarrow{\sim} \wedge^d \Omega^1_{T/k}$ (induced by $\mathcal{O}_T \otimes_{\mathbb{Z}} M \xrightarrow{\sim} \Omega^1_{T/k}$). The form $\omega_{\mathcal{B}}$ depends in fact only on the basis element $m_1 \wedge \cdots \wedge m_d$ of $\wedge^d M \simeq \mathbb{Z}$. We say that $\omega_{\mathcal{B}}$ is the *volume form induced by an orientation of* M.

Let $M' \subseteq M$ be a sublattice of finite index e. It corresponds to a finite surjective toric morphism $\varphi \colon T = \mathrm{Spec}\, k[M] \to T' = \mathrm{Spec}\, k[M']$. If $\mathcal{B}'$ is an ordered basis of M', then $\varphi^*\omega_{\mathcal{B}'} = (\pm e) \cdot \omega_{\mathcal{B}}$.

3.1.6 Affine toric log pairs

Let $T \subseteq X$ be a normal affine equivariant embedding of a torus. Thus $T = \operatorname{Spec} k[M]$ for some lattice M, and $X = \operatorname{Spec} k[M \cap \sigma]$ for a rationally polyhedral cone $\sigma \subseteq M_{\mathbb{R}}$ which generates $M_{\mathbb{R}}$. The complement $\Sigma_X = X \setminus T$ is called the toric boundary of X. We have $\Sigma_X = \cup_i E_i$, where E_i are the invariant codimension one subvarieties of X. Each E_i is of the form $\operatorname{Spec} k[M \cap \tau_i]$, where $\tau_i \prec \sigma$ is a codimension one face. Let $e_i \in N \cap \sigma^\vee$ be the primitive vector in the dual lattice N which cuts out τ_i, that is $\sigma^\vee \cap \tau_i^\perp \cap N = \mathbb{N} e_i$.

The volume form $\omega_{\mathcal{B}}$ on T, induced by an orientation of M, extends as a rational top differential form on X. Let E_i be an invariant prime divisor on X. As a subvariety, $E_i = \operatorname{Spec} k[M \cap \tau_i]$ is again toric and normal. Denote by M_i the lattice $M \cap \tau_i - M \cap \tau_i = M \cap (\tau_i - \tau_i)$. Let $\mathcal{B}_i = (m'_1, \ldots, m'_{d-1})$ be an ordered basis of M_i. Choose $u \in M$ such that $\langle e_i, u \rangle = 1$. Then $\mathcal{B}'_i = (u, m'_1, \ldots, m'_{d-1})$ becomes an ordered basis of M, and $\omega_{\mathcal{B}} = \epsilon_i \cdot \omega_{\mathcal{B}'_i}$ for some $\epsilon_i = \pm 1$. The sign ϵ_i does not depend on the choice of u. Since χ^u is a local parameter at the generic point of E_i, and $\omega_{\mathcal{B}'_i} = \frac{d\chi^u}{\chi^u} \wedge \omega_{\mathcal{B}_i}$, we obtain

$$\operatorname{Res}_{E_i} \omega_{\mathcal{B}} = \epsilon_i \cdot \omega_{\mathcal{B}_i}.$$

Therefore $\omega_{\mathcal{B}}$ has exactly logarithmic poles along the invariant prime divisors of X, and the induced Weil divisor on X is

$$(\omega_{\mathcal{B}}) = -\Sigma_X.$$

Lemma 3.1.3. *$(X/k, \Sigma_X)$ is a log variety with lc singularities.*

Proof. We have $\omega^{[1]}_{(X/k,\Sigma_X)} = \mathcal{O}_X \cdot \omega_{\mathcal{B}}$, so $\omega^{[1]}_{(X/k,\Sigma_X)} \simeq \mathcal{O}_X$. Let $\mu \colon X' \to X$ be a toric desingularization. Let $\Sigma_{X'} = X' \setminus T$ be the toric boundary of X', which is the union of its invariant codimension one subvarieties. Then X' is smooth, $\Sigma_{X'}$ is a simple normal crossings divisor, and $(\mu^* \omega_{\mathcal{B}}) + \Sigma_{X'} = 0$. Therefore $(X/k, \Sigma)$ has log canonical singularities. $\square$

The different of $(X/k, \Sigma_X)$ on E_i is Σ_{E_i}, and for every $n \in \mathbb{Z}$ we have residue isomorphisms

$$\operatorname{Res}^{[n]}_{E_i} \colon \omega^{[n]}_{(X/k,\Sigma_X)}|_{E_i} \xrightarrow{\sim} \omega^{[n]}_{(E_i/k,\Sigma_{E_i})}.$$

Choosing bases $\mathcal{B}, \mathcal{B}_i$ to trivialize the sheaves, the residue isomorphism becomes

$$\epsilon_i^n \cdot (\mathcal{O}_X|_{E_i} \xrightarrow{\sim} \mathcal{O}_{E_i}).$$

Let B be a T-invariant $\mathbb{Q}$-Weil divisor on X. That is $B = \sum_i b_i E_i$ with $b_i \in \mathbb{Q}$. We compute

$$\omega^{[n]}_{(X/k,B)} = \mathcal{O}_X(\lfloor -n\Sigma_X + nB \rfloor) \cdot \omega_{\mathcal{B}}^{\otimes n}.$$

Recall that X has a unique closed orbit, associated to the smallest face of σ, which is $\sigma \cap (-\sigma)$, or equivalently, the largest vector subspace contained in σ.

Lemma 3.1.4. *Let $n \in \mathbb{Z}$. The following properties are equivalent:*

a) $\omega^{[n]}_{(X/k,B)}$ *is invertible at some point* x*, which belongs to the closed orbit of* X*.*

b) $\omega^{[n]}_{(X/k,B)} \simeq \mathcal{O}_X$*.*

c) There exists $m \in M$ *such that* $(\chi^m) + \lfloor n(-\Sigma_X + B) \rfloor = 0$ *on* X*.*

Proof. *a)* $\Longrightarrow$ *c)* The T-equivariant sheaf $\mathcal{O}_X(\lfloor n(-\Sigma_X + B) \rfloor)$ is invertible near x. Since x belongs to the closed orbit of X, the sheaf is trivial, and there exists $m \in M$ with $(\chi^m) + \lfloor n(-\Sigma_X + B) \rfloor = 0$ on X [39].

c) $\Longrightarrow$ *b)* $\chi^m \omega_{\mathcal{B}}^{\otimes n}$ is a nowhere zero global section of $\omega^{[n]}_{(X/k,B)}$.

b) $\Longrightarrow$ *a)* is clear. $\square$

Proposition 3.1.5. $(X/k, B)$ *is a log pair if and only if and only if there exists* $\psi \in M_{\mathbb{Q}}$ *such that* $\langle e_i, \psi \rangle = 1 - \operatorname{mult}_{E_i}(B)$ *for all* i*. Moreover,* $(X/k, B)$ *has lc singularities if and only if the coefficients of* B *are at most* 1*, if and only if* $\psi \in \sigma$*.*

Proof. Suppose $(X/k, B)$ is a log pair. There exists $r \geq 1$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Then there exists $m \in M$ with $(\chi^m) + \lfloor r(-\Sigma_X + B) \rfloor = 0$ on X. That is $\langle e_i, m \rangle = r(1 - \operatorname{mult}_{E_i}(B))$ for every i. Then $\psi = \frac{1}{r} m$ satisfies the desired properties.

Conversely, let $\psi \in M_{\mathbb{Q}}$ with $\langle e_i, \psi \rangle = 1 - \operatorname{mult}_{E_i}(B)$ for all i. Let $r \geq 1$ with $r\psi \in M$. In particular, rB has integer coefficients. Since $(\chi^{r\psi}) + r(-\Sigma_X + B) = 0$, $\omega^{[r]}_{(X/k,B)} \simeq \mathcal{O}_X$.

The above proof also shows that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible if and only if $r\psi \in M$.

Suppose $\psi \in \sigma$. Since $\{\mathbb{R}_+ e_i\}_i$ are the extremal rays of $\sigma^\vee$, this is equivalent to $\langle e_i, \psi \rangle \geq 0$ for all i, which in turn is equivalent to $\operatorname{mult}_{E_i}(B) \leq 1$ for all i, that is $B \leq \Sigma_X$. Since $(X/k, \Sigma_X)$ has log canonical singularities, so does (X, B).

Conversely, suppose $(X/k, B)$ has log canonical singularities. Then the coefficients of B are at most 1, that is $\psi \in \sigma$. $\square$

We call $(X/k, B)$ a *toric (normal) log pair*, and $\psi \in M_{\mathbb{Q}}$ the *log discrepancy function* of the toric log pair $(X/k, B)$. The log discrepancy function is unique only up to an element of $M_{\mathbb{Q}} \cap \sigma \cap (-\sigma)$. It uniquely determines the boundary, by the formula $B = \sum_i (1 - \langle e_i, \psi \rangle) E_i$. The terminology derives from the following property:

Lemma 3.1.6. *Let* $(X/k, B)$ *be a toric log pair. Each* $e \in N^{prim} \cap \sigma^\vee$ *induces a toric valuation* E_e *over* X*, with log discrepancy* $a(E_e; X, B) = \langle e, \psi \rangle$*.*

Proof. Let Δ be a fan in N which is a subdivision of σ, and contains $\mathbb{R}_+ e$ as a face. Let $X' = T_N \operatorname{emb}(\Delta)$ be the induced toric variety. Then $\mu \colon X' \to X$ is a toric proper birational morphism, and e defines an invariant prime E_e on X'. Let $r\psi \in M$. Then $\chi^{r\psi} \omega_{\mathcal{B}}^{\otimes r}$ trivializes $\omega^{[r]}_{(X/k,B)}$, hence $\mu^* \chi^{r\psi} \omega_{\mathcal{B}}^{\otimes r}$ trivializes $\omega^{[r]}_{(X'/k,B_{X'})}$. Therefore $1 - \operatorname{mult}_{E_e}(B_{X'}) = \langle e, \psi \rangle$. $\square$

We have $(X/k, B)_{-\infty} = \cup_{b_i>1} E_i$. If non-empty, the non-lc locus has pure codimension one in X. If B is effective, the non-lc locus is the support of a natural subscheme structure [5], with ideal sheaf $I_{-\infty} = \oplus_m k \cdot \chi^m$, where the sum runs after all $m \in M \cap \sigma$ such that $\langle m, e\rangle \geq \max(0, -\langle \psi, e\rangle)$ for all $e \in N \cap \sigma^\vee$. From the existence of toric log resolutions, it follows that the lc centers of $(X/k, B)$ are the invariant subvarieties X_σ, where $\psi \in \sigma \prec \sigma_S$ and $\sigma \not\subset \tau_i$ if $b_i > 1$.

Let $(X/k, B)$ be a toric log pair, with log canonical singularities. That is $\psi \in M_\mathbb{Q} \cap \sigma$. The lc centers of $(X/k, B)$ are the invariant closed irreducible subvarieties $X_\tau = \operatorname{Spec} k[M \cap \tau]$, where τ is a face of σ which contains ψ. For $\tau = \sigma$, we obtain X as an lc center. For $\tau \neq \sigma$, we obtain lc centers defined by toric valuations of X. Each lc center is normal. Any union of lc centers is weakly normal. The intersection of two lc centers is again an lc center. With respect to inclusion, there exists a unique minimal lc center, namely $X_{\tau(\psi)}$ for $\tau(\psi) = \cap_{\psi \in \tau \prec \sigma} \tau$ (the unique face of σ which contains ψ in its relative interior). Note that X is the unique lc center of $(X/k, B)$ if and only if $(X/k, B)$ has klt singularities, if and only if the coefficients of B are strictly less than 1, if and only if ψ belongs to the relative interior of σ. Define the *LCS locus*, or *non-klt locus* of $(X/k, B)$, to be the union of all lc centers of positive codimension in X. We have $\max_i b_i \leq 1$ and $\operatorname{LCS}(X/k, B) = \cup_{b_i=1} E_i$. If non-empty, the LCS locus has pure codimension one in X.

Let $(X/k, B)$ be a toric log pair, let E_i be an invariant prime divisor with $\operatorname{mult}_{E_i}(B) = 1$. Let ψ be the log discrepancy function, let $r\psi \in M$. We have $E_i = \operatorname{Spec} k[M \cap \tau_i]$ for a codimension one face $\tau_i \prec \sigma$. The condition $\operatorname{mult}_{E_i}(B) = 1$ is equivalent to $r\psi \in M_i = M \cap \tau_i - M \cap \tau_i$. Then $\chi^{r\psi}\omega_\mathcal{B}^{\otimes r}$ trivializes $\omega^{[r]}_{(X/k,B)}$, and

$$\operatorname{Res}_{E_i}(\chi^{r\psi}\omega_\mathcal{B}^{\otimes r}) = \chi^{r\psi}(\operatorname{Res}_{E_i}\omega_\mathcal{B})^{\otimes r} = \epsilon_i^r \chi^{r\psi}\omega_{\mathcal{B}_i}^{\otimes r}$$

trivializes $\omega^{[r]}_{(E_i/k, B_{E_i})}$, where B_{E_i} is the different of $(X/k, B)$ on E_i, computed by the formula

$$(\chi^{r\psi}) = r(\Sigma_{E_i} - B_{E_i}) \text{ on } E_i.$$

Let $n \in \mathbb{Z}$. Then $\operatorname{Res}^{[n]}_{E_i}$ sends $\omega^{[n]}_{(X/k,B)}$ into $\omega^{[n]}_{(E_i/k,B_{E_i})}$. If $\omega^{[n]}_{(X/k,B)}$ is invertible (even if nB does not have integer coefficients), we obtain an isomorphism

$$\operatorname{Res}^{[n]}_{E_i} \colon \omega^{[n]}_{(X/k,B)}|_{E_i} \xrightarrow{\sim} \omega^{[n]}_{(E_i/k,B_{E_i})}.$$

The coefficients of the different B_{E_i} are controlled by those of B. Indeed, let $Q \subset E_i$ be an invariant prime divisor. The lattice dual to M_i is a quotient lattice N_i of N, and the cone in N_i dual to $\tau_i \subset (M_i)_\mathbb{R}$ is the image of $\sigma^\vee \subset N_\mathbb{R}$ under the quotient $\pi \colon N \to N_i$. Let $e_Q \in N_i$ be the primitive vector on the extremal ray of the cone dual to τ_i, which determines $Q \subset E_i$. There exists an extremal ray of $\sigma^\vee$ which maps onto $\mathbb{R}_+ e_Q$, and denote by e_j its primitive vector. Then $\pi(e_j) = q e_Q$ for some positive integer q. Since $\langle e_j, \psi\rangle = q\langle e_Q, \psi\rangle$, we obtain

$$\operatorname{mult}_Q(B_{E_i}) = 1 - \frac{1 - \operatorname{mult}_{E_j}(B)}{q}.$$

3.2 Serre's property S_2 for affine toric varieties

Let $X = \operatorname{Spec} k[\mathcal{M}]$ be the affine toric variety associated to a monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$. The torus $T = \operatorname{Spec} k[M]$ acts naturally on X. We give a combinatorial criterion for X to satisfy Serre's property S_2. Note that X is irreducible if and only if Δ has a unique maximal element, if and only if $X = \operatorname{Spec} k[S]$, where $S \subseteq M$ is a finitely generated semigroup such that $S - S = M$.

3.2.1 Irreducible case

Let $S \subseteq M$ be a finitely generated semigroup such that $S - S = M$. Let $k[S]$ be the induced semigroup ring, set $X = \operatorname{Spec} k[S]$. It is an equivariant embedding of T. Let $\sigma_S \subseteq M_\mathbb{R}$ be the cone generated by S. For a face $\sigma \prec \sigma_S$, denote $S_\sigma = S \cap \sigma$ and $X_\sigma = \operatorname{Spec} k[S_\sigma]$. Then X is the toric variety associated to the monoidal complex determined by M, the fan Δ consisting of faces of σ_S, and the collection of semigroups S_σ. The invariant closed subvarieties of X are X_σ.

If $S = M$, then $T = X$ is smooth, hence S_2. Else, $X \setminus T = \sum_i E_i$ is the sum of T-invariant codimension one subvarieties. We have $E_i = X_{\tau_i}$, where $(\tau_i)_i$ are the codimension one faces of σ_S. Set
$$S' = \cap_i (S - S \cap \tau_i).$$

Lemma 3.2.1. *$S \subseteq S' \subseteq \bar{S} = M \cap \sigma_S$.*

Proof. We only have to prove the inclusion $S' \subseteq M \cap \sigma_S$. Suppose by contradiction that $m \in S' \setminus \sigma_S$. Then there exists $\varphi \in \sigma_S^\vee$ such that $\sigma_S \cap \varphi^\perp$ is a codimension one face τ_i of σ_S, and $\langle \varphi, m \rangle < 0$. But $m + s_i \in S$ for some $s_i \in S \cap \tau_i$. Therefore $\langle \varphi, m \rangle = \langle \varphi, m + s_i \rangle \geq 0$, a contradiction. □

Theorem 3.2.2. *[29] The S_2-closure of X is* $\operatorname{Spec} k[S'] \to \operatorname{Spec} k[S]$.

Proof. Denote $R = \{f \in k(X); \text{regular in codimension one on } X\}$. If $f \in R$, then $f|_T$ is regular in codimension one on T. Since T is normal, f is regular on T. Therefore $R \subseteq \mathcal{O}(T) = \oplus_{m \in M} k \cdot \chi^m$. Now R is T-invariant. Therefore $R = \oplus_{m \in S_1} k \cdot \chi^m$, for a certain semigroup $S_1 \subseteq M$ which remains to be identified.

Let $m \in S_1$. Let $\tau_i \prec \sigma_S$ be a face of codimension one. Then χ^m is regular at the generic point of E_i. That is $m = s - s'$, for some $s \in S$ and $s' \in S \cap \tau_i$. We deduce that $S_1 \subseteq \cap_i (S - S \cap \tau_i) = S'$. For the converse, let $m \in S'$. Since χ^m is regular on T and at the generic points of $X \setminus T = \sum_i E_i$, it is regular in codimension one on X. Therefore $m \in S_1$. □

In particular, $\operatorname{Spec} k[S]$ is S_2 if and only if $S = \cap_i (S - S \cap \tau_i)$.

Recall [7, Proposition 2.10] that $X = \operatorname{Spec} k[S]$ is seminormal if and only if $S = \sqcup_{\sigma \prec \sigma_S} \Lambda_\sigma \cap \operatorname{relint} \sigma$, where $(\Lambda_\sigma)_{\sigma \prec \sigma_S}$ is a family of sublattices of finite index $\Lambda_\sigma \subseteq M \cap \sigma - M \cap \sigma$, such that $\Lambda_{\sigma_S} = M$ and $\Lambda_{\sigma'} \subseteq \Lambda_\sigma \cap \sigma'$ if $\sigma' \prec \sigma$. A family of sublattices defines S by the above formula, and S determines the family of sublattices $\Lambda_\sigma = S \cap \sigma - S \cap \sigma$.

Theorem 3.2.3. *[13]* $\operatorname{Spec} k[S]$ *is seminormal and* S_2 *if and only if* $\Lambda_\sigma = \cap_{\tau_i \supset \sigma} \Lambda_{\tau_i}$*, for every proper face* $\sigma \precneqq \sigma_S$*.*

Proof. Recall that $(\tau_i)_i$ are the codimension one faces of σ_S. For the proof, we may suppose $\operatorname{Spec} k[S]$ is seminormal.

Suppose $\operatorname{Spec} k[S]$ is not S_2. There exists $m \notin S$ and $s_i \in S_{\tau_i}$ $(1 \le i \le q)$ such that $m + s_i \in S$ for all i. It follows that $m \in \sigma_S$. Let $\sigma \prec \sigma_S$ be the unique face which contains m in its relative interior. Let τ_i be a codimension one face which contains σ. Then $m + s_i \in S_{\tau_i}$. Therefore $m \in S_{\tau_i} - S_{\tau_i} = \Lambda_{\tau_i}$. We obtain $m \in \cap_{\tau_i \supset \sigma} \Lambda_{\tau_i} \setminus \Lambda_\sigma$. Therefore Λ_σ is strictly contained in $\cap_{\tau_i \supset \sigma} \Lambda_{\tau_i}$.

Conversely, suppose $\operatorname{Spec} k[S]$ is S_2. Let $\sigma \precneqq \sigma_S$ be a proper face. We have an inclusion of lattices $\Lambda_\sigma \subseteq \cap_{\tau_i \supset \sigma} \Lambda_{\tau_i}$, both generating $\sigma - \sigma$. The inclusion of lattices is an equality, if it is so after restriction to $\operatorname{relint} \sigma$, by [7, Lemma 2.9].

Let $m \in \cap_{\tau_i \supset \sigma} \Lambda_{\tau_i} \cap \operatorname{relint} \sigma$. If $\tau_i \supseteq \sigma$, then $m \in \Lambda_{\tau_i} \subset S - S_{\tau_i}$. If $\tau_i \not\supseteq \sigma$, there exists $s_i \in S \cap \tau_i$ such that $m + s_i \in \operatorname{int} \sigma_S$. Therefore $m + s_i \in M \cap \operatorname{int} \sigma_S$, which is contained in S by seminormality. We obtain $m \in S'$. The S_2 property implies that $m \in S$. Therefore $m \in \Lambda_\sigma$. We obtain $\Lambda_\sigma = \cap_{\tau_i \supset \sigma} \Lambda_{\tau_i}$. □

So to give $X = \operatorname{Spec} k[S]$ which is seminormal and S_2, is equivalent to give (M, σ_S) (i.e. the normalization), the codimension one faces $(\tau_i)_i$ of σ_S, and finite index sublattices $\Lambda_i \subseteq M \cap \tau_i - M \cap \tau_i$, for each i. Moreover, X is weakly normal if and only if $\operatorname{char}(k)$ does not divide the index of the sublattice $\Lambda_i \subseteq M \cap \tau_i - M \cap \tau_i$ for all i, if and only if $\operatorname{char}(k)$ does not divide the incidence numbers $d_{Y \subset X}$ for every invariant subvariety Y of X (with the terminology of Definition 3.2.5).

The normalization of X is $\bar{X} = \operatorname{Spec} k[\bar{S}] \to X = \operatorname{Spec} k[S]$. If X is seminormal, the conductor subschemes $C \subset X$ and $\bar{C} \subset \bar{X}$ are reduced, described as follows.

Lemma 3.2.4. *Suppose* $X = \operatorname{Spec} k[S]$ *is seminormal. Let* Δ *be the fan consisting of the cones* $\sigma \prec \sigma_S$ *such that* $S_\sigma - S_\sigma \subsetneq M \cap \sigma - M \cap \sigma$*. Then* $C = \cup_{\sigma \in \Delta} X_\sigma$ *and* $\bar{C} = \cup_{\sigma \in \Delta} \bar{X}_\sigma$*.*

Proof. Note that $\bar{S}_\sigma = M \cap \sigma$ for $\sigma \prec \sigma_S$.

If $S_\sigma - S_\sigma \subsetneq M \cap \sigma - M \cap \sigma$, the same property holds for all faces $\tau \prec \sigma$. Therefore Δ is a fan. The conductor ideal is $I = \oplus_{m + \bar{S} \subseteq S} k \cdot \chi^m$. We claim

$$\{m \in S; m + \bar{S} \subseteq S\} = S \setminus \cup_{\sigma \in \Delta} \sigma.$$

For the inclusion $\subseteq$, let $m \in \sigma \prec \sigma_S$ with $m + \bar{S} \subset S$. Then $m + \bar{S}_\sigma \subseteq S_\sigma$. Since $m \in S_\sigma$, we obtain $S_\sigma - S_\sigma = \bar{S}_\sigma - \bar{S}_\sigma$. Therefore $\sigma \notin \Delta$.

For the inclusion $\supseteq$, let $m \in S$ with $m + \bar{S} \not\subseteq S$. There exists $\bar{s} \in \bar{S}$ such that $m + \bar{s} \notin S$. Let $\sigma \prec \sigma_S$ be the unique face with $m + \bar{s} \in \operatorname{relint} \sigma$. Then $m, \bar{s} \in \sigma$. Suppose by contradiction that $\sigma \notin \Delta$. Then $S_\sigma - S_\sigma = \bar{S}_\sigma - \bar{S}_\sigma$, and

$$m + \bar{s} \in \bar{S}_\sigma \cap \operatorname{relint} \sigma = (\bar{S}_\sigma - \bar{S}_\sigma) \cap \operatorname{relint} \sigma = (S_\sigma - S_\sigma) \cap \operatorname{relint} \sigma = S_\sigma \cap \operatorname{relint} \sigma,$$

where we have used that $\bar{X}$ and X are seminormal. Then $m + \bar{s} \in S$, a contradiction. Therefore $\sigma \in \Delta$. □

Definition 3.2.5. Let $X = \operatorname{Spec} k[S]$ and $Y \subset X$ an invariant closed irreducible subvariety. That is $Y = X_\tau$ for some face $\tau \prec \sigma_S$. Let $\pi \colon \bar{X} \to X$ be the normalization, let $\bar{Y} = \pi^{-1}(Y)$. Then $\bar{X} = \operatorname{Spec} k[M \cap \sigma_S]$, $\bar{Y} = (\bar{X})_\tau = \operatorname{Spec} k[M \cap \tau]$ and we obtain a cartesian diagram

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{Y} \arrow[l, hook'] \arrow[d, "\pi'"] \\ X & Y \arrow[l, hook'] \end{tikzcd}$$

The induced morphism $\pi' \colon \bar{Y} \to Y$ is finite, of degree $d_{Y \subset X}$, equal to the index of the sublattice $S_\tau - S_\tau \subseteq M \cap \tau - M \cap \tau$. We call $d_{Y \subset X}$ the *incidence number of* $Y \subset X$, sometimes denoted $d_{\tau \prec \sigma_S}$. Note that $d_{Y \subset X} > 1$ if and only if X is not normal at the generic point of Y.

3.2.2 Reducible case

Consider now the general case of an affine toric variety $X = \operatorname{Spec} k[\mathcal{M}]$. For $\sigma \in \Delta$, denote by X_σ the T-invariant closed irreducible subvariety of X coresponding to σ. The decomposition of X into irreducible components is $X = \cup_F X_F$, where $\{F\}$ are the facets (maximal cones) of Δ.

Lemma 3.2.6. *The sequence* $0 \to \mathcal{O}_X \to \oplus_F \mathcal{O}_{X_F} \to \oplus_{F \neq F'} \mathcal{O}_{X_{F \cap F'}}$ *is exact.*

Proof. Let $f_F \in \mathcal{O}(X_F)$ such that for every $F \neq F'$, f_F and $f_{F'}$ coincide on $X_{F \cap F'}$. We can write $f_F = \sum_m c^F_m \chi^m$. Let $m \in |\mathcal{M}|$. The map $F \ni m \mapsto c^F_m$ is constant. Denote by c_m this common value. Then $f = \sum_m c_m \chi^m \in \mathcal{O}(X)$ and $f|_{X_F} = f_F$ for every facet F. This shows that the sequence is exact in the middle. The map $\mathcal{O}(X) \to \oplus_F \mathcal{O}(X_F)$ is clearly injective. □

The S_2-closure of X is $\operatorname{Spec} R \to X$, where $R = \varinjlim_{\operatorname{codim}(Z \subset X) \geq 2} \mathcal{O}_X(X \setminus Z)$ is the ring of functions which are regular in codimension one points of X. We describe R explicitly. For $\sigma \in \Delta$, recall that $O_\sigma \subset X_\sigma$ is the open dense orbit. We have $\sqcup_F O_F \subset X$, with complement $\Sigma = \cup_{\operatorname{codim}(\sigma \in \Delta) > 0} X_\sigma$, the toric boundary of X.

Let $f \in R$. Then $f_F := f|_{O_F}$ is regular in codimension one. Since T_F is normal, hence S_2, $f_F \in \mathcal{O}(O_F)$. We can uniquely write $f_F = \sum_{m \in S_F - S_F} c^F_m \chi^m$, where the sum has finite support. Denote $\operatorname{Supp}(f_F) = \{m \in S_F - S_F; c^F_m \neq 0\}$.

Let $\sigma \in \Delta$ be a cone of codimension one. Equivalently, σ has codimension one in every facet containing it. Since f is regular at the generic point of X_σ, we obtain:

1) f_F is regular at the generic point of $X_\sigma \hookrightarrow X_F$. That is $\operatorname{Supp} f_F \subset S_F - S_\sigma$.

2) If F and F' are two facets containing σ, the restriction of f_F to $X_\sigma \hookrightarrow X_F$ coincides with the restriction of $f_{F'}$ to $X_\sigma \hookrightarrow X_{F'}$.

So $f \in R$ induces a family $(f_F)_F \in \prod_F \mathcal{O}(O_F)$ satisfying properties 1) and 2). This correspondence is bijective, by Lemma 3.2.6. Thus we may identify R with the collections $(f_F)_F \in \prod_F \mathcal{O}(O_F)$ satisfying properties 1) and 2), for every cone $\sigma \in \Delta$ of codimension one.

Definition 3.2.7. The fan Δ is called 1-*connected* if for every two facets $F \neq F'$ of Δ, there exists a sequence of facets $F_0 = F, F_1, \ldots, F_n = F'$ of Δ, which contain $F \cap F'$, and such that $F_i \cap F_{i+1}$ is a face of codimension one in both F_i and F_{i+1}, for all $0 \leq i < n$.

It is clear that for a 1-connected fan, every facet has the same dimension.

Lemma 3.2.8. *If X is S_2, then Δ is 1-connected.*

Proof. Let $F \neq F'$ be two facets of Δ. Define a graph Γ as follows: the vertices are the facets of Δ which contain $F \cap F'$, and two vertices are joined by an edge if their intersection has codimension one in both of them. Let $\{c\}$ be the connected components of Γ. Denote by X^c the union of the irreducible components of X which belong to c, and $Z = \cup_{c \neq c'} X^c \cap X^{c'}$. By construction, $\operatorname{codim}(Z \subset X) \geq 2$. Let Y be the union of the irreducible components of X which do not contain $X_{F \cap F'}$, set $U = X \setminus Y$.

If X is S_2, then $\mathcal{O}_X(U) \to \mathcal{O}_X(U \setminus Z)$ is an isomorphism. Since U is connected, it follows that $U \setminus Z = \sqcup_c (X^c \setminus Y)$ is connected, that is Γ is connected. Therefore F and F' can be joined by a chain with the desired properties. □

Lemma 3.2.9. *Suppose Δ is 1-connected. Denote by S'_F the S_2-closure of S_F. For $\sigma \in \Delta$, define $\tilde{S}_\sigma = \{m \in \sigma; m \in S'_F \ \forall F \ni m\}$. Then $\tilde{\mathcal{M}} = (M, \Delta, (\tilde{S}_\sigma)_{\sigma \in \Delta})$ is a monoidal complex, and $\operatorname{Spec} k[\tilde{\mathcal{M}}] \to \operatorname{Spec} k[\mathcal{M}]$ is the S_2-closure of X.*

Proof. Since Δ is 1-connected, the irreducible components of X have the same dimension. Therefore R is the ring of collections $(f_F) \in \prod_F \mathcal{O}(O_F)$ satisfying the following properties:

1') f_F is regular in codimension one on X_F. Since $X_F = \operatorname{Spec} k[S_F]$, this means that $\operatorname{Supp} f_F \subset S'_F$.

2') If F and F' are two facets intersecting in a codimension one face, the restriction of f_F to $X_{F \cap F'} \hookrightarrow X_F$ coincides with the restriction of $f_{F'}$ to $X_{F \cap F'} \hookrightarrow X_{F'}$.

Since Δ is 1-connected, 2') is equivalent to

2") If $F \neq F'$ are two facets, the restriction of f_F to $X_{F \cap F'} \hookrightarrow X_F$ coincides with the restriction of $f_{F'}$ to $X_{F \cap F'} \hookrightarrow X_{F'}$.

Let $m \in \cup_F \operatorname{Supp} f_F$. The map $F \ni m \mapsto c_m^F$ is constant, by 2"). And if the constant c_m is non-zero, then m belongs to $\cap_{F \ni m} S'_F$, by 1). If we set $f = \sum_m c_m \chi^m$, we have $f|_{X_F} = f_F$ for all m. We conclude that R identifies with the ring of finite sums $\sum_m c_m \chi^m$, such that $c_m \neq 0$ implies $m \in \cap_{F \ni m} S'_F$.

Denote $\mathcal{S} = \{\cap_{i=1}^n F_i; n \geq 1, F_i \in \Delta \text{ facets}\}$. The facets of Δ belong to $\mathcal{S}$, and if $\sigma, \tau \in \mathcal{S}$, then $\sigma \cap \tau \in \mathcal{S}$. Note that $\mathcal{S}$ may not contain faces of its cones. For $\sigma \in \mathcal{S}$, denote $B\sigma = \cup_{\sigma \supsetneq \tau \in \mathcal{S}} \tau$. We have

$$\cup_{\sigma \in \Delta} \sigma = \cup_F F = \sqcup_{\tau \in \mathcal{S}} \tau \setminus B\tau.$$

If $m \in \cup_F F$, then $\cap_{F \ni m} F$ is the unique element $\tau \in \mathcal{S}$ such that $m \in \tau \setminus B\tau$. If $\tau \in \mathcal{S}$ and $m \in \tau \setminus B\tau$, then $\{F; F \ni m\} = \{F; F \supseteq \tau\}$. Therefore R is the toric face ring of the monoidal complex $\tilde{\mathcal{M}} = (M, \Delta, (\tilde{S}_\sigma)_{\sigma \in \Delta})$, where

$$\tilde{S}_\sigma = \bigsqcup_{\tau \in \mathcal{S}} \sigma \cap (\tau \setminus B\tau) \cap \bigcap_{F \supseteq \tau} S'_F = \{m \in \sigma; m \in S'_F \ \forall F \ni m\}.$$

$\square$

Putting Lemmas 3.2.8 and 3.2.9 together, we obtain the S_2-criterion for $X = \operatorname{Spec} k[\mathcal{M}]$, which generalizes Terai's S_2-criterion for Stanley-Reisner rings associated to simplicial complexes [64].

Theorem 3.2.10. *X is S_2 if and only if the following properties hold:*

1) Δ is 1-connected, and

2) $S_\sigma = \{m \in \sigma; m \in S'_F \ \forall F \ni m\}$ for every $\sigma \in \Delta$, where S'_F is the S_2-closure of the semigroup S_F.

Corollary 3.2.11. *Suppose each irreducible component of X is S_2. Then X is S_2 if and only if Δ is 1-connected.*

Corollary 3.2.12. *Suppose X is seminormal, with lattice collection $\Lambda_\sigma = S_\sigma - S_\sigma$. Then X is S_2 if and only if the following properties hold:*

1) Δ is 1-connected.

2) $\Lambda_\sigma = \cap_{\sigma \subset \tau, \operatorname{codim}(\tau \in \Delta) = 1} \Lambda_\tau$ for every $\sigma \in \Delta$ of positive codimension.

So to give $X = \operatorname{Spec} k[\mathcal{M}]$ which is seminormal and S_2, it is equivalent to give the lattice M, a 1-connected fan Δ in M, sublattices of finite index $\Lambda_F \subseteq M \cap F - M \cap F$ for each facet F of Δ, and sublattices of finite index $\Lambda_\tau \subseteq M \cap \tau - M \cap \tau$ for each cone τ of Δ of codimension one (subject to the condition $\Lambda_\tau \subseteq \Lambda_F \cap \tau - \Lambda_F \cap \tau$ if $\tau \prec F$). Moreover, X/k is weakly normal if and only if $\operatorname{char}(k)$ does not divide the incidence numbers $d_{X_\tau \subset X_F}$ ($\tau \prec F$).

Let $\pi \colon \bar{X} \to X$ be the normalization. Then $\bar{X} = \sqcup_F \bar{X}_F$, where the direct sum is over all facets of Δ, $\bar{X}_F = \operatorname{Spec} k[\bar{S_F}]$ and $\bar{S_F} = (S_F - S_F) \cap F$. We compute the conductor ideal I of π. The normalization induces an inclusion of k-algebras

$$k[\cup_{\sigma \in \Delta} S_\sigma] \to \prod_F k[\bar{S_F}], f \mapsto (f|_{\bar{X_F}})_F.$$

The ideal I consists of $f \in \mathcal{O}(X)$ such that $f \cdot \mathcal{O}(\bar{X}) \subseteq \mathcal{O}(X)$. It is T-invariant, hence of the form

$$I = \oplus_{m \in \cup_\sigma S_\sigma \setminus A} k \cdot \chi^m$$

for a certain set A which it remains to identify. Now $\chi^m \in I$ if and only if $\chi^m \cdot (f_F, 0, \ldots, 0) \in \mathcal{O}(X)$ for every facet F and every $f_F \in \mathcal{O}(\overline{X_F})$; if and only if, for every facet F,

$$\chi^m \cdot (\chi^a, 0, \ldots, 0) = (\chi^m|_F \cdot \chi^a, 0, \ldots, 0) \in \mathcal{O}(X)$$

for every $a \in \bar{S_F}$; if and only if $m + a \in S_F \setminus \cup_{F' \neq F} F \cap F'$, for all $F \ni m$ and $a \in \bar{S_F}$. Therefore

$$\cup_\sigma S_\sigma \setminus A = \{m; m + \bar{S_F} \subset S_F \setminus BF \ \forall F \ni m\}.$$

If X is seminormal, the conductor subschemes $C \subset X$ and $\bar{C} \subset \bar{X}$ are reduced, described as follows.

Lemma 3.2.13. *Suppose X is seminormal. Let Δ' be the subfan of cones $\sigma \in \Delta$ which either are contained in at least two facets of Δ, or are contained in a unique facet F of Δ and $S_\sigma - S_\sigma \subsetneq (\bar{S_F})_\sigma - (\bar{S_F})_\sigma$. Then $C = \cup_{\sigma \in \Delta'} X_\sigma$ and $\bar{C} = \sqcup_F \cup_{\sigma \in \Delta', \sigma \prec F} (\bar{X}_F)_\sigma$.*

Proof. It suffices to show that the ideal I is radical, hence equal to the ideal of union $\cup_{\sigma \in \Delta'} X_\sigma \subset X$. Indeed, let $m + \bar{S_F} \subset S_F \setminus BF$ for all $F \ni m$. Assuming $m \in S_\sigma$ for some $\sigma \in \Delta'$, we derive a contradiction. We have two cases: suppose σ is contained in at least two facets $F \neq F'$. Then $m \in BF$, a contradiction. Suppose σ is contained in a unique facet F. Then $m + (\bar{S_F})_\sigma \subset (S_F)_\sigma = S_\sigma$. Then $S_\sigma - S_\sigma = (\bar{S_F})_\sigma - (\bar{S_F})_\sigma$, that is $\sigma \notin \Delta'$. Contradiction!

Conversely, let $m \in \cup_{\sigma \in \Delta} S_\sigma \setminus \cup_{\sigma \in \Delta'} \sigma$. Let $m \in F$ be a facet. We must show $m + \bar{S_F} \subset S_F \setminus BF$. Indeed, let $\bar{s} \in \bar{S_F}$. Then $m + \bar{s} \in \operatorname{relint} \sigma$ for a unique face $\sigma \prec F$. It follows that $m, \bar{s} \in \sigma$. If $m + \bar{s} \in BF$, then $m + \bar{s} \in F \cap F'$ for some $F' \neq F$. Then $m, \bar{s} \in F \cap F'$. Then $m \in F \cap F' \in \Delta'$, a contradiction. Therefore $m + \bar{s} \notin BF$. On the other hand, $\sigma \notin \Delta'$, that is $S_\sigma - S_\sigma = (\bar{S_F})_\sigma - (\bar{S_F})_\sigma$. As in the irreducible case, the seminormality of X_F and its normalization implies $m + \bar{s} \in S_\sigma \cap \operatorname{relint} \sigma$. Therefore $m + \bar{s} \in S_F$. □

3.2.3 The core

Let $X = \operatorname{Spec} k[\mathcal{M}]$ be seminormal and S_2. Define the *core of* X to be X if X is normal, and the intersection of the irreducible components of the non-normal locus C, if X is not normal.

Proposition 3.2.14. *The core of X is normal.*

Proof. Let $\{F\}$ and $\{\tau_i\}$ be the facets and codimension one faces of Δ, respectively. The core of X is the invariant closed subvariety $X_{\sigma(\Delta)}$, where

$$\sigma(\Delta) = \bigcap_F F \cap \bigcap_{X_{\tau_i} \subset C} \tau_i.$$

Indeed, if X is normal, Δ has a unique facet F and $C=\emptyset$, hence $\sigma(\Delta)=F$. If X is not normal, each facet contains some irreducible component of C, hence $\sigma(\Delta)=\cap_{X_{\tau_i}\subset C}\tau_i$.

We claim that $S_{\sigma(\Delta)}=\cap_F(S_F-S_F)\cap\bigcap_{X_{\tau_i}\subset C}(S_{\tau_i}-S_{\tau_i})\cap\sigma(\Delta)$. Indeed, the inclusion $\subseteq$ is clear. For the converse, let m be an element on the right hand side. Then $m\in\operatorname{relint}\tau$ for same $\tau\prec\sigma(\Delta)$. Let τ_i be a codimension one face which contains τ. If $X_{\tau_i}\subset C$, then $m\in S_{\tau_i}-S_{\tau_i}$ by assumption. If $X_{\tau_i}\not\subset C$, there exists a unique facet F which contains τ_i, and $S_{\tau_i}-S_{\tau_i}=(S_F-S_F)\cap\tau_i-(S_F-S_F)\cap\tau_i$. By assumption, m belongs to the right hand side. We conclude that $m\in\cap_{\tau_i\succ\tau}S_{\tau_i}-S_{\tau_i}$. By Corollary 3.2.12, this means $m\in S_\tau-S_\tau$. Then m belongs to $(S_\tau-S_\tau)\cap\operatorname{relint}\tau$, which is contained in S_τ since X is seminormal. Therefore $m\in S_\tau$, hence $m\in S_{\sigma(\Delta)}$.

From the claim, $S_{\sigma(\Delta)}$ is the trace on $\sigma(\Delta)$ of some lattice. Therefore

$$S_{\sigma(\Delta)}-S_{\sigma(\Delta)}=\cap_F(S_F-S_F)\cap\bigcap_{X_{\tau_i}\subset C}(S_{\tau_i}-S_{\tau_i}).$$

and $S_{\sigma(\Delta)}=(S_{\sigma(\Delta)}-S_{\sigma(\Delta)})\cap\sigma(\Delta)$, that is $X_{\sigma(\Delta)}$ is normal. □

Corollary 3.2.15. *Suppose the non-normal locus C of X is not empty. Then either C is irreducible and normal, or $C=\cup_i C_i$ is reducible and its non-normal locus is $\cup_{i\neq j}C_i\cap C_j$.*

Proof. Suppose C is irreducible. Then C is the core of X, hence normal by Proposition 3.2.14. Suppose C is reducible, with irreducible components C_i. If $C_i\neq C_j$, the intersection $C_i\cap C_j$ is contained in the non-normal locus of C. Therefore the non-normal locus of C contains $\cup_{i\neq j}C_i\cap C_j$. On the other hand, $C_i\setminus\cup_{j\neq i}C_j$ is normal (after localization, we obtain $C=C_i$ irreducible, hence normal by the above argument). Therefore the non-normal locus of C is $\cup_{i\neq j}C_i\cap C_j$. □

3.3 Weakly normal log pairs

Let X/k be an algebraic variety, weakly normal and S_2. Let $\pi\colon\bar X\to X$ be the normalization. The ideal sheaf $\{f\in\mathcal{O}_X; f\cdot\pi_*\mathcal{O}_{\bar X}\subseteq\mathcal{O}_X\}$ is also an ideal sheaf on $\bar X$, and cuts out the *conductor subschemes* $C\subset X$ and $\bar C\subset\bar X$. We obtain a cartesian diagram

$$\begin{tikzcd} \bar X \arrow[d, "\pi"'] & \bar C \arrow[l, hook'] \arrow[d, "\pi"] \\ X & C \arrow[l, hook'] \end{tikzcd}$$

Each irreducible component of X has the same dimension, equal to $d=\dim X$. Both $C\subset X$ and $\bar C\subset\bar X$ are reduced subschemes, of pure codimension one, and C *is the non-normal locus of* X. The morphism $\pi\colon\bar C\to C$ is finite, mapping irreducible components onto irreducible components. Denote by $Q(X)$ the k-algebra consisting of rational functions which are regular on an open dense subset of X. We have an isomorphism $\pi^*\colon Q(X)\xrightarrow{\sim}Q(\bar X)$ and a monomorphism $\pi^*\colon Q(C)\to Q(\bar C)$. Let B be the closure in X of a $\mathbb{Q}$-Cartier divisor on the smooth locus of X, and $\bar B$ the closure in $\bar X$.

Definition 3.3.1. For $n \in \mathbb{Z}$, define a coherent $\mathcal{O}_X$-module $\omega^{[n]}_{(X/k,B)}$ as follows: for an open subset $U \subseteq X$, let $\Gamma(U, \omega^{[n]}_{(X/k,B)})$ be the set of rational n-differential forms $\omega \in (\wedge^d \Omega^1_{Q(X)/k})^{\otimes n}$ satisfying the following properties

a) $(\pi^*\omega) + n(\bar{C} + \bar{B}) \geq 0$ on $\pi^{-1}(U)$.

b) If P is an irreducible component of $C \cap U$, there exists a rational n-differential form $\eta \in (\wedge^{d-1}\Omega^1_{Q(P)/k})^{\otimes n}$ such that $\operatorname{Res}_Q \pi^*\omega = \pi^*\eta$ for every irreducible component Q of $\bar{C}$ lying over P.

We have natural multiplication maps $\omega^{[m]}_{(X/k,B)} \otimes_{\mathcal{O}_X} \omega^{[n]}_{(X/k,B)} \to \omega^{[m+n]}_{(X/k,B)}$ $(m, n \in \mathbb{Z})$. By seminormality, $\omega^{[0]}_{(X/k,B)} = \mathcal{O}_X$.

Lemma 3.3.2. *Suppose rB has integer coefficients in a neighborhood of a codimension one point $P \in X$. Then in a neighborhood of P, $\omega^{[r]}_{(X/k,B)}$ is invertible and $(\omega^{[r]}_{(X/k,B)})^{\otimes n} \xrightarrow{\sim} \omega^{[rn]}_{(X/k,B)}$ for all $n \in \mathbb{Z}$.*

Proof. Suppose X/k is smooth at P. Let t be a local parameter at P and ω_0 a local generator of $(\wedge^d \Omega^1_{X/k})_P$, and $b = \operatorname{mult}_P(B)$. If $n \in \mathbb{Z}$, then $t^{-\lfloor nb \rfloor}\omega^{\otimes n}$ is a local generator of $\omega^{[n]}_{(X/k,B)}$. The claim follows.

Suppose X is singular at P. Let $(Q_j)_j$ be the finitely many prime divisors of $\bar{X}$ lying over P. We may localize at P and suppose $C = P$, $B = 0$, and $\bar{C} = \sum_j Q_j$. For every j, we have finite surjective maps $\pi|_{Q_j} \colon Q_j \to P$. By weak approximation [68, Chapter 10, Theorem 18], there exists an invertible rational function $t_1 \in Q(\bar{X})$ which induces a local parameter at Q_j, for every j. Let $f_2, \ldots, f_d$ be a separating transcendence basis of $k(C)/k$. For every $2 \leq i \leq d$, there exists $t_i \in Q(\bar{X})$, regular at each Q_j, such that $t_i|_{Q_j} = (\pi|_{Q_j})^*(f_i)$ for every j. Set

$$\omega = \frac{dt_1}{t_1} \wedge dt_2 \wedge \cdots \wedge dt_d \in \wedge^d \Omega^1_{Q(\bar{X})/k}.$$

Since $t_1\omega$ is regular, we have $(\omega) + \bar{C} \geq 0$. On the other hand,

$$\operatorname{Res}_{Q_j} \omega = (\pi|_{Q_j})^*(df_2 \wedge \cdots \wedge df_d).$$

The right hand side is non-zero, hence $(\omega) + \bar{C} = 0$. Property b) is also satisfied, so ω belongs to $\omega^{[1]}_{(X/k,B)}$. We claim that $\omega^{\otimes n}$ is a local generator of $\omega^{[n]}_{(X/k,B)}$ at P, for all $n \in \mathbb{Z}$. Indeed, let ω' be a local section of $\omega^{[n]}_{(X/k,B)}$ at P. There exists a regular function f on $\bar{X}$ such that $\pi^*\omega' = f \cdot \omega^{\otimes n}$. By assumption, there exists a rational n-differential $\eta \in (\wedge^{d-1}\Omega^1_{Q(P)/k})^{\otimes n}$ such that $\operatorname{Res}_{Q_j} \pi^*\omega' = (\pi|_{Q_j})^*(\eta)$ for every irreducible component Q_j. Let $\eta = h \cdot (df_2 \wedge \cdots \wedge df_d)^{\otimes n}$. We obtain $f|_{Q_j} = (\pi|_{Q_j})^*(h)$ for all Q_j. By seminormality, this means that $f \in \mathcal{O}_{X,P}$. $\square$

Corollary 3.3.3. *Let $r \geq 1$ such that rB has integer coefficients. There exists an open subset $U \subseteq X$ such that $\operatorname{codim}(X \setminus U, X) \geq 2$, $\omega^{[r]}_{(X/k,B)}|_U$ is invertible and $(\omega^{[r]}_{(X/k,B)}|_U)^{\otimes n} \xrightarrow{\sim} \omega^{[rn]}_{(X/k,B)}|_U$ for all $n \in \mathbb{Z}$.*

Lemma 3.3.4. *Let $U \subseteq X$ be an open subset and $\omega \in (\wedge^d \Omega^1_{k(X)/k})^{\otimes n} \setminus 0$. Then $1 \mapsto \omega$ induces an isomorphism $\mathcal{O}_U \xrightarrow{\sim} \omega^{[n]}_{(X/k,B)}|_U$ if and only if $(\pi^*\omega) + \lfloor n(\bar{C} + \bar{B}) \rfloor = 0$ on $\bar{U} = \pi^{-1}(U)$ and $\operatorname{Res}_{\bar{C} \cap \bar{U}}(\pi^*\omega) \in (\wedge^{d-1}\Omega^1_{Q(\bar{C} \cap \bar{U})/k})^{\otimes n}$ belongs to the image of $\pi^* \colon (\wedge^{d-1}\Omega^1_{Q(C \cap U)/k})^{\otimes n} \to (\wedge^{d-1}\Omega^1_{Q(\bar{C} \cap \bar{U})/k})^{\otimes n}$.*

Proof. The homomorphism is well defined if and only if $(\pi^*\omega) + \lfloor n(\bar{C} + \bar{B}) \rfloor \geq 0$ on $\bar{U} = \pi^{-1}(U)$ and $\operatorname{Res}_{\bar{C} \cap \bar{U}}(\pi^*\omega) = \pi^*\eta$ for $\eta \in (\wedge^{d-1}\Omega^1_{Q(C \cap U)/k})^{\otimes n}$. Suppose the homomorphism is an isomorphism. It follows that $(\pi^*\omega) + \lfloor n(\bar{C} + \bar{B}) \rfloor = 0$ on $\bar{U}$, since in the proof of Lemma 3.3.2 we constructed local generators with this property near each codimension one point of X. It follows that η is non-zero on each irreducible component of $C \cap U$.

Conversely, let $V \subseteq U$ be an open subset and $\omega' \in \Gamma(V, \omega^{[n]}_{(X/k,B)})$. Then $\omega' = f\omega$, with $f \in \Gamma(\pi^{-1}(V), \mathcal{O}_{\bar{X}})$. By definition, $\operatorname{Res}_{\bar{C}}(\omega') = \pi^*\eta'$. Since η is non-zero on each irreducible component of C, $h = \eta'/\eta$ is a well defined rational function on $C \cap V$. Comparing residues, we obtain that for every irreducible component P of $V \cap C$, for every prime divisor Q lying over P, we have $f|_Q = \pi^*h$. Since X is seminormal and S_2, this means that $f \in \Gamma(V, \mathcal{O}_X)$. Therefore ω generates $\omega^{[n]}_{(X/k,B)}$ on U. □

Corollary 3.3.5. *Suppose rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is an invertible $\mathcal{O}_X$-module. Then:*

a) $\omega^{[r]}_{(X/k,B)} \otimes_{\mathcal{O}_X} \omega^{[n]}_{(X/k,B)} \to \omega^{[r+n]}_{(X/k,B)}$ is an isomorphism, for every $n \in \mathbb{Z}$. In particular, the graded $\mathcal{O}_X$-algebra $\oplus_{n \in \mathbb{N}} \omega^{[n]}_{(X/k,B)}$ is finitely generated, and $(\omega^{[r]}_{(X/k,B)})^{\otimes n} \xrightarrow{\sim} \omega^{[rn]}_{(X/k,B)}$ for every $n \in \mathbb{Z}$.

b) $\pi^*\omega^{[r]}_{(X/k,B)} = \omega^{[r]}_{(\bar{X}/k, \bar{C}+\bar{B})}$.

Proof. a) Similar to normal case, using moreover the fact that residues commute with multiplication of pluri-differential forms.

b) It follows from Lemma 3.3.4. □

We may restate property b) as saying that the normalization $(\bar{X}/k, \bar{C} + \bar{B}) \to (X/k, B)$ is log crepant. Note that $\bar{X}$ is normal, but possibly disconnected.

Definition 3.3.6. A *weakly normal log pair* $(X/k, B)$ consists of an algebraic variety X/k, weakly normal and S_2, the (formal) closure B of a $\mathbb{Q}$-Weil divisor on the smooth locus of X/k, subject to the following property: there exists an integer $r \geq 1$ such that rB has integer coefficients and the $\mathcal{O}_X$-module $\omega^{[r]}_{(X/k,B)}$ is invertible.

If B is effective, we call $(X/k, B)$ a *weakly normal log variety.*

If X is normal, these notions coincide with *log pairs* and *log varieties.*

Let D be a $\mathbb{Q}$-Cartier divisor on X supported by primes at which X/k is smooth. If (X, B) is a weakly normal log pair, so is $(X, B + D)$.

3.3.1 Weakly log canonical singularities, lc centers

Suppose $\mathrm{char}(k) = 0$, or log resolutions exist (e.g. in the toric case). Note that any desingularization of X factors through the normalization of X. A *log resolution* $\mu\colon X' \to (X, B)$ is a composition $\mu = \pi \circ \bar{\mu}$, where $\bar{\mu}\colon X' \to (\bar{X}/k, \bar{C} + \bar{B})$ is a log resolution.

We say that $(X/k, B)$ has *weakly log canonical (wlc) singularities* if $(\bar{X}/k, \bar{C} + \bar{B})$ has log canonical singularities. The image $(X/k, B)_{-\infty} = \pi((\bar{X}/k, \bar{C} + \bar{B})_{-\infty})$ is called the *non-wlc locus* of $(X/k, B)$. It is the complement of the largest open subset of X where $(X/k, B)$ has weakly log canonical singularities. An *lc center* of $(X/k, B)$ is defined as the π-image of an lc center of $(\bar{X}/k, \bar{C} + \bar{B})$, which is not contained in $(X/k, B)_{-\infty}$. For example, the irreducible components of X are lc centers. From the normal case, it follows that $(X/k, B)$ has only finitely many lc centers.

Remark 3.3.7. If $(\bar{X}/k, \bar{C} + \bar{B})_{-\infty} = \pi^{-1}((X/k, B)_{-\infty})$, then π maps lc centers onto lc centers.

3.3.2 Residues in codimension one lc centers, different

Let $(X/k, B)$ be a weakly normal log pair. Suppose X is not normal. Let C be the non-normal locus of X, and $j\colon C^n \to C$ the normalization. We obtain a commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{C} \arrow[l, hook'] \arrow[d, "\pi"'] & \bar{C}^n \arrow[l, "i"'] \arrow[d, "g"] \\ X & C \arrow[l, hook'] & C^n \arrow[l, "j"'] \end{tikzcd}$$

Pick $l \in \mathbb{Z}$ such that lB has integer coefficients and $\omega^{[l]}_{(X/k,B)}$ is invertible. We will naturally define a structure of log pair $(C^n/k, B_{C^n})$ and isomorphisms

$$\mathrm{Res}^{[l]}\colon \omega^{[l]}_{(X/k,B)}|_{C^n} \xrightarrow{\sim} \omega^{[l]}_{(C^n/k,B_{C^n})}.$$

Indeed, suppose moreover that $\mathcal{O}_X \xrightarrow{\sim} \omega^{[l]}_{(X/k,B)}$. Let ω be the corresponding global generator. We have $(\pi^*\omega) + l(\bar{C} + \bar{B}) = 0$, and $\mathrm{Res}^{[l]}_{\bar{C}^n} \pi^*\omega = g^*\eta$ for some $\eta \in (\wedge^{d-1}\Omega^1_{Q(C)/k})^{\otimes l}$. It follows that η is non-zero on each component of C.

Note that $\eta = \eta(\omega)$ is uniquely determined by ω. If ω' is another global generator, it follows that $\omega' = f\omega$ for a global unit $f \in \Gamma(X, \mathcal{O}_X^\times)$. Therefore $\eta(\omega') = (f|_C) \cdot \eta(\omega)$ and $f|_C$ is a global unit on C. Therefore the $\mathbb{Q}$-Weil divisor on C^n

$$B_{C^n} = -\frac{1}{l}(\eta)$$

does not depend on the choice of a generator ω. It follows that the definition of B_{C^n} makes sense globally if $\omega^{[l]}_{(X/k,B)}$ is just locally free, since we can patch local trivializations. The definition does not depend on the choice of l either.

Denote by $i' \colon \bar{C}^n \to \bar{X}$ and $j' \colon C^n \to X$ the induced morphisms. Let $B_{\bar{C}^n}$ be the different of $(\bar{X}, \bar{C} + \bar{B})$ on (each connected component of) $\bar{C}^n$. We have isomorphisms

$$\pi^*\omega^{[l]}_{(X/k,B)} \tilde{\to} \omega^{[l]}_{(\bar{X}/k,\bar{C}+\bar{B})}, \operatorname{Res}^{[l]} \colon i'^*\omega^{[l]}_{(\bar{X}/k,\bar{C}+\bar{B})} \tilde{\to} \omega^{[l]}_{(\bar{C}^n/k,B_{\bar{C}^n})}, g^*\omega^{[l]}_{(C^n/k,B_{C^n})} \tilde{\to} \omega^{[l]}_{(\bar{C}^n/k,B_{\bar{C}^n})}.$$

In particular, we obtain an isomorphism $j'^*\omega^{[l]}_{(X/k,B)} \tilde{\to} \omega^{[l]}_{(C^n/k,B_{C^n})}$. We may say that in the following commutative diagram, all maps are log crepant:

$$\begin{tikzcd} (\bar{X}, \bar{C} + \bar{B}) \arrow[d, "\pi"'] & (\bar{C}^n, B_{\bar{C}^n}) \arrow[l, "i'"'] \arrow[d, "g"] \\ (X, B) & (C^n, B_{C^n}) \arrow[l, "j'"'] \end{tikzcd}$$

Lemma 3.3.8 (Inversion of adjunction)**.** *Suppose* $\operatorname{char}(k) = 0$ *and* $B \geq 0$*. Then* (X, B) *has wlc singularities near* C *if and only if* (C^n, B_{C^n}) *has lc singularities.*

Proof. We have $\bar{C} = \pi^{-1}(C)$. Therefore (X, B) has wlc singularities near C if and only if $(\bar{X}, \bar{C} + \bar{B})$ has lc singularities near $\bar{C}$. By [36], this holds if and only if $(\bar{C}^n, B_{\bar{C}^n})$ has lc singularities. Since g is a finite log crepant morphism, the latter holds if and only if (C^n, B_{C^n}) has lc singularities. □

If B is effective, then B_{C^n} is effective.

Let E be an lc center of $(X/k, B)$ of codimension one. Let $E^n \to E$ be the normalization. Then there exists a log pair structure (E^n, B_{E^n}) on the normalization of E, together with residue isomorphisms $\operatorname{Res}^{[r]}_E \colon \omega^{[r]}_{(X/k,B)}|_{E^n} \tilde{\to} \omega^{[r]}_{(E^n,B_{E^n})}$, for every $r \in \mathbb{Z}$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Indeed, if X is normal at E, we have the usual codimension one residue. Else, E is an irreducible component of C and E^n is an irreducible component of C^n, and the residue isomorphism and different was constructed above.

3.3.3 Semi-log canonical singularities

Suppose $\operatorname{char}(k) = 0$. We show that semi-log canonical pairs are exactly the weakly normal log varieties which have wlc singularities and are Gorenstein in codimension one. Recall [44, Definition-Lemma 5.10] that a *semi-log canonical pair* $(X/k, B)$ consists of an algebraic variety X/k which is S_2 and has at most nodal singularities in codimension one, and an effective $\mathbb{Q}$-Weil divisor B on X, supported by nonsingular codimension one points of X, such that the following properties hold:

1) There exists $r \geq 1$ such that rB has integer coefficients and the $\mathcal{O}_X$-module $\omega^{[r]}_X(rB)$ is invertible. This sheaf is constructed as follows: there exists an open subset $w \colon U \subseteq X$ such that $\operatorname{codim}(X \setminus U \subset X) \geq 2$, U has Gorenstein singularities and $rB|_U$ is Cartier. Let ω_U be a dualizing sheaf on U, which is invertible. Then $\omega^{[r]}_X(rB) = w_*(\omega_U^{\otimes r} \otimes \mathcal{O}_U(rB|_U))$.

If we consider the normalization of X and the conductor subschemes

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{C} \arrow[l, hook'] \arrow[d, "\pi"] \\ X & C \arrow[l, hook'] \end{tikzcd}$$

we obtain $\pi^*\omega_X^{[r]}(rB)\tilde{\to}\omega_{\bar{X}}^{[r]}(r\bar{C}+r\bar{B})$, where $\bar{B}=\pi^*B$ is the pullback as a $\mathbb{Q}$-Weil divisor.

2) $(\bar{X},\bar{C}+\bar{B})$ is a log variety (possibly disconnected) with at most log canonical singularities.

On the normal variety $\bar{X}$, we have $\omega_{\bar{X}}^{[r]}(r\bar{C}+r\bar{B})=\omega^{[r]}_{(\bar{X}/k,\bar{C}+\bar{B})}$. The normalizations of $\bar{C}$ and C induce a commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{C} \arrow[l, hook'] \arrow[d] & \bar{C}^n \arrow[l] \arrow[d, "g"] \\ X & C \arrow[l, hook'] & C^n \arrow[l] \end{tikzcd}$$

The assumption that the non-normal codimension one singularities of X are nodal means that g is $2\colon 1$. Equivalently, g is the quotient of $\bar{C}^n$ by an involution $\tau\colon \bar{C}^n\to\bar{C}^n$. If we further assume $2\mid r$, we obtain by [44, Proposition 5.8] that $\omega_X^{[r]}(rB)$ consists of the section ω of $\omega_{\bar{X}}^{[r]}(r\bar{C}+r\bar{B})$ whose residue ω' on $\bar{C}^n$ is τ-invariant, which is equivalent to ω' being pulled back from C^n. We obtain

$$\omega_X^{[r]}(rB)=\omega^{[r]}_{(X/k,B)}\ (2\mid r).$$

Since nodal singularities are weakly normal and Gorenstein, we conclude that $(X/k,B)$ is a weakly normal log variety with wlc singularities, which is Gorenstein in codimension one. Moreover, $\omega_X^{[r]}(rB)=\omega^{[r]}_{(X/k,B)}$ if $2\mid r$.

Conversely, let $(X/k,B)$ be a weakly normal log variety with wlc singularities, which is Gorenstein in codimension one. Among weakly normal points of codimension one, only smooth and nodal ones are Gorenstein. It follows that $(X/k,B)$ is a semi-log canonical pair, and $\omega_X^{[n]}(nB)=\omega^{[n]}_{(X/k,B)}$ for every $n\in 2\mathbb{Z}$.

Note that for a weakly normal log variety with wlc singularities $(X/k,B)$, the following are equivalent:

- $(X/k,B)$ is a semi-log canonical pair.
- X is Gorenstein in codimension one.
- If X is not normal, the induced morphism $g\colon \bar{C}^n\to C^n$ is $2:1$.

3.4 Toric weakly normal log pairs

3.4.1 Irreducible case

Let $X = \operatorname{Spec} k[S]$ be weakly normal and S_2. It is an equivariant embedding of the torus $T = \operatorname{Spec} k[M]$, where $M = S - S$ (see [7, Section 2]). Let $\pi\colon \bar{X} \to X$ be the normalization, with induced conductor subschemes $\pi\colon \bar{C} \to C$. Let $\{\tau_i\}_i$ be the codimension one faces of σ_S. Then $E_i = \operatorname{Spec} k[S_{\tau_i}]$ are the invariant codimension one subvarieties of X, and $\bar{E}_i = \operatorname{Spec} k[M \cap \tau_i]$ are the invariant codimension one subvarieties of $\bar{X}$. Each $\bar{E}_i$ is normal, and the following diagram is cartesian:

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{E}_i \arrow[l, hook'] \arrow[d, "\pi_i"] \\ X & E_i \arrow[l, hook'] \end{tikzcd}$$

Each morphism $\pi_i\colon \bar{E}_i \to E_i$ is finite surjective of degree d_i, the incidence number of $E_i \subset X$.

Let $X_{\sigma(\Delta)}$ be the core of X. We have $\sigma = \sigma_S$ if X is normal, and $\sigma(\Delta) = \cap_{d_i > 1} \tau_i$ otherwise. Denote $\Sigma_{\bar{X}} = \bar{X} \setminus T = \sum_i \bar{E}_i$.

Let $B = \sum_i b_i E_i$ be a $\mathbb{Q}$-Weil divisor on X supported by prime divisors in which X/k is smooth. Note that X/k is smooth at E_i if and only if $E_i \not\subset C$, if and only if $d_i = 1$.

Lemma 3.4.1. *Let $n \in \mathbb{Z}$. The following properties are equivalent:*

a) $\omega^{[n]}_{(X/k,B)}$ is invertible at some point x, which belongs to the closed orbit of X.

b) $\mathcal{O}_X \simeq \omega^{[n]}_{(X/k,B)}$.

c) There exists $m \in S_{\sigma(\Delta)} - S_{\sigma(\Delta)}$ such that $(\chi^m) + \lfloor n(-\Sigma_{\bar{X}} + \bar{C} + \bar{B}) \rfloor = 0$ on $\bar{X}$.

Proof. $a) \implies c)$ The torus T acts on $\omega^{[n]}_{(X/k,B)}$, hence on $\Gamma(X, \omega^{[n]}_{(X/k,B)})$. By the complete reducibility theorem, the space of global sections decomposes into one-dimensional invariant subspaces. Therefore the space of global sections is generated by semi-invariant pluri-differential forms. Since X is affine, $\omega^{[n]}_{(X/k,B)}$ is generated by its global sections. Suppose $\omega^{[n]}_{(X/k,B)}$ is invertible at x. Then there exists a semi-invariant global section $\omega \in \Gamma(X, \omega^{[n]}_{(X/k,B)})$ which induces a local trivialization near x.

Let $\bar{x}$ be a point of $\bar{X}$ lying over x. Then $\pi^*\omega$ is a local trivialization for $\omega^{[n]}_{(\bar{X}/k,\bar{C}+\bar{B})}$ near the point $\bar{x}$, which belongs to the closed orbit of $\bar{X}$. By Lemma 3.1.4, there exists $m \in M$ such that $(\chi^m) + \lfloor n(-\Sigma_{\bar{X}} + \bar{C} + \bar{B}) \rfloor = 0$ on $\bar{X}$. Then $\chi^m \omega_M^{\otimes n}$ becomes a nowhere zero global section of $\omega^{[n]}_{(\bar{X}/k,\bar{C}+\bar{B})}$, where ω_M is the volume form on T induced by an orientation of M.

Now $\pi^*\omega = f \cdot \chi^m \omega_{\mathcal{B}}^{\otimes n}$, for some $f \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}})$ which is a unit at $\bar{x}$. Since ω is semi-invariant, so is f. Therefore $f = c\chi^u$ for some $c \in k^\times$ and $u \in M$. Since f is a unit at $\bar{x}$, it is a global unit, that is $u \in \bar{S} \cap (-\bar{S})$. Replacing ω by ω/c and m by $m+u$, we may suppose

$$\pi^*\omega = \chi^m \omega_M^{\otimes n}.$$

Let $E_i \subseteq C$ be an irreducible component. The identity $(\chi^m) + \lfloor n(-\Sigma_{\bar{X}} + \bar{C} + \bar{B}) \rfloor = 0$ at $\bar{E}_i$ is equivalent to $m \in M \cap \tau_i - M \cap \tau_i$. We compute $\chi^m|_{\bar{E}_i} = \chi^m$ and

$$\operatorname{Res}^{[n]}_{\bar{E}_i} \pi^*\omega = \chi^m \cdot (\operatorname{Res}_{\bar{E}_i} \omega_M)^{\otimes n}.$$

Let ω_i be a volume form on the torus inside E_i induced by an orientation of $S_{\tau_i} - S_{\tau_i}$, let $\bar{\omega}_i$ be a volume form on the torus inside $\bar{E}_i$ induced by an orientation of $M \cap \tau_i - M \cap \tau_i$. Then $\pi^*\omega_i = (\pm d_i) \cdot \bar{\omega}_i$ and $\operatorname{Res}_{\bar{E}_i} \omega_M = (\pm 1) \cdot \bar{\omega}_i$. Since X/k is weakly normal, $\operatorname{char}(k) \nmid d_i$. Thus $\operatorname{Res}_{\bar{E}_i} \omega_M = \pi_i^*((\epsilon_i d_i)^{-1} \omega_i)$ for some $\epsilon_i = \pm 1$. Therefore $\operatorname{Res}_{\bar{E}_i} \pi^*\omega$ is pulled back from the generic point of E_i if and only if so is $\chi^m \in k(\bar{E}_i)$, which is equivalent to $m \in S_{\tau_i} - S_{\tau_i}$. In particular, m belongs to $M \cap \bigcap_{d_i > 1} (S_{\tau_i} - S_{\tau_i})$, which is $S_{\sigma(\Delta)} - S_{\sigma(\Delta)}$ by Proposition 3.2.14.

c) $\Longrightarrow$ *b)* $\chi^m \omega_M^{\otimes n}$ becomes a nowhere zero global section $\omega \in \Gamma(X, \omega^{[n]}_{(X/k,B)})$, with

$$\operatorname{Res}^{[n]}_{\bar{E}_i} \pi^*\omega = \pi_i^*((\epsilon_i d_i)^{-n} \chi^m \omega_i^{\otimes n}).$$

b) $\Longrightarrow$ *a)* is clear. □

Proposition 3.4.2. *$(X/k, B)$ is a weakly normal log pair if and only if $(\bar{X}/k, \bar{C} + \bar{B})$ is a log pair. Moreover:*

- *B is effective if and only if $\bar{C} + \bar{B}$ is effective.*
- *$(X/k, B)$ has wlc singularities if and only if $(\bar{X}/k, \bar{C} + \bar{B})$ has lc singularities, if and only if the coefficients of B are at most 1.*
- *$(X/k, B)$ has slc singularities if and only if $d_i \mid 2$ for all i.*

Proof. Denote $d = \operatorname{lcm}_i d_i$. Pick $r \geq 1$ such that rB has integer coefficients. If $\omega^{[r]}_{(X/k,B)}$ is invertible, so is $\pi^*\omega^{[r]}_{(X/k,B)} = \omega^{[r]}_{(\bar{X}/k, \bar{C}+\bar{B})}$. Conversely, the sheaf $\omega^{[r]}_{(\bar{X}/k, \bar{C}+\bar{B})}$ is invertible if and only if there exists $m \in M$ such that $(\chi^m) + r(-\Sigma_{\bar{X}} + \bar{C} + \bar{B}) = 0$ on $\bar{X}$. Let $E_i \subset \bar{C}$. Since $m \in \bar{S}_{\tau_i} - \bar{S}_{\tau_i}$, $dm \in S_{\tau_i} - S_{\tau_i}$. Since $(\chi^{dm}) + dr(-\Sigma_{\bar{X}} + \bar{C} + \bar{B}) = 0$ on $\bar{X}$, $\omega^{[dr]}_{(X/k,B)}$ is invertible by Lemma 3.4.1.

Note that $\psi = \frac{1}{r} m \in (S_{\sigma(\Delta)} - S_{\sigma(\Delta)})_{\mathbb{Q}}$ is a log discrepancy function of the toric log pair $(\bar{X}/k, \bar{C} + \bar{B})$. We deduce that $(X/k, B)$ has wlc singularities if and only if $(\bar{X}, \bar{C} + \bar{B})$ has lc singularities, if and only if the coefficients of B are at most 1, if and only if $\psi \in \sigma_S$. □

A log discrepancy function ψ is unique modulo the vector space $\sigma_S \cap (-\sigma_S)$, the largest vector space contained in σ_S, or equivalently, the smallest face of σ_S. We actually have $\psi \in \sigma(\Delta)$.

Lemma 3.4.3. *Suppose $(X/k, B)$ is a weakly normal log pair, with log discrepancy function ψ.*

1) $(X/k,B)_{-\infty} = \cup_{b_i>1} E_i$ *and* $(\bar{X}/k, \bar{C}+\bar{B})_{-\infty} = \cup_{b_i>1}\bar{E}_i = \pi^{-1}((X,B)_{-\infty})$.

2) The lc centers of $(X/k,B)$ *are* X_σ, *where* $\psi \in \sigma \prec \sigma_S$ *and* $\sigma \not\subset \tau_i$ *if* $b_i > 1$.

3) The correspondence $Z \mapsto \pi^{-1}(Z)$ *is one to one between lc centers of* $(X/k,B)$ *and lc centers of* $(\bar{X}/k, \bar{C}+\bar{B})$.

Suppose $(X/k,B)$ is wlc, with log discrepancy function $\psi \in \sigma_S$. The lc centers of $(X/k,B)$ are X_σ, where $\psi \in \sigma \prec \sigma_S$. For $\sigma = \sigma_S$, we obtain the lc center X, for $\sigma \neq \sigma_S$ we obtain lc centers defined by toric valuations. Any union of lc centers is weakly normal. The intersection of two lc centers is again an lc center. With respect to inclusion, there exists a unique minimal lc center, namely $X_{\sigma(\psi)}$ for $\sigma(\psi) = \cap_{\psi\in\sigma\prec\sigma_S}\sigma$ (the unique face of σ_S which contains ψ in its relative interior). Note that X is the unique lc center of $(X/k,B)$ if and only if X is normal and the coefficients of B are strictly less than 1.

Lemma 3.4.4. *Suppose* $(X/k,B)$ *is wlc. Then the minimal lc center of* $(X/k,B)$ *is normal.*

Proof. Let $X_{\sigma(\Delta)}$ be the core of X. It is an intersection of lc centers of $(X/k,B)$, hence an lc center itself. Equivalently, $\sigma(\psi) \prec \sigma(\Delta)$ and the minimal lc center $X_{\sigma(\psi)}$ is an invariant closed subvariety of $X_{\sigma(\Delta)}$

By Proposition 3.2.14, the core is normal. Then so is each invariant closed irreducible subvariety of the core. Therefore $X_{\sigma(\psi)}$ is normal. □

Example 3.4.5. Let X/k be an irreducible affine toric variety, weakly normal and S_2. Let Σ be the sum of codimension one subvarieties at which X/k is smooth. Then $\mathcal{O}_X \tilde{\to} \omega^{[1]}_{(X/k,\Sigma)}$ and $(X/k,\Sigma)$ is a weakly normal log variety with wlc singularities.

Indeed, let ω be the volume form on $T = \operatorname{Spec} k[M]$ induced by an orientation of M. Then $(\omega) + \Sigma_{\bar{X}} = 0$ on $\bar{X}$. Its residues descend by weak normality (cf. the proof of Lemma 3.4.1), so ω becomes a nowhere zero global section of $\omega^{[1]}_{(X/k,\Sigma)}$. Since $\bar{C} + \bar{\Sigma} = \Sigma_{\bar{X}}$ and $(\bar{X}, \Sigma_{\bar{X}})$ has lc singularities, the claim holds.

The lc centers of $(X/k,B)$ of codimension one are the invariant primes E_i such that either $\operatorname{mult}_{E_i} B = 1$, or E_i is an irreducible component of C. The normalization of E_i is $E_i^n = \operatorname{Spec} k[(S_{\tau_i} - S_{\tau_i}) \cap \tau_i]$, the different $B_{E_i^n}$ is induced by the log discrepancy function ψ of $(X/k,B)$, and the residue of $\chi^{r\psi}\omega^{\otimes r}$ is $(\epsilon_i d_i)^{-1}\chi^{r\psi}\omega^{\otimes r}_{\mathcal{B}_i}$.

3.4.2 Reducible case

Let $X = \operatorname{Spec} k[\mathcal{M}]$ be weakly normal and S_2. Let $\{F\}$ and $\{\tau_i\}$ be the facets and codimension one faces of Δ, respectively. The normalization $\pi\colon \bar{X} \to X$ is $\sqcup_F \bar{X}_F \to \cup_F X_F$, where $\bar{X}_F = \operatorname{Spec} k[\bar{S}_F]$ and $\bar{S}_F = (S_F - S_F) \cap F$. The invariant codimension one subvarieties of X are $E_i = \operatorname{Spec} k[S_{\tau_i}]$ (either irreducible components of C, or invariant prime divisors at which X/k is smooth). Note that $\pi^{-1}(E_i) = \sqcup_F (\bar{X}_F)_{\tau_i \cap F}$ may have components of different dimension. The primes of $\bar{X}$ over E_i are $\bar{E}_{i,F} = (\bar{X}_F)_{\tau_i}$, one for each facet F containing τ_i. For $F \succ \tau_i$, $\bar{E}_{i,F} = \operatorname{Spec} k[\bar{S}_F \cap \tau_i]$ (note $\bar{S}_F \cap \tau_i = (S_F - S_F) \cap \tau_i$), and the morphism $\pi_{i,F} \colon \bar{E}_{i,F} \to E_i$ is finite of degree $d_{\tau_i \prec F}$, equal to the incidence number of $E_i \subset X_F$. Since X/k is weakly normal, $\operatorname{char}(k) \nmid d_{\tau_i \prec F}$, that is $d_{\tau_i \prec F}$ is invertible in $k^\times$. Let $X_{\sigma(\Delta)}$ be the core of X.

Lemma 3.4.6. *Let ω_F be a volume form on the torus inside X_F, induced by an orientation of the lattice $S_F - S_F$. Let ω_i be a volume form on the torus inside E_i, induced by an orientation of the lattice $S_{\tau_i} - S_{\tau_i}$. For $\tau_i \prec F$, there exists $\epsilon_{\tau_i \prec F} = \pm 1$ such that $\pi_{i,F}^* \omega_i = \epsilon_{\tau_i \prec F} d_{\tau_i \prec F} \cdot \operatorname{Res}_{\bar{E}_{i,F}} \omega_F$.*

Let $n \in \mathbb{Z}$. The following properties are equivalent:

a) There exist $c_F, c_i \in k^\times$ such that $\operatorname{Res}^{[n]}_{\bar{E}_{i,F}}(c_F \omega_F^{\otimes n}) = \pi_{i,F}^*(c_i \omega_i^{\otimes n})$ for every $\tau_i \prec F$.

b) For every cycle $F_0, F_1, \ldots, F_l, F_{l+1} = F_0$ of facets of Δ such that $F_i \cap F_{i+1}$ $(0 \le i < l)$ has codimension one, the following identity holds in $k^\times$:

$$(\prod_{i=0}^{l} \frac{\epsilon_{F_i \cap F_{i+1} \prec F_{i+1}} d_{F_i \cap F_{i+1} \prec F_{i+1}}}{\epsilon_{F_i \cap F_{i+1} \prec F_i} d_{F_i \cap F_{i+1} \prec F_i}})^n = 1.$$

Proof. Denote $e_{i,F} = (\epsilon_{\tau_i \prec F} d_{\tau_i \prec F})^n$. Property a) is equivalent to $c_F = c_i \cdot e_{i,F}$ for every $\tau_i \prec F$.

a) $\Longrightarrow$ *b)* Suppose a) holds. If (F, F') is a pair of facets which intersect in a codimenson one face, then c_F determines $c_{F'}$, by the formula

$$c_{F'} = c_F \cdot \frac{e_{F \cap F' \prec F'}}{e_{F \cap F' \prec F}}.$$

Let $F_0, F_1, \ldots, F_l, F_{l+1} = F_0$ be a cycle such that $F_i \cap F_{i+1}$ $(0 \le i < l)$ has codimension one. Multiplying the above formulas for each pair (F_i, F_{i+1}) $(0 \le i < l)$, and factoring out the nonzero constants c_{F_i}, we obtain

$$\prod_{i=0}^{l} \frac{e_{F_i \cap F_{i+1} \prec F_{i+1}}}{e_{F_i \cap F_{i+1} \prec F_i}} = 1.$$

b) $\Longrightarrow$ *a)* Fix a facet F_0, set $c_{F_0} = 1$. Since Δ is 1-connected, each facet F is the end of a chain of facets $F_0, F_1, \ldots, F_l = F$ such that $F_i \cap F_{i+1}$ has codimension one for every $0 \le i < l$. Define

$$c_F = \prod_{0 \le i < l} \frac{e_{F_i \cap F_{i+1} \prec F_{i+1}}}{e_{F_i \cap F_{i+1} \prec F_i}} \in k^\times.$$

The definition is independent of the choice of the chain reaching F, by the cycle condition b) applied to the concatenation of two chains. For each τ_i, choose a facet F containing it, and define

$$c_i = \frac{c_F}{e_{i,F}}.$$

The definition is independent of the choice of F. Indeed, if F, F' are two facets which contain τ_i, then $\tau_i = F \cap F'$. Forming a cycle with a chain from F_0 to F, followed by F', and by the reverse of a chain from F_0 to F', we obtain from b) that

$$\frac{c_F}{e_{i,F}} = \frac{c_{F'}}{e_{i,F'}}.$$

Property a) holds by construction. □

We say that X/k is *n-orientable* if the equivalent properties of Lemma 3.4.6 hold. If n is even, this property is independent on the choice of orientations, and becomes

$$(\prod_{i=0}^{l} \frac{d_{F_i \cap F_{i+1} \prec F_{i+1}}}{d_{F_i \cap F_{i+1} \prec F_i}})^n = 1 \in k^\times.$$

We say that X/k is *$\mathbb{Q}$-orientable* if it is n-orientable for some $n \geq 1$.

Lemma 3.4.7. *Suppose $d_{\tau_i \prec F}$ does not depend on F. Then X/k is n-orientable, for every $n \in 2\mathbb{Z}$. In particular, X is $\mathbb{Q}$-orientable.*

Proof. Since $\frac{d_{F_i \cap F_{i+1} \prec F_{i+1}}}{d_{F_i \cap F_{i+1} \prec F_i}} = 1$ in this case. □

Example 3.4.8. Some examples where the incidence numbers $d_{\tau_i \prec F}$ do not depend on F are:

1) X is irreducible.

2) X has normal irreducible components (equivalent to X_σ normal for every $\sigma \in \Delta$). Then $d_{\tau_i \prec F} = 1$ for all $\tau_i \prec F$.

3) X is nodal in codimension one. Equivalently, for each codimension one face $\tau_i \in \Delta$, either τ_i is contained in a unique facet F and $d_{\tau_i \prec F} \mid 2$, or τ_i is contained in exactly two facets F, F' and $d_{\tau_i \prec F} = d_{\tau_i \prec F'} = 1$.

Let $B = \sum_i b_i E_i$ be a $\mathbb{Q}$-Weil divisor supported by invariant codimension one subvarieties of X at which X/k is smooth. Note that X/k is smooth at E_i if and only if E_i is contained in a unique irreducible component X_F of X, and $d_{E_i \subset X_F} = 1$.

Lemma 3.4.9. *Let $n \in \mathbb{Z}$. The following properties are equivalent:*

a) $\omega^{[n]}_{(X/k,B)}$ is invertible at some point x, which belongs to the closed orbit of X.

b) $\mathcal{O}_X \simeq \omega^{[n]}_{(X/k,B)}$.

c) X is n-orientable and there exists $m \in S_{\sigma(\Delta)} - S_{\sigma(\Delta)}$ such that $(\chi^m) + \lfloor n(-\Sigma_{\bar{X}} + \bar{C} + \bar{B}) \rfloor = 0$ on $\bar{X}_F$ for every F.

Proof. We use the definitions and notations of Lemma 3.4.6.

a) $\Longrightarrow$ *c*) As in the proof of Lemma 3.4.1, there exists a semi-invariant form $\omega \in \Gamma(X, \omega^{[n]}_{(X/k,B)})$ which induces a local trivialization at x. Let F be a facet of Δ. Let $\bar{x}_F \in \bar{X}_F$ be a point lying over x. Then $\pi^*\omega|_{\bar{X}_F} \in \Gamma(\bar{X}_F, \omega^{[n]}_{(\bar{X}/k,\bar{C}+\bar{B})})$ induces a local trivialization at $\bar{x}_F$, which belongs to the closed orbit of $\bar{X}_F$. By Lemma 3.4.1, there exists $m_F \in S_F - S_F$ such that $(\chi^{m_F}) + \lfloor n(-\Sigma_{\bar{X}} + \bar{C} + \bar{B})\rfloor = 0$ on $\bar{X}_F$, so that $\chi^{m_F}\omega_F^{\otimes n} \in \Gamma(\bar{X}_F, \omega^{[n]}_{(\bar{X}/k,\bar{C}+\bar{B})})$ is a nowhere zero section. Since ω is T-semi-invariant, we obtain $\pi^*\omega|_{\bar{X}_F} = c_F \cdot \chi^{u_F}\chi^{m_F}\omega_F^{\otimes n}$, where $c_F \in k^\times$ and χ^{u_F} is a global unit on $\bar{X}_F$. Replacing m_F by $u_F + m_F$, we obtain

$$\pi^*\omega|_{\bar{X}_F} = c_F \cdot \chi^{m_F}\omega_F^{\otimes n}.$$

By assumption, there exists $\eta_i \in \omega^{\otimes n}_{k(E_i)/k}$ such that for every $E_i \subset C$, and every inclusion $E_i \subset X_F$, we have

$$\operatorname{Res}^{[n]}_{\bar{E}_{i,F}} \pi^*\omega = \pi^*_{i,F}\eta_i.$$

We have $\eta_i = f_i\omega_i^{\otimes n}$ for some $f_i \in k(E_i)^\times$. The residue formula becomes

$$c_F\chi^{m_F} = (\epsilon_{\tau_i \prec F} d_{\tau_i \prec F})^n \pi^*_{i,F} f_i.$$

Then f_i is a unit on the torus inside E_i, hence $f_i = c_i\chi^{m_i}$ for some $c_i \in k^\times$ and $m_i \in S_{\tau_i} - S_{\tau_i}$. We obtain

$$c_F\chi^{m_F} = c_i(\epsilon_{\tau_i \prec F} d_{\tau_i \prec F})^n \chi^{m_i}.$$

That is $c_F = (\epsilon_{\tau_i \prec F} d_{\tau_i \prec F})^n$ and $m_F = m_i$. Since Δ is 1-connected, the latter means that $m_F = m_i = m$ for all F and i, for some $m \in S_{\sigma(\Delta)} - S_{\sigma(\Delta)}$. The former means that X is n-orientable.

c) $\Longrightarrow$ *b*) By Lemma 3.4.6, there exist $c_F, c_i \in k^\times$ with $\operatorname{Res}^{[n]}_{\bar{E}_{i,F}}(c_F\omega_F^{\otimes n}) = \pi^*_{i,F}(c_i\omega_i^{\otimes n})$ if $\tau_i \prec F$. The pluridifferential forms $\{c_F\chi^m\omega_F^{\otimes n}\}_F$ on the normalization of X glue to a nowhere zero global section ω of $\omega^{[n]}_{(X/k,B)}$. Moreover, $\operatorname{Res}^{[n]}_{\bar{E}_{i,F}} \pi^*\omega = \pi^*_{i,F}(c_i\omega_i^{\otimes n})$.

b) $\Longrightarrow$ *a*) is clear. □

Proposition 3.4.10. *$(X/k, B)$ is a weakly normal log pair if and only if X is $\mathbb{Q}$-orientable, and the components of the normalization $(\bar{X}/k, \bar{C} + \bar{B})$ are toric normal log pairs with the same log discrepancy function ψ. Moreover,*

- *B is effective if and only if $\bar{C} + \bar{B}$ is effective.*
- *$(X/k, B)$ has wlc singularities if and only if $(\bar{X}/k, \bar{C} + \bar{B})$ has lc singularities, if and only if the coefficients of B are at most 1, if and only if $\psi \in \sigma(\Delta)$.*

Proof. Suppose $(X/k, B)$ is a weakly normal log pair. There exists an even integer $r \geq 1$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Then $\pi^*\omega^{[r]}_{(X/k,B)} = \omega^{[r]}_{(\bar{X}/k,\bar{C}+\bar{B})}$ is invertible, hence each irreducible component of $(\bar{X}/k, \bar{C} + \bar{B})$ is a toric log pair. By Lemma 3.4.9, X is r-orientable and there exists $m \in S_{\sigma(\Delta)} - S_{\sigma(\Delta)}$ such that $(\chi^m) + \lfloor r(-\Sigma_{\bar{X}} + \bar{C} + \bar{B})|_{\bar{X}_F} \rfloor = 0$ for every facet F of Δ. Therefore $\psi = \frac{1}{r} m \in (S_{\sigma(\Delta)} - S_{\sigma(\Delta)})_{\mathbb{Q}}$ is a log discrepancy function for $(\bar{X}/k, \bar{C} + \bar{B})|_{\bar{X}_F}$, for each facet F. We call ψ a *log discrepancy function* of $(X/k, B)$.

Conversely, suppose that X is $\mathbb{Q}$-orientable, and that the irreducible (connected) components of the normalization $(\bar{X}/k, \bar{C} + \bar{B})$ are toric normal log pairs with the same log discrepancy function $\psi \in \cap_F (S_F - S_F)_{\mathbb{Q}}$. We have $(\chi^\psi) + (-\Sigma_{\bar{X}} + \bar{C} + \bar{B})|_{\bar{X}_F} = 0$ for every facet F of Δ. Choose an even integer $r \geq 1$ such that $r\psi \in \cap_F (S_F - S_F)$. Let τ_i be a codimension one face of Δ. Choose $F \succ \tau_i$. The $\mathbb{Q}$-divisor $(\chi^\psi) + (-\Sigma_{\bar{X}} + \bar{C} + \bar{B})|_{\bar{X}_F}$ is zero at $\bar{E}_{i,F}$, that is $r\psi \in (\bar{S}_F)_{\tau_i} - (\bar{S}_F)_{\tau_i}$. Therefore $d_{\tau_i \prec F} r\psi \in S_{\tau_i} - S_{\tau_i}$.

Let d be a positive integer such that X is d-orientable, and $d_{\tau_i \prec F} \mid d$ for all $\tau_i \prec F$. Then X is dr-orientable and $m = dr\psi$ satisfies the properties of Lemma 3.4.9.c), hence $\omega^{[dr]}_{(X/k,B)}$ is invertible.

Suppose $(X/k, B)$ is a weakly normal log pair. It has wlc singularities if and only if each irreducible component of $(\bar{X}/k, \bar{C} + \bar{B})$ has lc singularities. This holds if and only if the coefficients of B are at most 1, or equivalently, $\psi \in F$ for every facet F. $\square$

Corollary 3.4.11. *$(X/k, B)$ has slc singularities if and only X has at most nodal singularities in codimension one, and the components of the normalization $(\bar{X}/k, \bar{C} + \bar{B})$ are toric normal log pairs with lc singularities having the same log discrepancy function.*

Proof. See Example 3.4.8.3) for the combinatorial criterion for X to be at most nodal in codimension one. In particular, X is 2-orientable. We may apply Proposition 3.4.10. $\square$

Lemma 3.4.12. *Suppose $(X/k, B)$ is a weakly normal log pair, with log discrepancy function ψ.*

1) *$(X/k, B)_{-\infty} = \cup_{b_i > 1} E_i$ and $(\bar{X}/k, \bar{C} + \bar{B})_{-\infty} = \sqcup_F \cup_{E_i \subset F, b_i > 1} \bar{E}_{i,F} = \pi^{-1}((X, B)_{-\infty})$. In particular, π maps lc centers onto lc centers.*

2) *The lc centers of $(X/k, B)$ are X_σ, where $\psi \in \sigma \in \Delta$ and $\sigma \not\subset \tau_i$ if $b_i > 1$.*

3) *Suppose $(X/k, B)$ is wlc. Let $Z = X_\sigma$ be an lc center of $(X/k, B)$. Then $\pi^{-1}(Z)$ is a disjoint union of lc centers, one for each irreducible component of $\bar{X}$:*

$$\pi^{-1}(Z) = \sqcup_F (\bar{X}_F)_{F \cap \sigma}.$$

Some components of $\pi^{-1}(Z)$ may not dominate Z.

Proof. 1) We have $(\bar{X}/k, \bar{C} + \bar{B})_{-\infty} = \sqcup_F \cup_{E_i \subset F, b_i > 1} \bar{E}_{i,F}$. Its image $(X/k, B)_{-\infty}$ on X equals $\cup_{b_i > 1} E_i$. The inclusion $(\bar{X}/k, \bar{C} + \bar{B})_{-\infty} \subseteq \pi^{-1}((X, B)_{-\infty})$ is clear, while the converse may be restated as follows: if $(\bar{X}, \bar{C} + \bar{B})$ is lc at a closed point $\bar{x}$, then (X, B) is wlc at $\pi(\bar{x})$. To prove this, we may localize and suppose $\pi(\bar{x})$ belongs to the closed orbit of X. If F is the facet such that $\bar{x} \in \bar{X}_F$, it follows that $\bar{x}$ belongs to the closed orbit of $\bar{X}_F$. We know that the toric log pair $(\bar{X}/k, \bar{C} + \bar{B})|_{\bar{X}_F}$ has lc singularities at $\bar{x}$, a point belonging to its closed orbit. Then $(\bar{X}/k, \bar{C} + \bar{B})|_{\bar{X}_F}$ has lc singularities. That is $\psi \in F$.

Let F' be a facet of Δ. Since Δ is 1-connected, there exists a chain of facets $F = F_0, F_1, \ldots, F_l = F'$ such that $F_i \cap F_{i+1}$ $(0 \le i < l)$ has codimension one. We know $\psi \in F_0$. The codimension one face $\tau = F_0 \cap F_1$ defines an irreducible component X_τ of C. Therefore $(\bar{X}_{F_0})_\tau$ appears as an irreducible component of $\bar{C}$. It is an lc center of $(\bar{X}/k, \bar{C} + \bar{B})|_{\bar{X}_{F_0}}$, that is $\psi \in \tau$. Therefore $\psi \in F_1$. Repeating this argument along the chain, we obtain $\psi \in F'$.

We conclude that $\psi \in F$ for every facet F, that is $(\bar{X}/k, \bar{C} + \bar{B})$ has lc singularities. Therefore $(X/k, B)$ has wlc singularities.

2) This follows from 1) and the description of the lc centers on the normalization.

3) This is clear. $\square$

Suppose $(X/k, B)$ has wlc singularities. The lc centers of $(X/k, B)$ are X_σ, where $\psi \in \sigma \in \Delta$. Any union of lc centers is weakly normal. The intersection of two lc centers is again an lc center. With respect to inclusion, there exists a unique minimal lc center, namely $X_{\sigma(\psi)}$ for $\sigma(\psi) = \cap_{\psi \in \sigma \in \Delta} \sigma$ (the unique cone of Δ which contains ψ in its relative interior).

Lemma 3.4.13. *Suppose $(X/k, B)$ is wlc. Then the minimal lc center of $(X/k, B)$ is normal.*

Proof. Same as for Lemma 3.4.4. $\square$

Example 3.4.14. Let $X = \operatorname{Spec} k[\mathcal{M}]$ be weakly normal and S_2. Let $B \subset X$ be the reduced sum of invariant codimension one subvarieties at which X/k is smooth (i.e. $B = \Sigma_X - C$). Then $(X/k, B)$ is a weakly normal log variety with wlc singularities if and only if X is $\mathbb{Q}$-orientable. Moreover, $\omega^{[2r]}_{(X/k,B)} \simeq \mathcal{O}_X$ if and only if X is $2r$-orientable.

Indeed, suppose X is $2r$-orientable. The log discrepancy function ψ is zero. The forms $\{c_F \omega_F^{\otimes 2r}\}_F$ glue to a nowhere zero global section $\omega \in \Gamma(X, \omega^{[2r]}_{(X/k,B)})$, and the log crepant structure $(\bar{X}, \bar{C} + \bar{B} = \Sigma_{\bar{X}})$ induced on the normalization has log canonical singularities.

The lc centers of $(X/k, B)$ of codimension one are the invariant primes E_i such that either $\operatorname{mult}_{E_i} B = 1$, or E_i is an irreducible component of C. The normalization of E_i is $E_i^n = \operatorname{Spec} k[(S_{\tau_i} - S_{\tau_i}) \cap \tau_i]$, the different $B_{E_i^n}$ is induced by the log discrepancy function ψ of $(X/k, B)$, and the residue of $\{c_F \chi^{r\psi} \omega_F^{\otimes r}\}_F$ is $c_i \chi^{r\psi} \omega_i^{\otimes r}$.

3.4.3 The LCS locus

Let $(X/k, B)$ be a toric weakly normal log pair, with wlc singularities. Let ψ be its log discrepancy function. The *LCS locus*, or *non-klt locus* of $(X/k, B)$, is the union Y of all lc centers of positive codimension in X. The zero codimension lc centers are exactly the irreducible components of X. Therefore Y is the union of all X_σ such that $\psi \in \sigma \in \Delta$, and σ is strictly contained in some facet of Δ.

Proposition 3.4.15. *Y is weakly normal and S_2, of pure codimension one in X. Moreover, Y is Cohen Macaulay if so is X.*

Proof. Let $\pi\colon \bar{X} \to X$ be the normalization. Let $\bar{Y} = \pi^{-1}(Y)$. Then $\bar{Y} = \mathrm{LCS}(\bar{X}/k, \bar{C} + \bar{B})$. Since Y contains C, the cartesian diagram

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{Y} \arrow[l, hook'] \arrow[d, "\pi"] \\ X & Y \arrow[l, hook'] \end{tikzcd}$$

is also a push-out. Equivalently, we have a Mayer-Vietoris short exact sequence

$$0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\bar{X}} \oplus \mathcal{O}_Y \to \pi_*\mathcal{O}_{\bar{Y}} \to 0.$$

The subvariety Y is weakly normal, since X is. It is of pure codimension one in X, since $Y = C \cup \mathrm{Supp}(B^{=1})$. We verify Serre's property in two steps.

Step 1: If $(X/k, B)$ is a *normal* toric log pair with lc singularities, then $Y = \mathrm{LCS}(X/k, B)$ is Cohen Macaulay.

Indeed, let $X = \mathrm{Spec}\, k[M \cap \sigma]$ and $\psi \in \sigma$ be the log discrepancy function. Let $\tau \prec \sigma$ be the unique face which contains ψ in its relative interior. In particular, a face of σ contains ψ if and only if it contains τ. Then $Y = \cup_{\tau \prec \tau' \subsetneq \sigma} X_{\tau'}$. Consider the quotient $M \to M' = M/(M \cap \tau - M \cap \tau)$, let σ' be the image of σ, denote $X' = \mathrm{Spec}\, k[M' \cap \sigma']$ and $T'' = \mathrm{Spec}\, k[M \cap \tau - M \cap \tau]$. Then X' is a normal affine variety with a fixed point P, and $Y \simeq T'' \times \Sigma_{X'}$ (using the construction in [7, Remark 2.19], we reduced to the case $\psi = 0$). Since T'' is smooth, it is Cohen Macaulay. By [15, Lemma 3.4.1], $\mathrm{depth}_P(\Sigma_{X'}) = \dim \Sigma_{X'}$, that is X' is Cohen Macaulay. Therefore Y is Cohen Macaulay.

Step 2: The disjoint union of normal affine toric varieties $\bar{X}$ is Cohen Macaulay by [30], and $\bar{Y}$ is Cohen Macaulay by Step 1). The Mayer-Vietoris short exact sequence and the cohomological interpretation of Serre's property, give that Y is S_2 (respectively Cohen Macaulay) if so is X. □

Note that $\mathrm{LCS}(X/k, B)$ becomes the union of codimension one lc centers. The normalizations of $\bar{Y}$ and Y induce a commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{Y} \arrow[l, hook'] \arrow[d] & \bar{Y}^n \arrow[l] \arrow[d, "g"] \\ X & Y \arrow[l, hook'] & Y^n \arrow[l, "n"'] \end{tikzcd}$$

Let $X = \cup_F X_F$ and $Y = \cup_j E_j$ be the irreducible decompositions. We have $\bar{X} = \sqcup_F \bar{X}_F$, $\bar{Y} = \sqcup_F \mathrm{LCS}(\bar{X}_F, (\bar{C} + \bar{B})|_{\bar{X}_F})$ and $\mathrm{LCS}(\bar{X}_F, (\bar{C} + \bar{B})|_{\bar{X}_F}) = (\bar{C} \cup \mathrm{Supp}(\bar{B}^{=1}))|_{\bar{X}_F}$. The irreducible components of $\bar{Y}$ are normal. Therefore $\bar{Y}^n = \sqcup_F \sqcup_{\psi \in \tau_j \prec F} \bar{E}_{j,F} = \sqcup_{\tau_j \ni \psi} \sqcup_{F \succ \tau_j} \bar{E}_{j,F}$. The normalization of Y decomposes as $Y^n = \sqcup_j E_j^n$, with $E_j^n = \mathrm{Spec}\, k[(S_{\tau_j} - S_{\tau_j}) \cap \tau_j]$.

Pick $r \geq 1$ such that such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Equivalently, $r\psi \in S_{\sigma(\Delta)}$ and there exists a nowhere zero global section $\omega \in \Gamma(X, \omega^{[r]}_{(X/k,B)})$

such that $\pi^*\omega|_{\bar{X}_F} = c_F\chi^{r\psi}\omega_F^{\otimes r}$ and $\operatorname{Res}^{[r]}_{\bar{E}_{i,F}} \pi^*\omega = \pi_{i,F}^*(c_i\chi^{r\psi}\omega_i^{\otimes r})$. Let η be the rational pluridifferential form on Y^n whose restriction to E_j^n is $c_j\chi^{r\psi}\omega_j^{\otimes r}$. Then

$$\operatorname{Res}^{[r]}_{\bar{Y}^n} \pi^*\omega = g^*\eta.$$

Let $B_{\bar{Y}^n} = -\frac{1}{r}(g^*\eta)$ and $B_{Y^n} = -\frac{1}{r}(\eta)$. Then $B_{\bar{Y}^n}$ is the discriminant of $(\bar{X}, \bar{C} + \bar{B})$ after codimension one adjunction to the components of $\bar{Y}^n$, which is effective if B is effective. Moreover, $g\colon (\bar{Y}^n, B_{\bar{Y}^n}) \to (Y^n, B_{Y^n})$ is log crepant. In particular $B_{Y^n} = g_*(B_{\bar{Y}^n})$ is effective if B is effective. All normal toric log pair structures induced on the irreducible components of $(\bar{X}, \bar{C} + \bar{B})$, $(\bar{Y}^n, B_{\bar{Y}^n})$ and (Y^n, B_{Y^n}) have the same log discrepancy function, namely ψ. The correspondence $\omega \mapsto \eta$ induces the residue isomorphism

$$\operatorname{Res}^{[r]}_{X \to Y^n} \colon \omega^{[r]}_{(X/k,B)}|_{Y^n} \xrightarrow{\sim} \omega^{[r]}_{(Y^n/k, B_{Y^n})}$$

Proposition 3.4.16. *Let $r \in 2\mathbb{Z}$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. The following are equivalent:*

1) *There exists an invariant boundary B_Y on Y such that $(Y/k, B_Y)$ becomes a weakly normal log pair with the same log discrepancy function ψ, with induced log structure (Y^n, B_{Y^n}) on the normalization, and such that codimension one residues onto the components of Y^n glue to a (residue) isomorphism*

$$\operatorname{Res}^{[r]}_{X \to Y} \colon \omega^{[r]}_{(X/k,B)}|_Y \xrightarrow{\sim} \omega^{[r]}_{(Y/k,B_Y)}.$$

   Moreover, rB_Y has integer coefficients, and B_Y is effective if so is B.

2) *$(d_{Q\subset E_1} d_{E_1 \subset X_F})^r = (d_{Q\subset E_2} d_{E_2\subset X_F})^r$ in $k^\times$, if Q is an irreducible component of the non-normal locus of Y, X_F is an irreducible component of X containing Q, and E_1, E_2 are the (only) codimension one invariant subvarieties of X_F containing Q.*

Proof. If Y is normal, there is nothing to prove. Suppose Y is not normal. Let Q be an irreducible component of the non-normal locus of Y. Then $Q = X_\gamma$ for some cone $\gamma \in \Delta$ of codimension two. The primes of Y^n over Q are $Q_{\gamma,j} = \operatorname{Spec} k[(S_{\tau_j} - S_{\tau_j}) \cap \gamma] \subset E_j^n$, one for each τ_j which contains γ. The induced morphism $n_{\gamma,j}\colon Q_{\gamma,j} \to Q$ is finite surjective, of degree $d_{Q\subset E_j}$. Let ω_Q be a volume form on the torus inside Q, induced by an orientation of $S_\gamma - S_\gamma$.

We have $Q = E_1 \cap E_2$ for some irreducible components E_1, E_2 of Y (by the argument of the proof of Corollary 3.2.15). Since E_1, E_2 are lc centers of $(X/k, B)$, so is their intersection Q. That is $\psi \in \gamma$. Therefore $\operatorname{mult}_{Q_{j,\gamma}} B_{Y^n} = 1$ for every $E_j \supset Q$. Since r is even, we compute

$$\operatorname{Res}^{[r]}_{Q_{j,\gamma}} \eta = c_j\chi^{r\psi}(\operatorname{Res}_{Q_{j,\gamma}} \omega_j)^{\otimes r} = n_{j,\gamma}^*(c_j d_{Q\subset E_j}^{-r}\omega_Q^{\otimes r}).$$

Property 1) holds if and only if $\operatorname{Res}^{[r]}_{Q_{j,\gamma}} \eta$ does not depend on j, that is $c_i d^{-r}_{Q\subset E_i} = c_Q$ for every $E_i \supset Q$ (it follows that E_i is an lc center, hence an irreducible component of Y).

Since $c_i = c_F d_{E_i \subset X_F}^{-r}$, property 1) holds if and only if $c_F(d_{Q\subset E_i} d_{E_i \subset X_F})^{-r} = c_Q$ for every $Q \subset E_i \subset X_F$.

1) $\implies$ 2): $c_F(d_{Q\subset E_1} d_{E_1\subset X_F})^{-r} = c_Q = c_F(d_{Q\subset E_2} d_{E_2 \subset X_F})^{-r}$. Therefore 2) holds.

2) $\implies$ 1): We claim that $c_F(d_{Q\subset E_i} d_{E_i\subset X_F})^{-r}$ depends only on Q. By 2), it does not depend on the choice of E_i, once F is chosen. It remains to verify independence on F as well. Since Δ is 1-connected, we may only consider two facets F, F' which contain γ, and intersect in codimension one. Let $\tau_i = F \cap F'$. From $c_F d_{E_i\subset X_F}^{-r} = c_i = c_{F'} d_{E_i \subset X_{F'}}^{-r}$, we obtain $c_F(d_{Q\subset E_i} d_{E_i\subset X_F})^{-r} = c_F(d_{Q\subset E_i} d_{E_i \subset X_{F'}})^{-r}$. Therefore $c_F(d_{Q\subset E_i} d_{E_i\subset X_F})^{-r}$ does not depend on F either, say equal to c_Q. We obtain

$$\operatorname{Res}^{[r]}_{Q_{j,\gamma}} \eta = n^*_{j,\gamma}(c_Q \omega_Q^{\otimes r}).$$

Therefore $(Y/k, B_Y = n_*(B_{Y^n} - \operatorname{Cond}(n)))$ is a weakly normal log pair, rB_Y has integer coefficients and $\omega^{[r]}_{(Y/k,B_Y)}$ is trivialized by a nowhere zero global section such that $n^*\omega' = \eta$. The map $\omega \mapsto \omega'$ induces an isomorphism $\operatorname{Res}^{[r]}_{X\to Y} \colon \omega^{[r]}_{(X/k,B)}|_Y \xrightarrow{\sim} \omega^{[r]}_{(Y/k,B_Y)}$. $\square$

3.5 Residues to lc centers of higher codimension

Definition 3.5.1. We say that $X = \operatorname{Spec} k[\mathcal{M}]$ has *normal components* if each irreducible component X_F of X is normal.

Suppose X has normal components. If F is a facet of Δ and $\sigma \prec F$, then $S_\sigma = (S_F - S_F) \cap \sigma$. Therefore each invariant closed irreducible subvariety X_σ ($\sigma \in \Delta$) is normal. Moreover, X/k is weakly normal, and it is S_2 if and only if Δ is 1-connected.

For the rest of this section, let $(X/k, B)$ be a toric weakly normal log pair with wlc singularities, such that X *has normal components*. Under the latter assumption (which implies that X is 2-orientable), $(X/k, B)$ is a wlc log pair if and only if the toric log structures induced on the irreducible components of the normalization of X have the same log discrepancy function $\psi \in \cap_F F$. Let $r \in 2\mathbb{Z}$. Suppose $r\psi \in \cap_F S_F$, that is rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. The lc centers of $(X/k, B)$ are $\{X_\sigma; \psi \in \sigma \in \Delta\}$. Let X_σ be an lc center. Let B_{X_σ} be the invariant boundary induced by $\psi \in \sigma$. Then $(X_\sigma/k, B_{X_\sigma})$ becomes a *normal* toric log pair with lc singularities, rB_{X_σ} has integer coefficients (effective if so is B) and $\omega^{[r]}_{(X_\sigma/k,B_{X_\sigma})}$ is trivial, and the lc centers of $(X_\sigma/k, B_{X_\sigma})$ are exactly the lc centers of $(X/k, B)$ which are contained in X_σ. Let ω_σ be a volume form on the torus inside X_σ induced by some orientation of the lattice $S_\sigma - S_\sigma$. The forms $\{\chi^{r\psi}\omega_F^{\otimes r}\}_F$ glue to a nowhere zero global section of $\omega^{[r]}_{(X/k,B)}$.

Let Z be an lc center of $(X/k, B)$. On an irreducible toric variety, any proper invariant closed irreducible subvariety is contained in some invariant codimension one subvariety. Therefore we can construct a chain of invariant closed irreducible subvarieties

$$X \supset X_0 \supset X_1 \supset \cdots \supset X_{c-1} \supset X_c = Z$$

such that X_0 is an irreducible component of X and $\operatorname{codim}(X_j \subset X_{j-1}) = 1$ $(0 < j \leq c)$. Let $X_i = X_{\sigma_i}$. Since σ_c contains ψ, each σ_i contains ψ. Therefore each X_i is an lc center of $(X/k, B)$, and X_j becomes a codimension one lc center of $(X_{j-1}/k, B_{X_{j-1}})$. Define the codimension zero residue $\operatorname{Res}^{[r]}_{X\to X_0} \colon \omega^{[r]}_{(X/k,B)}|_{X_0} \xrightarrow{\sim} \omega^{[r]}_{(X_0/k,B_{X_0})}$ as the pullback to the normalization of X, followed by the restriction to the irreducible component X_0 of $\bar{X}$. We have

$$\operatorname{Res}^{[r]}_{X\to X_0}\{\chi^{r\psi}\omega_F^{\otimes r}\}_F = \chi^{r\psi}\omega_{\sigma_0}^{\otimes r}.$$

For $0 < j \leq c$, let $\operatorname{Res}^{[r]}_{X_{j-1}\to X_j} \colon \omega^{[r]}_{(X_{j-1}/k,B_{X_{j-1}})}|_{X_j} \xrightarrow{\sim} \omega^{[r]}_{(X_j/k,B_{X_j})}$ be the usual codimension one residue. We have $\operatorname{Res}_{X_{j-1}\to X_j} \omega_{\sigma_j} = \epsilon_{j-1,j}\omega_{\sigma_j}$ for some $\epsilon_{j-1,j} = \pm 1$. Since r is even, we obtain

$$\operatorname{Res}^{[r]}_{X_{j-1}\to X_j} \chi^{r\psi}\omega_{\sigma_{j-1}}^{\otimes r} = \chi^{r\psi}\omega_{\sigma_j}^{\otimes r}.$$

The composition $\operatorname{Res}^{[r]}_{X_{c-1}\to X_c}|_Z \circ \cdots \circ \operatorname{Res}^{[r]}_{X_0\to X_1}|_Z \circ \operatorname{Res}^{[r]}_{X\to X_0}|_Z$ is an isomorphism

$$\omega^{[r]}_{(X/k,B)}|_Z \xrightarrow{\sim} \omega^{[r]}_{(Z/k,B_Z)}$$

which maps $\{\chi^{r\psi}\omega_F^{\otimes r}\}_F$ onto $\chi^{r\psi}\omega_{\sigma_c}^{\otimes r}$. It does not depend on the choice of the chain from X to Z, so we can denote it

$$\operatorname{Res}^{[r]}_{X\to Z} \colon \omega^{[r]}_{(X/k,B)}|_Z \xrightarrow{\sim} \omega^{[r]}_{(Z/k,B_Z)},$$

and call it the *residue from* $(X/k, B)$ *to the lc center* Z.

Lemma 3.5.2. *Let Z' be an lc center of $(Z/k, B_Z)$. Then Z' is also an lc center of $(X/k, B)$, and the following diagram is commutative:*

$$\begin{tikzcd}
\omega^{[r]}_{(X/k,B)}|_{Z'} \arrow[rr, "\operatorname{Res}^{[r]}_{X\to Z'}"] \arrow[dr, "(\operatorname{Res}^{[r]}_{X\to Z})|_{Z'}"'] & & \omega^{[r]}_{(Z'/k,B_{Z'})} \\
& \omega^{[r]}_{(Z/k,B_Z)}|_{Z'} \arrow[ur, "\operatorname{Res}^{[r]}_{Z\to Z'}"'] &
\end{tikzcd}$$

Proof. Let $Z = X_\sigma$ and $Z' = X_{\sigma'}$. Then $\sigma' \prec \sigma$, and the generators are mapped as follows

$$\begin{tikzcd}
\{\chi^{r\psi}\omega_F^{\otimes r}\}_F \arrow[rr] \arrow[dr] & & \chi^{r\psi}\omega_{\sigma'}^{\otimes r} \\
& \chi^{r\psi}\omega_\sigma^{\otimes r} \arrow[ur] &
\end{tikzcd}$$

Therefore the triangle of isomorphisms commutes. $\square$

We may define residues onto lc centers in a more invariant fashion.

Proposition 3.5.3. *Suppose $Y = \mathrm{LCS}(X/k, B)$ is non-empty. Then $(Y/k, B_Y)$ is a toric weakly normal log pair with wlc singularities, such that Y has normal components, and the codimension one residues onto the components of Y glue to a residue isomorphism*

$$\mathrm{Res}^{[r]}_{X\to Y} : \omega^{[r]}_{(X/k,B)}|_Y \xrightarrow{\sim} \omega^{[r]}_{(Y/k,B_Y)}.$$

Moreover, the lc centers of $(Y/k, B_Y)$ are exactly the lc centers of $(X/k, B)$ which are not maximal with respect to inclusion.

Proof. Since X has normal components, so does Y. In particular, Y/k is weakly normal. It is S_2 by Proposition 3.4.15. Since X has normal components, the incidence numbers $d_{E_i\subset X_F}$ are all 1. Therefore the condition 2) of Proposition 3.4.16 holds, and the codimension one residues glue to a residue onto Y. □

Iteration of the restriction to LCS-locus induces a chain $X = X_0 \supset X_1 \supset \cdots \supset X_c = W$ with the following properties:

- $(X_0/k, B_{X_0}) = (X/k, B)$.
- $X_i = \mathrm{LCS}(X_{i-1}/k, B_{X_{i-1}})$ and B_{X_i} is the different of $(X_{i-1}/k, B_{X_{i-1}})$ on X_i.
- $\mathrm{LCS}(W/k, B_W) = \emptyset$. That is W/k is normal and the coefficients of B_W are strictly less than 1.

The irreducible components of X_i are the lc centers of $(X/k, B)$ of codimension i, and W is the (unique) minimal lc center of $(X/k, B)$. We compute

$$\mathrm{Res}^{[r]}_{X\to W} = \mathrm{Res}^{[r]}_{X_{c-1}\to X_c}|_W \circ \cdots \circ \mathrm{Res}^{[r]}_{X_0\to X_1}|_W.$$

If Z is an lc center of $(X/k, B)$ of codimension i, then Z is an irreducible component of X_i, and

$$\mathrm{Res}^{[r]}_{X\to Z} = \mathrm{Res}^{[r]}_{X_i\to Z} \circ \mathrm{Res}^{[r]}_{X_{i-1}\to X_i}|_Z \circ \cdots \circ \mathrm{Res}^{[r]}_{X_0\to X_1}|_Z,$$

where $\mathrm{Res}^{[r]}_{X_i\to Z}$ is defined as the pullback to the normalization of X_i, followed by the restriction to the irreducible component Z.

Lemma 3.5.4. *Let X' be a union of lc centers of $(X/k, B)$, such that X' is S_2. Then $(X', B_{X'})$ is a toric log pair with wlc singularities and the same log discrepancy function ψ, and residues onto the components of X' glue to a residue isomorphism*

$$\mathrm{Res}^{[r]}_{X\to X'} : \omega^{[r]}_{(X/k,B)}|_{X'} \xrightarrow{\sim} \omega^{[r]}_{(X'/k,B_{X'})}.$$

Proof. Note that X' has normal components, hence it is weakly normal. Since X' is S_2, all irreducible components have the same codimension, say i, in X. Then X' is a union of some irreducible components of X_i. Define

$$\mathrm{Res}^{[r]}_{X\to X'} = \mathrm{Res}^{[r]}_{X_i\to X'} \circ \mathrm{Res}^{[r]}_{X_{i-1}\to X_i}|_{X'} \circ \cdots \circ \mathrm{Res}^{[r]}_{X_0\to X_1}|_{X'}.$$

The codimension zero residue $\mathrm{Res}^{[r]}_{X_i\to X'}$ is defined as the pullback to the normalization of X_i, followed by restriction to the union of irreducible components consisting of the normalization of X', followed by descent to X'. □

Example 3.5.5. Let $X = \operatorname{Spec} k[\mathcal{M}]$ be S_2, with normal components. Let $B = \Sigma_X - C_X$, the reduced sum of invariant prime divisors at which X/k is smooth. Then $(X/k, B)$ is a toric weakly normal log variety, with log discrepancy function $\psi = 0$, and $\operatorname{LCS}(X/k, B) = \Sigma_X$.

Indeed, X is 2-orientable since it has normal components. The 2-forms $\{\omega_F^{\otimes 2}\}_F$ glue to a nowhere zero global section of $\omega^{[2]}_{(X/k,B)}$. Since $\psi = 0$, the lc centers are the invariant closed irreducible subvarieties of X. Therefore $\operatorname{LCS}(X/k, B) = \Sigma_X$.

Proposition 3.5.6. *Let $X = \operatorname{Spec} k[\mathcal{M}]$ be S_2, with normal components. Let X_i be the union of codimension i invariant subvarieties of X. Then X_i is S_2 with normal components, $X_{i+1} \subset X_i$ has pure codimension one if non-empty, and coincides with the non-normal locus of X_i if $i > 0$, and the following properties hold:*

- *$(X/k, \Sigma_X - C)$ is a wlc log variety, with zero log discrepancy function, and* LCS-*locus X_1. The induced boundary on X_1 is zero, and we have a residue isomorphism*

$$\operatorname{Res}^{[2]} \colon \omega^{[2]}_{(X/k,\Sigma_X - C)}|_{X_1} \xrightarrow{\sim} \omega^{[2]}_{(X_1/k,0)}.$$

- *For $i > 0$, $(X_i/k, 0)$ is a wlc log variety, with zero log discrepancy function, and* LCS-*locus X_{i+1}. The induced boundary on X_{i+1} is zero, and we have a residue isomorphism*

$$\operatorname{Res}^{[2]} \colon \omega^{[2]}_{(X_i/k,0)}|_{X_{i+1}} \xrightarrow{\sim} \omega^{[2]}_{(X_{i+1}/k,0)}.$$

Proof. By iterating the construction of Example 3.5.5 and Proposition 3.5.3, we obtain for all $i \geq 0$ that $(X_i/k, B_{X_i})$ is a wlc log variety, with zero log discrepancy function, and LCS-locus X_{i+1}, and the boundary induced on X_{i+1} by codimension one residues is $B_{X_{i+1}}$.

If X_i is a torus (i.e. X contains no invariant prime divisors), then $X_{i+1} = \emptyset$. If X_i is not a torus, then X_{i+1} has pure codimension one in X_i.

Let $i > 0$. We claim that $B_{X_i} = 0$ and X_{i+1} is the non-normal locus of X_i. Suppose X_i contains an invariant prime divisor Q. Since $i > 0$, there exists an irreducible component Q' of X_{i-1} which contains Q. Then Q has codimension two in Q'. Therefore Q' has exactly two invariant prime divisors which contain Q, say Q_1, Q_2. Then $Q_1 \neq Q_2$ are irreducible components of X_i, and $Q = Q_1 \cap Q_2$. Therefore Q is contained in C_{X_i}, the non-normal locus. We deduce $C_{X_i} = \Sigma_{X_i} = X_{i+1}$. In particular, $B_{X_i} = 0$. □

3.5.1 Higher codimension residues for normal crossings pairs

Let $(X/k, B)$ be a wlc log pair, let $x \in X$ be a closed point. We say that $(X/k, B)$ is *n-wlc at x* if there exists an affine toric variety $X' = \operatorname{Spec} k[\mathcal{M}]$ *with normal components*, associated to some monoidal complex $\mathcal{M}$, an invariant boundary B' on X' and a closed point x' in the closed orbit of X', together with an isomorphism of complete local k-algebras $\mathcal{O}^\wedge_{X,x} \simeq \mathcal{O}^\wedge_{X',x'}$, and such that $(\omega^{[r]}_{(X/k,B)})^\wedge_x$ corresponds to $(\omega^{[r]}_{(X'/k,B')})^\wedge_{x'}$ for r sufficiently divisible. By [10], this is equivalent to the existence of a common étale neighborhood

$$\begin{tikzcd}
 & (U,y) \arrow[dl, "i"'] \arrow[dr, "i'"] & \\
(X,x) & & (X',x')
\end{tikzcd}$$

and a wlc pair structure (U,B_U) on U such that $i^*\omega^{[n]}_{(X/k,B)}=\omega^{[n]}_{(U/k,B_U)}=i'^*\omega^{[n]}_{(X'/k,B')}$ for all $n\in\mathbb{Z}$. It follows that X'/k must be weakly normal and S_2, and $(X'/k,B')$ is wlc.

Being n-wlc at a closed point is an open property. We say that $(X/k,B)$ *is n-wlc* if it so at every closed point. For the rest of this section, let $(X/k,B)$ be n-wlc. Let $r\in 2\mathbb{Z}$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible.

Proposition 3.5.7. *Suppose $Y=\operatorname{LCS}(X/k,B)$ is non-empty. Then Y is weakly normal and S_2, of pure codimension one in X. There exists a unique boundary B_Y such that $(Y/k,B_Y)$ is n-wlc, and codimension one residues onto the irreducible components of the normalization of Y glue to a residue isomorphism*

$$\operatorname{Res}^{[r]}_{X\to Y}\colon \omega^{[r]}_{(X/k,B)}|_Y \xrightarrow{\sim} \omega^{[r]}_{(Y/k,B_Y)}.$$

Moreover, the lc centers of $(Y/k,B_Y)$ are exactly the lc centers of $(X/k,B)$ which are not maximal with respect to inclusion.

Proof. By Proposition 3.5.3 for a local analytic model. □

Iteration of the restriction to LCS-locus induces a chain $X=X_0\supset X_1\supset\cdots\supset X_c=W$ with the following properties:

- $(X_0/k,B_{X_0})=(X/k,B)$.
- $(X_i/k,B_{X_i})$ is a n-wlc pair, $X_i=\operatorname{LCS}(X_{i-1}/k,B_{X_{i-1}})$ and B_{X_i} is the different on X_i of $(X_{i-1}/k,B_{X_{i-1}})$.
- $\operatorname{LCS}(W/k,B_W)=\emptyset$. That is W/k is normal and the coefficients of B_W are strictly less than 1.

The irreducible components of X_i are the lc centers of $(X/k,B)$ of codimension i, and W is the union of lc centers of $(X/k,B)$ of largest codimension.

Let Z be an lc center of $(X/k,B)$, of codimension i. Then Z is an irreducible component of X_i. Let $Z^n\to Z$ be the normalization. Then Z^n is an irreducible component of the normalization of X_i. Let B_{Z^n} be the induced boundary. Define the zero codimension residue

$$\operatorname{Res}^{[r]}_{X_i\to Z^n}\colon \omega^{[r]}_{(X_i/k,B_{X_i})}|_{Z^n} \xrightarrow{\sim} \omega^{[r]}_{(Z^n/k,B_{Z^n})}$$

as the pullback from X_i to its normalization, followed by the restriction to the irreducible component Z^n. Define $\operatorname{Res}^{[r]}_{X\to Z^n}=\operatorname{Res}^{[r]}_{X_i\to Z^n}\circ\operatorname{Res}^{[r]}_{X_{i-1}\to X_i}|_{Z^n}\circ\cdots\circ\operatorname{Res}^{[r]}_{X_0\to X_1}|_{Z^n}$. We obtain:

Theorem 3.5.8. *Let $(X/k, B)$ be n-wlc. Let $r \in 2\mathbb{Z}$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Let Z be an lc center, with normalization $Z^n \to Z$. Then there exists a log pair structure (Z^n, B_{Z^n}) on Z^n, and a higher codimension residue isomorphism*

$$\operatorname{Res}^{[r]}_{X\to Z^n} \colon \omega^{[r]}_{(X/k,B)}|_{Z^n} \xrightarrow{\sim} \omega^{[r]}_{(Z^n/k,B_{Z^n})}.$$

Moreover, B_{Z^n} is effective if so is B, and rB_{Z^n} has integer coefficients.

Definition 3.5.9. A *normal crossings pair* $(X/k, B)$ is an n-wlc pair with local analytic models of the following special type: $0 \in (X'/k, B')$, where $X' = \cup_{i\in I} H_i \subset \mathbb{A}^n_k$ for some $I \subseteq \{1, \ldots, n\}$, and $H_i = \{z_i = 0\} \subset \mathbb{A}^n_k$ is the i-th standard hyperplane. It follows that $B' = \sum_{i\notin I} b_i H_i|_{X'}$ for some $b_i \in \mathbb{Q}_{\le 1}$.

Corollary 3.5.10. *Let $(X/k, B)$ be normal crossings pair. Let $r \in 2\mathbb{Z}$ such that rB has integer coefficients and $\omega^{[r]}_{(X/k,B)}$ is invertible. Let Z be an lc center, with normalization $Z^n \to Z$. Then there exists a log pair structure (Z^n, B_{Z^n}) on Z^n, with log smooth support, and a higher codimension residue isomorphism*

$$\operatorname{Res}^{[r]}_{X\to Z^n} \colon \omega^{[r]}_{(X/k,B)}|_{Z^n} \xrightarrow{\sim} \omega^{[r]}_{(Z^n/k,B_{Z^n})}.$$

Moreover, B_{Z^n} is effective if so is B, and rB_{Z^n} has integer coefficients.

Example 3.5.11. Let $(X/\mathbb{C}, \Sigma)$ be a log smooth pair, that is $X/\mathbb{C}$ is smooth and Σ is a divisor with normal crossings in X. Let Z be an lc center of $(X/\mathbb{C}, \Sigma)$, let $Z^n \to Z$ be the normalization. Deligne [19] defines a residue isomorphism $\operatorname{Res}\colon \omega_X(\log \Sigma)|_{Z^n} \xrightarrow{\sim} \omega_{Z^n}(\log \Sigma_{Z^n}) \otimes \epsilon_{Z^n}$, where ϵ_{Z^n} is a local system (orientations of the local analytic branches of Σ through Z) such that $\epsilon^{\otimes 2}_{Z^n} \simeq \mathcal{O}_{Z^n}$. Then $\operatorname{Res}^{\otimes 2}$ coincides with $\operatorname{Res}^{[2]}\colon \omega^{[r]}_{(X/\mathbb{C},\Sigma)}|_{Z^n} \xrightarrow{\sim} \omega^{[r]}_{(Z^n/k,\Sigma_{Z^n})}$ defined above.

Chapter 4

Esnault-Viehweg injectivity

We are interested in the following *lifting problem*: given a Cartier divisor L on a complex variety X and a closed subvariety $Y \subset X$, when is the restriction map

$$\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L))$$

surjective? The standard method is to consider the short exact sequence

$$0 \to \mathcal{I}_Y(L) \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L) \to 0,$$

which induces a long exact sequence in cohomology

$$0 \to \Gamma(X, \mathcal{I}_Y(L)) \to \Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L)) \to H^1(X, \mathcal{I}_Y(L)) \stackrel{\alpha}{\to} H^1(X, \mathcal{O}_X(L)) \cdots$$

The restriction is surjective if and only if α is injective. In particular, if $H^1(X, \mathcal{I}_Y(L)) = 0$.

If X is a nonsingular proper curve, Serre duality answers completely the lifting problem: the restriction map is not surjective if and only if $L \sim K_X + Y - D$ for some effective divisor D such that $D - Y$ is not effective. In particular, $\deg L \leq \deg(K_X + Y)$. If $\deg L > \deg(K_X + Y)$, then $H^1(X, \mathcal{I}_Y(L)) = 0$, and therefore lifting holds.

If X is a nonsingular projective surface, only sufficient criteria for lifting are known (see [67]). If H is a general hyperplane section induced by a Veronese embedding of sufficiently large degree (depending on L), then $\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(H, \mathcal{O}_H(L))$ is an isomorphism (Enriques-Severi-Zariski). If H is a hyperplane section of X, then $H^i(X, \mathcal{O}_X(K_X + H)) = 0$ $(i > 0)$ (Picard-Severi).

These classical results were extended by Serre [57] as follows: if X is affine and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F}) = 0$ $(i > 0)$. If X is projective, H is ample and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F}(mH)) = 0$ $(i > 0)$ for m sufficiently large.

Kodaira [40] extended Picard-Severi's result as follows: if X is a projective complex manifold, and H is an ample divisor, then $H^i(X, \mathcal{O}_X(K_X + H)) = 0$ $(i > 0)$. This vanishing remains true over a field of characteristic zero, but may fail in positive characteristic (Raynaud [55]). Kodaira's vanishing is central in the classification theory of complex algebraic varieties, but one has to weaken the positivity of H to apply it successfully: it still holds if H is only semiample and big (Mumford [49], Ramanujam [53]), or if $K_X + H$

is replaced by $\lceil K_X + H \rceil$ for a $\mathbb{Q}$-divisor H which is nef and big, whose fractional part is supported by a normal crossings divisor (Ramanujam [54], Miyaoka [48], Kawamata [37], Viehweg [66]). Recall that the round up of a real number x is $\lceil x \rceil = \min\{n \in \mathbb{Z}; x \leq n\}$, and the round up of a $\mathbb{Q}$-divisor $D = \sum_E d_E E$ is $\lceil D \rceil = \sum_E \lceil d_E \rceil E$.

The first lifting criterion in the absence of bigness is due to Tankeev [62]: if X is proper nonsingular and $Y \subset X$ is the general member of a free linear system, then the restriction

$$\Gamma(X, \mathcal{O}_X(K_X + 2Y)) \to \Gamma(Y, \mathcal{O}_Y(K_X + 2Y))$$

is surjective. Kollár [41] extended it to the following injectivity theorem: if H is a semi-ample divisor and $D \in |m_0 H|$ for some $m_0 \geq 1$, then the homomorphism

$$H^q(X, \mathcal{O}_X(K_X + mH)) \to H^q(X, \mathcal{O}_X(K_X + mH + D))$$

is injective for all $m \geq 1, q \geq 0$. Esnault and Viehweg [24, 25] removed completely the positivity assumption, to obtain the following injectivity result: let L be a Cartier divisor on X such that $L \sim_{\mathbb{Q}} K_X + \sum_i b_i E_i$, where $\sum_i E_i$ is a normal crossings divisor and $0 \leq b_i \leq 1$ are rational numbers. If D is an effective divisor supported by $\sum_{0 < b_i < 1} E_i$, then the homomorphism

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

is injective, for all q. The original result [25, Theorem 5.1] was stated in terms of roots of sections of powers of line bundles, and restated in this logarithmic form in [4, Corollary 3.2]. It was used in [3, 4] to derive basic properties of log varieties and quasi-log varieties.

The main result of this chapter (Theorem 4.2.3) is that Esnault-Viehweg's injectivity remains true even if some components E_i of D have $b_i = 1$. In fact, it reduces to the special case when all $b_i = 1$, which has the following geometric interpretation:

Theorem 4.0.12. *Let X be a proper nonsingular variety, defined over an algebraically closed field of characteristic zero. Let Σ be a normal crossings divisor on X, let $U = X \setminus \Sigma$. Then the restriction homomorphism*

$$H^q(X, \mathcal{O}_X(K_X + \Sigma)) \to H^q(U, \mathcal{O}_U(K_U))$$

is injective, for all q.

Combined with Serre vanishing on affine varieties, it gives:

Corollary 4.0.13. *Let X be a proper nonsingular variety, defined over an algebraically closed field of characteristic zero. Let Σ be a normal crossings divisor on X such that $X \setminus \Sigma$ is contained in an affine open subset of X. Then*

$$H^q(X, \mathcal{O}_X(K_X + \Sigma)) = 0$$

for $q > 0$.

If $X \setminus \Sigma$ itself is affine, this vanishing is due to Esnault and Viehweg [25, page 5]. It implies the Kodaira vanishing theorem.

We outline the structure of this chapter. After some preliminaries in Section 1, we prove the main injectivity result in Section 2. The proof is similar to that of Esnault-Viehweg, except that we do not use duality. It is an immediate consequence of the Atiyah-Hodge Lemma and Deligne's degeneration of the logarithmic Hodge to de Rham spectral sequence. In Section 3, we obtain some vanishing theorems for sheaves of logarithmic forms of intermediate degree. The results are the same as in [25], except that the complement of the boundary is only contained in an affine open subset, instead of being itself affine. They suggest that injectivity may extend to forms of intermediate degree (Question 3.2.5). In section 4, we introduce the *locus of totally canonical singularities* and the *non-log canonical locus* of a log variety. The latter has the same support as the subscheme structure for the non-log canonical locus introduced in [3], but the scheme structure usually differ (see Remark 4.4.4). In Section 5, we partially extend the injectivity theorem to the category of log varieties. The open subset to which we restrict is the locus of totally canonical singularities of some log structure. We can only prove the injectivity for the first cohomology group. The idea is to descend injectivity from a log resolution, and to make this work for higher cohomology groups one needs vanishing theorems or at least the degeneration of the Leray spectral sequence for a certain resolution. We do not pursue this here. In Section 6, we establish the *lifting property of* $\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L))$ *for a Cartier divisor* $L \sim_{\mathbb{R}} K_X + B$, *with* Y *the non-log canonical locus of* X (Theorem 4.6.2). We give two applications for this unexpected property. For a proper generalized log Calabi-Yau variety, we show that the non-log canonical locus is connected and intersects every lc center (Theorem 4.6.3). And we obtain an extension theorem from a union of log canonical centers, in the log canonical case (Theorem 4.6.4). We expect this extension to play a key role in the characterization of the restriction of log canonical rings to lc centers. In Section 7 we list some questions that appeared naturally during this work.

4.1 Preliminaries

4.1.1 Directed limits

A *directed family* of abelian groups $(A_m)_{m \in \mathbb{Z}}$ consists of homomorphisms of abelian groups $\varphi_{mn} \colon A_m \to A_n$, for $m \leq n$, such that $\varphi_{mm} = \mathrm{id}_{A_m}$ and $\varphi_{np} \circ \varphi_{mn} = \varphi_{mp}$ for $m \leq n \leq p$. The *directed limit* $\varinjlim_m A_m$ of $(A_m)_{m \in \mathbb{Z}}$ is defined as the quotient of $\oplus_{m \in \mathbb{Z}} A_m$ modulo the subgroup generated by $x_m - \varphi_{mn}(x_m)$ for all $m \leq n$ and $x_m \in A_m$. The homomorphisms $\mu_m \colon A_m \to \varinjlim_n A_n, a_m \mapsto [a_m]$ are compatible with φ_{mn}, and satisfy the following universal property: if B is an abelian group and $f_n \colon A_m \to B$ are homomorphisms compatible with φ_{mn}, then there exists a unique homomorphism $f \colon \varinjlim_m A_m \to B$ such that $f_m = f \circ \mu_m$ for all m. From the explicit description of the directed limit, the following properties hold: $\varinjlim_n A_n = \cup_m \mu_m(A_m)$, and $\mathrm{Ker}(A_m \to \varinjlim_n A_n) = \cup_{m \leq n} \mathrm{Ker}(A_m \to A_n)$. In particular, we obtain

Lemma 4.1.1. *Let $(A_m)_{m\in\mathbb{Z}}$ be a directed system of abelian groups.*

1) *$A_m \to \varinjlim_n A_n$ is injective if and only if $A_m \to A_n$ is injective for all $n \geq m$.*

2) *Let $(B_m)_{m\in\mathbb{Z}}$ be another directed family of abelian groups, let $f_m \colon A_m \to B_m$ be a sequence of compatible homomorphisms. They induce a homomorphism $f \colon \varinjlim_m A_m \to \varinjlim_m B_m$. If f_m is injective for $m \geq m_0$, then f is injective.*

4.1.2 Homomorphisms induced in cohomology

For standard notations and results, see Grothendieck [27, 12.1.7,12.2.5]. Let $f \colon X' \to X$ and $\pi \colon X \to S$ be morphisms of ringed spaces. Denote $\pi' = \pi \circ f \colon X' \to S$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $\mathcal{F}'$ an $\mathcal{O}_{X'}$-module. A homomorphism of $\mathcal{O}_X$-modules $u \colon \mathcal{F} \to f_*\mathcal{F}'$ induces functorial homomorphisms of $\mathcal{O}_S$-modules

$$R^q u \colon R^q\pi_*\mathcal{F} \to R^q\pi'_*(\mathcal{F}') \ (q \geq 0).$$

Grothendieck-Leray constructed a spectral sequence

$$E_2^{pq} = R^p\pi_*(R^q f_*\mathcal{F}') \Longrightarrow R^{p+q}\pi'_*(\mathcal{F}').$$

Lemma 4.1.2. *The homomorphism $R^1\pi_*(f_*\mathcal{F}') \to R^1\pi'_*(\mathcal{F}')$, induced by $\mathrm{id} \colon f_*\mathcal{F}' \to f_*\mathcal{F}'$, is injective.*

Proof. The exact sequence of terms of low degree of the Grothendieck-Leray spectral sequence is

$$0 \to R^1\pi_*(f_*\mathcal{F}') \to R^1\pi'_*(\mathcal{F}') \to \pi_*(R^1 f_*\mathcal{F}') \to R^2\pi_*(f_*\mathcal{F}') \to R^2\pi'_*(\mathcal{F}'),$$

and $R^1\pi_*(f_*\mathcal{F}') \to R^1\pi'_*(\mathcal{F}')$ is exactly the homomorphism induced by the identity of $f_*\mathcal{F}'$. □

The other maps $R^p\pi_*(f_*\mathcal{F}') \to R^p\pi'_*(\mathcal{F}')$ ($p \geq 2$), appearing in the spectral sequence as the edge maps $E_2^{p,0} \to H^p$, may not be injective.

Example 4.1.3. Let $f \colon X \to Y$ be the blow-up at a point of a proper smooth complex surface Y, let E be the exceptional divisor. Then the map

$$H^2(Y, f_*\mathcal{O}_X(K_X + E)) \to H^2(X, \mathcal{O}_X(K_X + E))$$

is not injective. In particular, the Leray spectral sequence for f and $\mathcal{O}_X(K_X + E)$ does not degenerate. Indeed, consider the commutative diagram

$$\begin{tikzcd}
H^2(X, \mathcal{O}_X(K_X)) \arrow[r, "\gamma"] & H^2(X, \mathcal{O}_X(K_X + E)) \\
H^2(Y, f_*\mathcal{O}_X(K_X)) \arrow[u, "\alpha"] \arrow[r, "\beta"] & H^2(Y, f_*\mathcal{O}_X(K_X + E)) \arrow[u, "\delta"']
\end{tikzcd}$$

We have $R^i f_* \mathcal{O}_X(K_X) = 0$ for $i = 1, 2$. Therefore α is an isomorphism, from the Leray spectral sequence. The natural map $f_*\mathcal{O}_X(K_X) \to f_*\mathcal{O}_X(K_X + E)$ is an isomorphism. Therefore β is an isomorphism. By Serre duality, the dual of γ is the inclusion $\Gamma(X, \mathcal{O}_X(-E)) \to \Gamma(X, \mathcal{O}_X)$. Since X is proper, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Therefore $\Gamma(X, \mathcal{O}_X(-E)) = 0$. We obtain $\gamma^\vee = 0$. Therefore $\gamma = 0$.

Since α, β are isomorphisms and $\gamma = 0$, we deduce $\delta = 0$. But $H^2(Y, f_*\mathcal{O}_X(K_X + E))$ is non-zero, being isomorphic to $H^2(X, \mathcal{O}_X(K_X))$, which is dual to $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Therefore δ is not injective.

4.1.3 Weil divisors

Let X be a normal algebraic variety defined over k, an algebraically closed field. A *prime on* X is a reduced irreducible cycle of codimension one. An $\mathbb{R}$-*Weil divisor* D on X is a formal sum

$$D = \sum_E d_E E,$$

where the sum runs after all primes on X, and d_E are real numbers such that $\{E; d_E \neq 0\}$ has at most finitely many elements. It can be viewed as an $\mathbb{R}$-valued function defined on all primes, with finite support. By restricting the values to $\mathbb{Q}$ or $\mathbb{Z}$, we obtain the notion of $\mathbb{Q}$-*Weil divisor* and *Weil divisor*, respectively.

Let $f \in k(X)$ be a rational function. For a prime E on X, let t be a local parameter at the generic point of E. Define $v_E(f)$ as the supremum of all $m \in \mathbb{Z}$ such that ft^{-m} is regular at the generic point of E. If $f = 0$, then $v_E(f) = +\infty$. Else, $v_E(f)$ is a well defined integer. We have $v_E(fg) = v_E(f) + v_E(g)$ and $v_E(f + g) \geq \min(v_E(f), v_E(g))$.

For non-zero $f \in k(X)$ define $(f) = \sum_E v_E(f)E$, where the sum runs after all primes on X. The sum has finite support, so (f) is a Weil divisor. A Weil divisor D on X is *linearly trivial*, denoted $D \sim 0$, if there exists $0 \neq f \in k(X)$ such that $D = (f)$.

Definition 4.1.4. Let D be an $\mathbb{R}$-Weil divisor on X. We call D

- $\mathbb{R}$-*linearly trivial*, denoted $D \sim_\mathbb{R} 0$, if there exist finitely many $r_i \in \mathbb{R}$ and $0 \neq f_i \in k(X)$ such that $D = \sum_i r_i(f_i)$.
- $\mathbb{Q}$-*linearly trivial*, denoted $D \sim_\mathbb{Q} 0$, if there exist finitely many $r_i \in \mathbb{Q}$ and $0 \neq f_i \in k(X)$ such that $D = \sum_i r_i(f_i)$.

Lemma 4.1.5 ([58], page 97)**.** *Let $E_1, \ldots, E_l$ be distinct prime divisors on X, and D a $\mathbb{Q}$-Weil divisor on X. If not empty, the set $\{(x_1, \ldots, x_l) \in \mathbb{R}^l; \sum_{i=1}^l x_i E_i \sim_\mathbb{R} D\}$ is an affine subspace of $\mathbb{R}^l$ defined over $\mathbb{Q}$.*

Proof. Case $D = 0$: the set $V_0 = \{x \in \mathbb{R}^l; \sum_{i=1}^l x_i E_i \sim_\mathbb{R} 0\}$ is an $\mathbb{R}$-vector subspace of $\mathbb{R}^l$. Let $x \in V_0$. This means that there exist finitely many non-zero rational functions $f_\alpha \in k(X)^\times$ and finitely many real numbers $r_\alpha \in \mathbb{R}$ such that

$$\sum_{i=1}^l x_i E_i = \sum_\alpha r_\alpha (f_\alpha).$$

This equality of divisors is equivalent to the system of linear equations

$$\mathrm{mult}_E(\sum_{i=1}^l x_i E_i) = \sum_\alpha r_\alpha \,\mathrm{mult}_E(f_\alpha),$$

one equation for each prime divisor E which may appear in the support of f_α, for some α. We have $\mathrm{mult}_E(f_\alpha) \in \mathbb{Z}$. If we fix the f_α, this means that r_α are the solutions of a linear system defined over $\mathbb{Q}$, and the corresponding x's belong to an $\mathbb{R}$-vector subspace of $\mathbb{R}^l$ defined over $\mathbb{Q}$.

The above argument shows that V_0 is a union of vector subspaces defined over $\mathbb{Q}$. Let $v_1, \ldots, v_k$ be a basis for V_0 over $\mathbb{R}$. Each v_a belongs to some subspace of V_0 defined over $\mathbb{Q}$. That is, there exist $(w_{ab})_b$ in $V_0 \cap \mathbb{Q}^l$ such that $v_a \in \sum_b \mathbb{R} w_{ab}$. It follows that the elements $w_{ab} \in V_0 \cap \mathbb{Q}^l$ generate V_0 as an $\mathbb{R}$-vector space. Therefore V_0 is defined over $\mathbb{Q}$.

Case D arbitrary: suppose $V = \{x \in \mathbb{R}^l; \sum_{i=1}^l x_i E_i \sim_\mathbb{R} D\}$ is nonempty. Let $x \in V$. Then $\sum_{i=1}^l x_i E_i = D + \sum_\alpha r_\alpha(f_\alpha)$ for finitely many r_α, f_α as above. Since D has rational coefficients, the same argument used above shows that once f_α are fixed, there exists another representation $\sum_{i=1}^l x'_i E_i = D + \sum_\alpha r'_\alpha(f_\alpha)$, with $x'_i, r_\alpha \in \mathbb{Q}$. In particular, $x' \in V \cap \mathbb{Q}^l$. We have $V = x' + V_0$. Since V_0 is defined over $\mathbb{Q}$, we conclude that V is an affine subspace of $\mathbb{R}^l$ defined over $\mathbb{Q}$. □

If $D \sim_\mathbb{Q} 0$, then D has rational coefficients. If D has rational coefficients, then $D \sim_\mathbb{Q} 0$ if and only if $D \sim_\mathbb{R} 0$ (by Lemma 4.1.5).

Let D be an $\mathbb{R}$-divisor on X. Denote $D^{=1} = \sum_{d_E=1} E$, $D^{\neq 1} = \sum_{d_E \neq 1} d_E E$, $D^{<0} = \sum_{d_E<0} d_E E$, $D^{>0} = \sum_{d_E>0} d_E E$. The round up (down) of D is defined as $\lceil D \rceil = \sum_E \lceil d_E \rceil E$ ($\lfloor D \rfloor = \sum_E \lceil d_E \rceil E$), where for $x \in \mathbb{R}$ we denote $\lfloor x \rfloor = \max\{n \in \mathbb{Z}; n \le x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{Z}; x \le n\}$. The fractional part of D is defined as $\{D\} = \sum_E \{d_E\} E$, where for $x \in \mathbb{R}$ we denote $\{x\} = x - \lfloor x \rfloor$.

Definition 4.1.6. Let D be an $\mathbb{R}$-Weil divisor on X. We call D $\mathbb{R}$-*Cartier* ($\mathbb{Q}$-*Cartier*, *Cartier*) if there exists an open covering $X = \cup_i U_i$ such that $D|_{U_i} \sim_\mathbb{R} 0$ ($D|_{U_i} \sim_\mathbb{Q} 0$, $D|_{U_i} \sim 0$) for all i.

4.1.4 Complements of effective Cartier divisors

Lemma 4.1.7. *Let D be an effective Cartier divisor on a Noetherian scheme X. Let $U = X \setminus \operatorname{Supp} D$ and consider the open embedding $w \colon U \subseteq X$. Then*

1) *w is an affine morphism.*

2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The natural inclusions $\mathcal{F}(mD) \subset \mathcal{F}(nD)$, for $m \le n$, form a directed family of $\mathcal{O}_X$-modules $(\mathcal{F}(mD))_{m \in \mathbb{Z}}$, and*

$$\varinjlim_m \mathcal{F}(mD) = w_*(\mathcal{F}|_U).$$

3) Let $\pi\colon X \to S$ be a morphism and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then

$$\varinjlim_m R^q\pi_*\mathcal{F}(mD)\tilde{\to}R^q(\pi|_U)_*(\mathcal{F}|_U)$$

for all q.

Proof. Let $X = \cup_\alpha V_\alpha$ be an affine open covering such $D = (f_\alpha)_\alpha$, for non-zero divisors $f_\alpha \in \Gamma(V_\alpha, \mathcal{O}_{V_\alpha})$ such that $f_\alpha f_\beta^{-1} \in \Gamma(V_\alpha \cap V_\beta, \mathcal{O}_X^\times)$ for all α, β.

The set $w^{-1}(V_\alpha) = U \cap V_\alpha = D(f_\alpha)$ is affine, so 1) holds. Statement 2) is local, equivalent to the known property

$$\Gamma(D(f_\alpha), \mathcal{F}) = \Gamma(V_\alpha, \mathcal{F})_{f_\alpha} = \varinjlim_m \Gamma(V_\alpha, \mathcal{F}(mD)) = \Gamma(V_\alpha, \varinjlim_m \mathcal{F}(mD)).$$

For 3), directed limits commute with cohomology on quasi-compact topological spaces. Therefore

$$\varinjlim_m R^q\pi_*\mathcal{F}(mD)\tilde{\to}R^q\pi_*(\varinjlim_m \mathcal{F}(mD)) = R^q\pi_*(w_*(\mathcal{F}|_U)).$$

Since w is affine, the Leray spectral sequence for w degenerates to isomorphisms

$$R^q\pi_*(w_*(\mathcal{F}|_U))\tilde{\to}R^q(\pi|_U)_*(\mathcal{F}|_U).$$

Therefore 3) holds. □

4.1.5 Convention on algebraic varieties

Throughout this chapter, a variety is a reduced scheme of finite type over an algebraically closed field k of characteristic zero.

4.1.6 Explicit Deligne-Du Bois complex for normal crossing varieties

Let X be a variety with at most *normal crossing singularities*. That is, for every point $P \in X$, there exist $n \geq 1$, $I \subseteq \{1, \ldots, n\}$, and an isomorphism of complete local k-algebras

$$\frac{k[[T_1, \ldots, T_n]]}{(\prod_{i \in I} T_i)}\tilde{\to}\widehat{\mathcal{O}}_{X,P}.$$

Let $\pi\colon \bar{X} \to X$ be the normalization. For $p \geq 0$, define the $\mathcal{O}_X$-module $\tilde{\Omega}^p_{X/k}$ to be the image of the natural map $\Omega^p_{X/k} \to \pi_*\Omega^p_{\bar{X}/k}$. We have induced differentials $d\colon \tilde{\Omega}^p_{X/k} \to \tilde{\Omega}^{p+1}_{X/k}$, and $\tilde{\Omega}^\bullet_{X/k}$ becomes a differential complex of $\mathcal{O}_X$-modules. We call the hypercohomology group $\mathbb{H}^r(X, \tilde{\Omega}^\bullet_{X/k})$ the r-th *de Rham cohomology group* of X/k, and denote it by

$$H^r_{DR}(X/k).$$

If the base field is understood, we usually drop it from notation. Let $X_\bullet$ be the simplicial algebraic variety induced by π (see [20]). Its components are $X_n = (\bar{X}/X)^{\Delta_n}$, and the simplicial maps are naturally induced. We have a natural augmentation

$$\epsilon \colon X_\bullet \to X.$$

We have $X_0 = \bar{X}$, $X_1 = X_0 \times_X X_0$, $\epsilon_0 = \pi$ and $\delta_0, \delta_1 \colon X_1 \to X_0$ are the natural projections. For $p \geq 0$, let $\Omega^p_{X_\bullet}$ be the simplicial $\mathcal{O}_{X_\bullet}$-module with components $\Omega^p_{X_n}$ $(n \geq 0)$. The $\mathcal{O}_X$-module $\epsilon_*(\Omega^p_{X_\bullet})$ is defined as the kernel of the homomorphism

$$\delta_1^* - \delta_0^* \colon \epsilon_{0*}\Omega^p_{X_0} \to \epsilon_{1*}\Omega^p_{X_1}.$$

By [21, Lemme 2], ϵ is a smooth resolution, and $R^i\epsilon_*(\Omega^p_{X_\bullet}) = 0$ for $i > 0, p \geq 0$.

Lemma 4.1.8. *For every p, $\tilde{\Omega}^p_X = \epsilon_*(\Omega^p_{X_\bullet})$.*

Proof. Since $\pi \circ \delta_0 = \pi \circ \delta_1$, we obtain an inclusion $\tilde{\Omega}^p_X \subseteq \epsilon_*(\Omega^p_{X_\bullet})$. The opposite inclusion may be checked locally, in an étale neighborhood of each point. Therefore we may suppose

$$X : (\prod_{i=1}^c z_i = 0) \subset \mathbb{A}^{d+1}.$$

Then X has c irreducible components $X_1, \ldots, X_c$, each of them isomorphic to $\mathbb{A}^d$. The normalization $\bar{X}$ is the disjoint union of the X_i. Therefore $\Gamma(X, \epsilon_*(\Omega^p_{X_\bullet}))$ consists of c-uples $(\omega_1, \ldots, \omega_c)$ where $\omega_i \in \Gamma(X_i, \Omega^p_{X_i})$ satisfy the cycle condition $\omega_i|_{X_i \cap X_j} = \omega_j|_{X_i \cap X_j}$ for every $i < j$.

By induction on c, we show that $\Gamma(X, \epsilon_*(\Omega^p_{X_\bullet}))$ is the image of the homomorphism $\Gamma(\mathbb{A}^{d+1}, \Omega^p_{\mathbb{A}^{d+1}}) \to \Gamma(\bar{X}, \Omega^p_{\bar{X}})$. The case $c = 1$ is clear. Suppose $c \geq 2$. Let $\alpha = (\omega_1, \ldots, \omega_c)$ be an element of $\Gamma(X, \epsilon_*(\Omega^p_{X_\bullet}))$. There exists $\omega \in \Gamma(\mathbb{A}^{d+1}, \Omega^p_{\mathbb{A}^{d+1}})$ such that $\omega_c = \omega|_{X_c}$. Then we may replace α by $\alpha - \omega|_X$, so that

$$\alpha = (\omega_1, \ldots, \omega_{c-1}, 0).$$

The cycle conditions for pairs $i < c$ give $\omega_i = z_c \eta_i$, for some $\eta_i \in \Gamma(X_i, \Omega^p_{X_i})$. The other cycle conditions are equivalent to the fact that $(\eta_1, \ldots, \eta_{c-1}) \in \Gamma(X', \epsilon_*(\Omega^p_{X'_\bullet}))$, where $X' : (\prod_{i=1}^{c-1} z_i = 0) \subset \mathbb{A}^{d+1}$. By induction, there exists $\eta \in \Gamma(\mathbb{A}^{d+1}, \Omega^p_{\mathbb{A}^{d+1}})$ such that $\eta_i = \eta|_{X_i}$ for $1 \leq i \leq c-1$. Then $\alpha = z_c \eta|_X$.

The map $\Gamma(\mathbb{A}^{d+1}, \Omega^p_{\mathbb{A}^{d+1}}) \to \Gamma(\bar{X}, \Omega^p_{\bar{X}})$ factors through the *surjection* $\Gamma(\mathbb{A}^{d+1}, \Omega^p_{\mathbb{A}^{d+1}}) \to \Gamma(X, \Omega^p_X)$. Therefore its image is the same as the image of $\Gamma(X, \Omega^p_X) \to \Gamma(\bar{X}, \Omega^p_{\bar{X}})$. $\square$

It follows that $\tilde{\Omega}^\bullet_X \to R\epsilon_*(\Omega^\bullet_{X_\bullet})$ is a quasi-isomorphism. From [28, 20] (see [22, Théorème 4.5]), we deduce

Theorem 4.1.9. *The filtered complex $(\tilde{\Omega}^\bullet_X, F)$, where F is the naive filtration, induces a spectral sequence in hypercohomology*

$$E_1^{pq} = H^q(X, \tilde{\Omega}^p_X) \Longrightarrow \mathbb{H}^{p+q}(X, \tilde{\Omega}^\bullet_X) = H^{p+q}_{DR}(X/k).$$

If X is proper, this spectral sequence degenerates at E_1.

Note $\tilde{\Omega}^0_X = \mathcal{O}_X$. If $d = \dim X$, then $\tilde{\Omega}^d_X = \pi_*\Omega^d_{\bar{X}}$, which is a locally free $\mathcal{O}_X$-module if and only if X has no singularities. If X is non-singular, the natural surjections $\Omega^p_X \to \tilde{\Omega}^p_X$ are isomorphisms, for all p. So our definition of de Rham cohomology for varieties with at most normal crossing singularities is consistent with Grothendieck's definition [28] for nonsingular varieties.

4.1.7 Differential forms with logarithmic poles

Let (X, Σ) be a *log smooth pair*, that is X is a nonsingular variety and Σ is an effective divisor with at most normal crossing singularities. Denote $U = X \setminus \Sigma$. Let $w\colon U \to X$ be the inclusion. Then $w_*(\Omega^\bullet_U)$ is the complex of rational differentials on X which are regular on U. We identify it with the union of $\Omega^\bullet_X \otimes \mathcal{O}_X(m\Sigma)$, after all $m \geq 0$.

Let $p \geq 0$. The *sheaf of germs of differential p-forms on X with at most logarithmic poles along* Σ, denoted $\Omega^p_X(\log \Sigma)$ (see [18]), is the sheaf whose sections on an open subset V of X are

$$\Gamma(V, \Omega^p_X(\log \Sigma)) = \{\omega \in \Gamma(V, \Omega^p_X \otimes \mathcal{O}_X(\Sigma)); d\omega \in \Gamma(V, \Omega^{p+1}_X \otimes \mathcal{O}_X(\Sigma))\}.$$

It follows that $\{\Omega^p_X(\log \Sigma), d^p\}_p$ becomes a subcomplex of $w_*(\Omega^\bullet_U)$. It is called the *logarithmic de Rham complex of* (X, Σ), denoted by $\Omega^\bullet_X(\log \Sigma)$.

Let $n = \dim X$. Then $\Omega^p_X(\log \Sigma) = 0$ if $p \notin [0, n]$. And $\Omega^n_X(\log \Sigma) = \Omega^p_X \otimes \mathcal{O}_X(\Sigma) = \mathcal{O}_X(K_X + \Sigma)$, where K_X is the canonical divisor of X.

Lemma 4.1.10. *Let $0 \leq p \leq n$. Then $\Omega^p_X(\log \Sigma)$ is a coherent locally free extension of Ω^p_U to X. Moreover, $\Omega^0_X(\log \Sigma) = \mathcal{O}_X$, $\wedge^p \Omega^1_X(\log \Sigma) = \Omega^p_X(\log \Sigma)$, and the wedge product induces a perfect pairing*

$$\Omega^p_X(\log \Sigma) \otimes_{\mathcal{O}_X} \Omega^{n-p}_X(\log \Sigma) \to \Omega^n_X(\log \Sigma).$$

Proof. The $\mathcal{O}_X$-module $\Omega^p_X(\log \Sigma)$ is coherent, being a subsheaf of $\Omega^p_X \otimes \mathcal{O}_X(\Sigma)$. The statements may be checked near a fixed point, after passing to completion. Therefore it suffices to verify the statements at the point $P = 0$ for $X = \mathbb{A}^n_k$ and $\Sigma = (\prod_{i \in J} z_i)$. As in [25, Properties 2.2] for example, it can be checked that in this case $\Omega^p_X(\log \Sigma)_P$ is the free $\mathcal{O}_{X,P}$-module with basis

$$\{\frac{dz^I}{\prod_{i \in J \cap I} z_i}; I \subseteq \{1, \ldots, n\}, |I| = p\},$$

where for $I = \{i_1 < \cdots < i_p\}$, dz^I denotes $dz_{i_1} \wedge \cdots \wedge dz_{i_p}$. And $\prod_{i \in \emptyset} z_i = 1$. All the statements follow in this case. □

Theorem 4.1.11. *[11, 28, 18, 25] The inclusion $\Omega^\bullet_X(\log \Sigma) \subset w_*(\Omega^\bullet_U)$ is a quasi-isomorphism.*

Proof. We claim that $\Omega^\bullet_X(\log \Sigma) \otimes \mathcal{O}_X(D)$ is a subcomplex of $w_*(\Omega^\bullet_U)$, for every divisor D supported by Σ. Indeed, the sheaves in question are locally free, so it suffices to check the statement over the open subset $X \setminus \operatorname{Sing}\Sigma$, whose complement has codimension at least two in X. Therefore we may suppose Σ is non-singular. After passing to completion at a fixed point, it suffices to check the claim at $P = 0$ for $X = \mathbb{A}^1_k$ and $\Sigma = (z)$. This follows from the formula

$$d(1 \otimes z^m) = m \cdot \frac{dz}{z} \otimes z^m \ (m \in \mathbb{Z}).$$

We obtain an increasing filtration of $w_*(\Omega^\bullet_U)$ by sub-complexes

$$\mathcal{K}_m = \Omega^*_X(\log \Sigma) \otimes \mathcal{O}_X(m\Sigma) \ (m \geq 0).$$

We claim that the quotient complex $\mathcal{K}_m/\mathcal{K}_{m-1}$ is acyclic, for every $m > 0$. Since $\mathcal{K}_0 = \Omega^\bullet_X(\log \Sigma)$ and $\cup_{m\geq 0}\mathcal{K}_m = w_*(\Omega^\bullet_U)$, this implies that the quotient complex $w_*(\Omega^\bullet_U)/\Omega^\bullet_X(\log \Sigma)$ is acyclic, or equivalently $\Omega^\bullet_X(\log \Sigma) \subset w_*(\Omega^\bullet_U)$ is a quasi-isomorphism.

To prove that $\mathcal{K}_m/\mathcal{K}_{m-1}$ $(m > 0)$ is acyclic, note that we may work locally near a fixed point, and we may also pass to completion (since the components of the two complexes are coherent). Therefore it suffices to verify the claim at $P = 0$ for $X = \mathbb{A}^n_k$ and $\Sigma = (\prod_{i\in J} z_i)$. If we denote $H_j = (z_j)$, the claim in this case follows from the stronger statement of [25, Lemma 2.10]: the inclusion $\Omega^*_X(\log \Sigma) \otimes \mathcal{O}_X(D) \subset \Omega^*_X(\log \Sigma) \otimes \mathcal{O}_X(D + H_j)$ is a quasi-isomorphism, for every effective divisor D supported by Σ and every $j \in J$. □

Theorem 4.1.12. *[19] The filtered complex $(\Omega^\bullet_X(\log \Sigma), F)$, where F is the naive filtration, induces a spectral sequence in hypercohomology*

$$E_1^{pq} = H^q(X, \Omega^p_X(\log \Sigma)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log \Sigma)).$$

If X is proper, this spectral sequence degenerates at E_1.

Proof. If $k = \mathbb{C}$, the claim follows from [19] and GAGA. By the Lefschetz principle, the claim extends to the case when k is a field of characteristic zero. □

Lemma 4.1.13. *For each $p \geq 0$, we have a short exact sequence*

$$0 \to \mathcal{I}_\Sigma \otimes \Omega^p_X(\log \Sigma) \to \Omega^p_X \to \tilde{\Omega}^p_\Sigma \to 0.$$

Proof. Let $\pi \colon \bar{\Sigma} \to \Sigma$ be the normalization. We claim that we have an exact sequence

$$0 \to \mathcal{I}_\Sigma \otimes \Omega^p_X(\log \Sigma) \to \Omega^p_X \to \pi_*\Omega^p_{\bar{\Sigma}},$$

where the second arrow is induced by the inclusion $\Omega^p_X(\log \Sigma) \subseteq \Omega^p_X \otimes \mathcal{O}_X(\Sigma)$, and the third arrow is the restriction homomorphism $\omega \mapsto \omega|_{\bar{\Sigma}}$. Indeed, denote $\mathcal{K} = \operatorname{Ker}(\Omega^p_X \to \pi_*\Omega^p_{\bar{\Sigma}})$. We have to show that $\mathcal{I}_\Sigma \otimes \Omega^p_X(\log \Sigma) = \mathcal{K}$. This is a local statement which can be checked locally near each point, and since the sheaves are coherent, we may also pass to completion. Therefore it suffices to check the equality at $P = 0$ in the special case $X = \mathbb{A}^n_k$, $\Sigma = (\prod_{j\in J} z_j)$. From the explicit description of local bases for the logarithmic sheaves, the claim holds in this case.

Finally, we compute the image of the restriction. The restriction factors through the *surjection* $\Omega^p_X \to \Omega^p_\Sigma$. Therefore the image coincides with the image of $\Omega^p_\Sigma \to \pi_*\Omega^p_{\bar{\Sigma}}$, which by definition is $\tilde{\Omega}^p_\Sigma$. □

4.1.8 The cyclic covering trick

Let X be an irreducible normal variety, let T be a $\mathbb{Q}$-Weil divisor on X such that $T \sim_{\mathbb{Q}} 0$. Let $r \geq 1$ be minimal such that $rT \sim 0$. Choose a rational function $\varphi \in k(X)^\times$ such that $(\varphi) = rT$. Denote by

$$\tau' \colon X' \to X$$

the normalization of X in the field extension $k(X) \subseteq k(X)(\sqrt[r]{\varphi})$. The normal variety X' is irreducible, since r is minimal. Choose $\psi \in k(X')^\times$ such that $\psi^r = \tau'^*\varphi$. One computes

$$\tau'_*\mathcal{O}_{X'} = \oplus_{i=0}^{r-1}\mathcal{O}_X(\lfloor iT \rfloor)\psi^i.$$

The finite morphism τ' is Galois, with Galois group cyclic of order r. Moreover, τ' is étale over $X \setminus \operatorname{Supp}\{T\}$.

Suppose now that (X, Σ) is a log smooth pair structure on X, and the fractional part $\{T\}$ is supported by Σ. Then τ' is flat, X' has at most quotient singularities (in the étale topology), and $X' \setminus \tau'^{-1}\Sigma$ is nonsingular. Let $\mu \colon Y \to X'$ be an embedded resolution of singularities of $(X', \tau'^{-1}\Sigma)$. If we denote $\tau = \tau' \circ \mu$, then $\tau^{-1}(\Sigma) = \Sigma_Y$ is a normal crossings divisor and $\mu \colon Y \setminus \Sigma_Y \to X' \setminus \tau'^{-1}\Sigma$ is an isomorphism. We obtain a commutative diagram

$$\begin{tikzcd} X' \arrow[d, "\tau'"'] & Y \arrow[l, "\mu"'] \arrow[dl, "\tau"] \\ X & \end{tikzcd}$$

From Theorems 1.0.1 and 1.0.2 we obtain

Lemma 4.1.14. *$R^q\tau_*\Omega^p_Y(\log \Sigma_Y) = 0$ for $q \neq 0$, and*

$$\begin{aligned} \tau_*\Omega^p_Y(\log \Sigma_Y) &= \Omega^p_X(\log \Sigma_X) \otimes \tau'_*\mathcal{O}_{X'} \\ &\simeq \oplus_{i=0}^{r-1}\Omega^p_X(\log \Sigma_X) \otimes \mathcal{O}_X(\lfloor iT \rfloor). \end{aligned}$$

This statement was proved in [23, Lemme 1.2, 1.3] with two extra assumptions: X is projective, and Σ is a *simple* normal crossing divisor, that is it has normal crossing singularities and its irreducible components are smooth.

4.2 Injectivity for open embeddings

Let (X, Σ) be a log smooth pair, with X proper. Denote $U = X \setminus \Sigma$.

Theorem 4.2.1. *The restriction homomorphism $H^q(X, \mathcal{O}_X(K_X + \Sigma)) \to H^q(U, \mathcal{O}_U(K_U))$ is injective, for all q.*

Proof. Consider the inclusion of filtered differential complexes of $\mathcal{O}_X$-modules

$$(\Omega^\bullet_X(\log \Sigma), F) \subset (w_*(\Omega^\bullet_U), F),$$

where F is the naive filtration of a complex. Let $n = \dim X$. The inclusion $F^n \subseteq F^0$ induces a commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^{q+n}(X, F^n\Omega^\bullet_X(\log\Sigma)) & \xrightarrow{\beta} & \mathbb{H}^{q+n}(X, \Omega^\bullet_X(\log\Sigma)) \\
\downarrow{\scriptstyle \alpha^n} & & \downarrow{\scriptstyle \alpha} \\
\mathbb{H}^{q+n}(X, F^n w_*(\Omega^\bullet_U)) & \longrightarrow & \mathbb{H}^{q+n}(X, w_*(\Omega^\bullet_U))
\end{array}$$

By Theorem 4.1.11, α is an isomorphism. Theorem 4.1.12 implies that β is injective. Therefore $\alpha \circ \beta$ injective. Therefore α^n is injective.

But $F^n\Omega^\bullet_X(\log\Sigma) = \Omega^n_X(\log\Sigma)[-n]$ and $F^n w_*(\Omega^\bullet_U) = w_*(\Omega^n_U)[-n]$. Therefore α^n becomes

$$\alpha^n \colon H^q(X, \Omega^n_X(\log\Sigma)) \to H^q(X, w_*(\Omega^n_U))$$

The morphism $w \colon U \subset X$ is affine, so $H^q(X, w_*(\Omega^n_U)) \to H^q(U, \Omega^n_U)$ is an isomorphism. Therefore α^n becomes the restriction map

$$\alpha^n \colon H^q(X, \Omega^n_X(\log\Sigma)) \to H^q(U, \Omega^n_U).$$

□

Corollary 4.2.2. *Let T be a $\mathbb{Q}$-divisor on X such that $T \sim_{\mathbb{Q}} 0$ and* $\operatorname{Supp}\{T\} \subseteq \Sigma$. *In particular, $T|_U$ has integer coefficients. Then the restriction homomorphism*

$$H^q(X, \mathcal{O}_X(K_X + \Sigma + \lfloor T \rfloor)) \to H^q(U, \mathcal{O}_U(K_U + T|_U))$$

is injective, for all q.

Proof. We use the notations of paragraph 1.H. Denote $V = \tau^{-1}(U) = Y \setminus \Sigma_Y$. By Theorem 4.2.1, the restriction

$$H^q(Y, \mathcal{O}_Y(K_Y + \Sigma_Y)) \to H^q(V, \mathcal{O}_V(K_V))$$

is injective. By the Leray spectral sequence and Lemma 4.1.14, the restriction

$$H^q(X, \tau_*\mathcal{O}_Y(K_Y + \Sigma_Y)) \to H^q(U, \tau_*\mathcal{O}_V(K_V))$$

is injective. Equivalently, the direct sum of restrictions

$$\oplus_{i=0}^{r-1}(H^q(X, \mathcal{O}_X(K_X + \Sigma + \lfloor iT \rfloor)) \to H^q(U, \mathcal{O}_U(K_U + iT|_U))$$

is injective. For $i = 1$, we obtain the claim. □

Theorem 4.2.3. *Let X be a proper non-singular variety. Let U be an open subset of X such that $X \setminus U$ is a normal crossings divisor with irreducible components $(E_i)_i$. Let L be a Cartier divisor on X such that $L \sim_{\mathbb{R}} K_X + \sum_i b_i E_i$, with $0 < b_i \leq 1$ for all i. Then the restriction homomorphism*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(U, \mathcal{O}_U(L|_U))$$

is injective, for all q.

Proof. Choose a labeling of the components, say $E_1, \ldots, E_l$. Since $L - K_X$ has integer coefficients, it follows by Lemma 4.1.5 that the set

$$V = \{x \in \mathbb{R}^l ; L \sim_{\mathbb{R}} K_X + \sum_{i=1}^{l} x_i E_i\}$$

is a non-empty affine linear subspace of $\mathbb{R}^l$ defined over $\mathbb{Q}$. Then $(b_1, \ldots, b_l) \in V \cap (0,1]^l$ can be approximated by $(b'_1, \ldots, b'_l) \in V \cap (0,1]^l \cap \mathbb{Q}^l$, such that $b'_i = b_i$ if $b_i \in \mathbb{Q}$. Since $0 \sim_{\mathbb{R}} -L + K_X + \sum_i b'_i E_i$ and the right hand side has rational coefficients, it follows that $0 \sim_{\mathbb{Q}} -L + K_X + \sum_i b'_i E_i$.

In conclusion, $L \sim_{\mathbb{Q}} K_X + \sum_i b'_i E_i$ and $0 < b'_i \le 1$ for all i. Set $\Sigma = \sum_i E_i$ and $T = L - K_X - \sum_i b'_i E_i$. Then $T \sim_{\mathbb{Q}} 0$, $\{T\} = \sum_i \{-b'_i\} E_i$ and $L = K_X + \Sigma + \lfloor T \rfloor$. Corollary 4.2.2 gives the claim. □

Remark 4.2.4. Let $U \subseteq U' \subseteq X$ be another open subset. From the commutative diagram

$$\begin{tikzcd}
H^q(X, \mathcal{O}_X(L)) \arrow[rr] \arrow[dr] & & H^q(U, \mathcal{O}_U(L|_U)) \\
& H^q(U', \mathcal{O}_{U'}(L|_{U'})) \arrow[ur] &
\end{tikzcd}$$

it follows that $H^q(X, \mathcal{O}_X(L)) \to H^q(U', \mathcal{O}_{U'}(L|_{U'}))$ is injective for all q.

Remark 4.2.5. Recall that for an $\mathcal{O}_X$-module $\mathcal{F}$, $\Gamma_\Sigma(X, \mathcal{F})$ is defined as the kernel of $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}|_U)$. The functor $\Gamma_\Sigma(X, \cdot)$ is left exact. Its derived functors, denoted $(H^i_\Sigma(X, \mathcal{F}))_{i \ge 0}$, are called the cohomology of X modulo U, with coefficients in $\mathcal{F}$. For every $\mathcal{F}$ we have long exact sequences

$$0 \to \Gamma_\Sigma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}|U) \to H^1_\Sigma(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}|U) \to \cdots$$

Therefore Theorem 4.2.3 says that the homomorphism $H^q_\Sigma(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L))$ is zero for all q. Equivalently, $\Gamma_\Sigma(X, \mathcal{O}_X(L)) = 0$, and for all q we have short exact sequences

$$0 \to H^q(X, \mathcal{O}_X(L)) \to H^q(U, \mathcal{O}_U(L|_U)) \to H^{q+1}_\Sigma(X, \mathcal{O}_X(L)) \to 0.$$

Remark 4.2.6. Theorem 4.2.3 is also equivalent to the following statement, which generalizes the original result of Esnault and Viehweg [25, Theorem 5.1]: let D be an effective Cartier divisor supported by Σ. Then the long exact sequence induced in cohomology by the short exact sequence $0 \to \mathcal{O}_X(L) \to \mathcal{O}_X(L + D) \to \mathcal{O}_D(L + D) \to 0$ breaks up into short exact sequences

$$0 \to H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)) \to H^q(D, \mathcal{O}_D(L + D)) \to 0 \ (q \ge 0).$$

Indeed, let D be as above. We have a commutative diagram

$$\begin{tikzcd}
H^q(X, \mathcal{O}_X(L)) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H^q(X, \mathcal{O}_X(L + D)) \arrow[d] \\
H^q(U, \mathcal{O}_U(L|_U)) \arrow[r, "\gamma"] & H^q(U, \mathcal{O}_U((L + D)|_U))
\end{tikzcd}$$

Since D is disjoint from U, γ is an isomorphism. By Theorem 4.2.3, β is injective. Therefore $\gamma \circ \beta$ is injective. It follows that α is injective. Conversely, suppose $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ is injective for all divisors D supported by $X \setminus U$. Then $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + m\Sigma))$ is injective for every $m \geq 0$. Lemma 4.1.1 implies the injectivity of

$$H^q(X, \mathcal{O}_X(L)) \to \varinjlim_m H^q(X, \mathcal{O}_X(L + m\Sigma)).$$

By Lemma 4.1.7, this is isomorphic to the homomorphism $H^q(X, \mathcal{O}_X(L)) \to H^q(U, \mathcal{O}_U(L|_U))$.

Corollary 4.2.7. *Let D be an effective Cartier divisor supported by Σ. Then*

$$0 \to H^q(X, \mathcal{O}_X(K_X + \Sigma)) \to H^q(X, \mathcal{O}_X(K_X + \Sigma + D)) \to H^q(D, \mathcal{O}_D(K_X + \Sigma + D)) \to 0$$

is a short exact sequence, for all q.

Proof. By Remark 4.2.6 for $L = K_X + \Sigma$. $\square$

Corollary 4.2.8. *The homomorphism $\Gamma(X, \mathcal{O}_X(K_X + 2\Sigma)) \to \Gamma(\Sigma, \mathcal{O}_\Sigma(K_X + 2\Sigma))$ is surjective.*

If Σ is the general member of a base point free linear system, this is the original result of Tankeev [62, Proposition 1].

4.3 Differential forms of intermediate degree

Let (X, Σ) be a log smooth pair such that X is proper and $U = X \setminus \Sigma$ is contained in an affine open subset of X.

Theorem 4.3.1. *$H^q(X, \Omega^p_X(\log \Sigma)) = 0$ for $p+q > \dim X$. In particular, $H^q(X, \mathcal{O}_X(K_X + \Sigma)) = 0$ for $q > 0$.*

Proof. Consider the logarithmic de Rham complex $\Omega^\bullet_X(\log \Sigma)$. Let U' be an affine open subset of X containing U. The inclusions $U \subseteq U' \subset X$ induce a commutative diagram

$$\begin{tikzcd}
\mathbb{H}^r(X, \Omega^\bullet_X(\log \Sigma)) \arrow[rr] \arrow[dr] & & \mathbb{H}^r(U, \Omega^\bullet_U) \\
& \mathbb{H}^r(U', \Omega^\bullet_X(\log \Sigma)|_{U'}) \arrow[ur] &
\end{tikzcd}$$

Since U' is affine, $H^q(U', \Omega^p_X(\log \Sigma)|_{U'}) = 0$ for $q > 0$. Therefore $\mathbb{H}^r(U', \Omega^\bullet_X(\log \Sigma)|_{U'})$ is the r-th homology of the differential complex $\Gamma(U', \Omega^\bullet_X(\log \Sigma))$. Since $\Omega^p_X(\log \Sigma) = 0$ for $p > \dim X$, we obtain

$$\mathbb{H}^r(U', \Omega^\bullet_X(\log \Sigma)|_{U'}) = 0 \text{ for } r > \dim X.$$

Let $r > \dim X$. It follows that the horizontal map is zero. But it is an isomorphism by Theorem 4.1.11. Therefore

$$\mathbb{H}^r(X, \Omega_X^\bullet(\log \Sigma)) = 0.$$

By Theorem 4.1.12, we have a non-canonical isomorphism

$$\mathbb{H}^r(X, \Omega_X^\bullet(\log \Sigma)) \simeq \oplus_{p+q=r} H^q(X, \Omega_X^p(\log \Sigma))$$

Therefore $H^q(X, \Omega_X^p(\log \Sigma)) = 0$ for all $p + q = r$. □

Let T be a $\mathbb{Q}$-divisor on X such that $T \sim_{\mathbb{Q}} 0$ and $\operatorname{Supp}\{T\} \subseteq \Sigma$. In particular, $T|_U$ has integer coefficients.

Theorem 4.3.2. *$H^q(X, \Omega_X^p(\log \Sigma) \otimes \mathcal{O}_X(\lfloor T \rfloor)) = 0$ for $p + q > \dim X$. In particular, $H^q(X, \mathcal{O}_X(K_X + \Sigma + \lfloor T \rfloor)) = 0$ for $q > 0$.*

Proof. We use the notations of paragraph 1.H. Let $X \setminus \Sigma \subseteq U'$, with U' an affine open subset of X. Let $V' = \tau^{-1}(U')$. By Lemma 4.1.14, the Leray spectral sequence associated to $\tau|_{V'} \colon V' \to U'$ and $\Omega_Y^p(\log \Sigma_Y)|_{V'}$ degenerates into isomorphisms

$$H^q(U', (\tau|_{V'})_* \Omega_Y^p(\log \Sigma_Y)|_{V'}) \xrightarrow{\sim} H^q(V', \Omega_Y^p(\log \Sigma_Y)|_{V'}).$$

Since U' is affine, the left hand side is zero for $q > 0$. Therefore

$$H^q(V', \Omega_Y^p(\log \Sigma_Y)|_{V'}) = 0 \text{ for } q > 0.$$

In particular, the spectral sequence

$$E_1^{pq} = H^q(V', \Omega_Y^p(\log \Sigma_Y)|_{V'}) \Longrightarrow \mathbb{H}^q(V', \Omega_Y^\bullet(\log \Sigma_Y)|_{V'})$$

degenerates into isomorphisms

$$h^r(\Gamma(V', \Omega_Y^\bullet(\log \Sigma_Y))) \simeq \mathbb{H}^r(V', \Omega_Y^\bullet(\log \Sigma_Y)|_{V'}),$$

where the first term is the r-th homology group of the differential complex $\Gamma(V', \Omega_Y^\bullet(\log \Sigma_Y))$. Since $\Omega_Y^p(\log \Sigma_Y) = 0$ for $p > \dim Y$, we obtain

$$\mathbb{H}^r(V', \Omega_Y^\bullet(\log \Sigma_Y)|_{V'}) = 0 \text{ for } r > \dim Y.$$

Let $V = \tau^{-1}(U) = Y \setminus \Sigma_Y$. The restriction map

$$\mathbb{H}^r(Y, \Omega_Y^\bullet(\log \Sigma_Y)) \to \mathbb{H}^r(V, \Omega_Y^\bullet(\log \Sigma_Y)|_V)$$

is an isomorphism by Theorem 4.1.11. It factors through $\mathbb{H}^r(V', \Omega_Y^\bullet(\log \Sigma_Y)|_{V'})$, hence it is zero for $r > \dim Y$. Therefore

$$\mathbb{H}^r(Y, \Omega_Y^\bullet(\log \Sigma_Y)) = 0 \text{ for } r > \dim Y.$$

By Theorem 4.1.12, $\mathbb{H}^r(Y, \Omega_Y^\bullet(\log \Sigma_Y)) \simeq \oplus_{p+q=r} H^q(Y, \Omega_Y^p(\log \Sigma_Y))$. Therefore

$$H^q(Y, \Omega_Y^p(\log \Sigma_Y)) = 0 \text{ for } p + q > \dim Y.$$

The cyclic group of order r acts on $H^q(Y, \Omega_Y^p(\log \Sigma_Y))$, with eigenspace decomposition

$$\oplus_{i=0}^{r-1} H^q(X, \Omega_X^p(\log \Sigma) \otimes \mathcal{O}_X(\lfloor iT \rfloor)).$$

Therefore $H^q(X, \Omega_X^p(\log \Sigma) \otimes \mathcal{O}_X(\lfloor T \rfloor)) = 0$. □

4.3.1 Applications

Corollary 4.3.3. *$H^q(X, \Omega^p_X(\log \Sigma) \otimes \mathcal{O}_X(-\Sigma - \lfloor T \rfloor)) = 0$ for $p+q < \dim X$. In particular, $H^q(X, \mathcal{O}_X(-\Sigma - \lfloor T \rfloor)) = 0$ for all $q < \dim X$.*

Proof. This is the dual form of Theorem 4.3.2, using Serre duality and the isomorphism $(\Omega^p_X(\log \Sigma))^\vee \simeq \Omega^{\dim X - p}_X(\log \Sigma) \otimes \mathcal{O}_X(-K_X - \Sigma)$. □

For $T = 0$, we obtain $H^q(X, \mathcal{I}_\Sigma \otimes \Omega^p_X(\log \Sigma)) = 0$ for all $p + q < \dim X$. In particular, $H^q(X, \mathcal{I}_\Sigma) = 0$ for all $q < \dim X$.

Corollary 4.3.4. *The homomorphism $H^q(X, \Omega^p_X \otimes \mathcal{O}_X(-\lfloor T \rfloor)) \to H^q(\Sigma, \tilde{\Omega}^p_\Sigma \otimes \mathcal{O}_\Sigma(-\lfloor T \rfloor))$ is bijective for $p + q < \dim \Sigma$ and injective for $p + q = \dim \Sigma$.*

Proof. Denote $K^{pq} = H^q(X, \Omega^p_X(\log \Sigma) \otimes \mathcal{O}_X(-\Sigma - \lfloor T \rfloor))$. The short exact sequence of Lemma 4.1.13 induces a long exact sequence in cohomology

$$K^{pq} \to H^q(X, \Omega^p_X \otimes \mathcal{O}_X(-\lfloor T \rfloor)) \stackrel{\alpha^{qp}}{\to} H^q(\Sigma, \tilde{\Omega}^p_\Sigma \otimes \mathcal{O}_\Sigma(-\lfloor T \rfloor)) \to K^{p,q+1}$$

By Corollary 4.3.3, α^{qp} is bijective for $q + 1 < \dim X - p$, and injective for $q + 1 = \dim X - p$. □

Corollary 4.3.5 (Weak Lefschetz)**.** *The restriction homomorphism $H^r_{DR}(X/k) \to H^r_{DR}(\Sigma/k)$ is bijective for $r < \dim \Sigma$ and injective for $r = \dim \Sigma$.*

Proof. Set $T = 0$. The homomorphism $H^q(X, \Omega^p_X) \to H^q(\Sigma, \tilde{\Omega}^p_\Sigma)$ is bijective for $p + q < \dim \Sigma$ and injective for $p+q = \dim \Sigma$. The Hodge to de Rham spectral sequence degenerates at E_1, for X/k by [17, Theorem 5.5] and for Σ/k by Theorem 4.1.9, and is compatible with the maps above. □

Corollary 4.3.6. *Suppose $\operatorname{Supp}\{T\} = \Sigma$. Then $H^q(X, \Omega^p_X(\log \Sigma) \otimes \mathcal{O}_X(\lfloor T \rfloor)) = 0$ for all $p + q \neq \dim X$.*

Proof. For $p + q > \dim X$, this follows from above. For $p + q < \dim X$, apply the dual form to $-T$, using $-\Sigma - \lfloor -T \rfloor = \lfloor T \rfloor$. □

Corollary 4.3.7. *Suppose $X \setminus \operatorname{Supp}\{T\}$ is contained in an affine open subset of X. Then $H^q(X, \mathcal{O}_X(\lfloor T \rfloor)) = 0$ for $q < \dim X$.*

Theorem 4.3.8 (Akizuki-Nakano)**.** *Let X be projective non-singular variety. Let L be an ample divisor. Then $H^q(X, \Omega^p_X(L)) = 0$ for $p + q > \dim X$. Dually, $H^q(X, \Omega^p_X(-L)) = 0$ for $p + q < \dim X$.*

Proof. There exists $r \geq 1$ such that the general member $Y \in |rL|$ is non-singular. Set $T = L - \frac{1}{r}Y$ and $\Sigma = Y$. Then $T \sim_{\mathbb{Q}} 0$, $\operatorname{Supp}\{T\} = \Sigma$ and $X \setminus \Sigma$ is affine. We also have $\lfloor T \rfloor = L - Y$. By Theorem 4.3.2, we obtain

$$H^q(X, \Omega^p_X(\log Y) \otimes \mathcal{O}_X(L - Y)) = 0 \text{ for } p + q > \dim Y.$$

The short exact sequence of Lemma 4.1.13, tensored by L, gives an exact sequence

$$H^q(X, \Omega^p_X(\log Y)(L - Y)) \to H^q(X, \Omega^p_X(L)) \to H^q(Y, \Omega^p_Y(L)).$$

Let $p + q > \dim X$. The first term is zero from above, and the third is zero by induction. Therefore $H^q(X, \Omega^p_X \otimes \mathcal{O}_X(L)) = 0$. □

Corollary 4.3.9 (Kodaira)**.** *Let X be projective non-singular variety. Let L be an ample divisor on X. Then $H^q(X, \mathcal{O}_X(K_X + L)) = 0$ for $q > 0$.*

4.4 Log pairs

A *log pair* (X, B) consists of a normal algebraic variety X, endowed with an $\mathbb{R}$-Weil divisor B such that $K_X + B$ is $\mathbb{R}$-Cartier. If B is effective, we call (X, B) a *log variety*.

A contraction $f : X \to Y$ is a proper morphism such that the natural homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

4.4.1 Totally canonical locus

Let (X, B) be a log pair. Let $\mu : X' \to X$ be a birational contraction such that $(X', \operatorname{Exc}(\mu) \cup \operatorname{Supp} \mu_*^{-1} B)$ is log smooth. Let

$$\mu^*(K_X + B) = K_{X'} + B_{X'}$$

be the induced log pair structure on X'. We say that $\mu : (X', B_{X'}) \to (X, B)$ is a log crepant birational contraction.

For a prime divisor E on X', $1 - \operatorname{mult}_E(B_{X'})$ is called the log discrepancy of (X, B) in the valuation of $k(X)$ defined by E, denoted $a(E; X, B)$ (see [4] for example).

Define an open subset of X by the formula $U = X \setminus \mu(\operatorname{Supp}(B_{X'})^{>0})$. The definition of U does not depend on the choice of μ, by the following

Lemma 4.4.1. *Let $\mu : (X', B') \to (X, B)$ be a log crepant proper birational morphism of log pairs with log smooth support. Then $\mu(\operatorname{Supp} B'^{>0}) = \operatorname{Supp} B^{>0}$.*

Proof. First, we claim that $B' \le \mu^* B$. Indeed, X is non-singular, so $K_{X'} - \mu^* K_X$ is effective μ-exceptional. From $\mu^*(K_X + B) = K_{X'} + B'$ we obtain

$$\mu^* B - B' = K_{X'} - \mu^* K_X \ge 0.$$

To prove the statement, denote $U = X \setminus \operatorname{Supp}(B^{>0})$. Then $B|_U \le 0$. The claim for $\mu|_{\mu^{-1}(U)} : (\mu^{-1}(U), B'|_{\mu^{-1}(U)}) \to (U, B|_U)$ gives $B'|_{\mu^{-1}(U)} \le 0$. Therefore $\mu(\operatorname{Supp} B'^{>0}) \subseteq \operatorname{Supp} B^{>0}$. For the opposite inclusion, note that $\operatorname{Supp} B^{>0}$ has codimension one. Let E be a prime in $\operatorname{Supp} B^{>0}$. Since μ is an isomorphism in a neighbourhood of the generic point of E, E also appears as a prime on X' and $\operatorname{mult}_E(B') = \operatorname{mult}_E(B) > 0$. Therefore $E \subseteq \mu(\operatorname{Supp} B'^{>0})$. □

We call U the *totally canonical locus* of (X,B). It is the largest open subset U of X with the property that every geometric valuation over U has log discrepancy at least 1 with respect to $(U,B|_U)$. We have

$$X \setminus (\mathrm{Sing}(X) \cup \mathrm{Supp}(B^{>0})) \subseteq U \subseteq X \setminus \mathrm{Supp}(B^{>0}).$$

The first inclusion implies that U is dense in X. The second inclusion is an equality if $(X, \mathrm{Supp}\, B)$ is log smooth.

4.4.2 Non-log canonical locus

Let (X,B) be a log pair with log smooth support. Write $B = \sum_E b_E E$, where the sum runs after the prime divisors of X. Define

$$N(B) = \sum_{b_E<0} \lfloor b_E \rfloor E + \sum_{b_E>1} (\lceil b_E \rceil - 1)E.$$

Then $N(B)$ is a Weil divisor. There exists a unique decomposition $N(B) = N^+ - N^-$, where N^+, N^- are effective divisors with no components in common. Then $\mathrm{Supp}(N^+) = \mathrm{Supp}(B^{>1})$ and $\mathrm{Supp}(N^-) = \mathrm{Supp}(B^{<0})$. We have

$$\lfloor B^{>1} \rfloor - N^+ = \sum_{0<b_E\in\mathbb{Z}} E.$$

In particular $N^+ \le \lfloor B^{>1} \rfloor$, and the two divisors have the same support. Denote

$$\Delta(B) = B - N(B).$$

We have $\Delta(B) = \sum_{b_E<0}\{b_E\}E + \sum_{b_E>0}(b_E + 1 - \lceil b_E \rceil)E$. The following properties hold:

1) The coefficients of $\Delta(B)$ belong to the interval $[0,1]$. They are rational if and only if the coefficients of B are.

2) $\mathrm{Supp}(\Delta(B)) = \mathrm{Supp}(B^{>0}) \cup \cup_{0>b_E\notin\mathbb{Z}} E$. In particular, $(X,\Delta(B))$ is a log variety with log canonical singularities and log smooth support.

3) $\mathrm{mult}_E\, \Delta(B) = 1$ if and only if $\mathrm{mult}_E\, B \in \mathbb{Z}_{>0}$.

Lemma 4.4.2. *Let $\mu\colon (X',B') \to (X,B)$ be a log crepant birational contraction of log pairs with log smooth support. Then $\mu^*N(B) - N(B')$ is an effective μ-exceptional divisor. In particular,*

$$\mathcal{O}_X(-N(B)) = \mu_*\mathcal{O}_{X'}(-N(B')).$$

Proof. The operation $B \mapsto N(B)$ is defined componentwise, so $\mu^*N(B) - N(B')$ is clearly μ-exceptional. Decompose $B = \Delta + N$ and $B' = \Delta' + N'$. From $\mu^*(K+B) = K_{X'} + B'$ we deduce

$$\mu^*N - N' = K_{X'} + \Delta' - \mu^*(K + \Delta).$$

In particular, let E be a prime divisor on X'. And $m_E = \operatorname{mult}_E(\mu^* N - N')$. Then

$$m_E = a(E; X, \Delta) - a(E; X', \Delta').$$

Since (X, Δ) has log canonical singularities and Δ' is effective, we obtain

$$m_E \geq 0 - 1 \geq -1.$$

If $m_E > -1$, then $m_E \geq 0$, as it is an integer. Else, $m_E = -1$. In this case $a(E; X, \Delta) = 0$ and $a(E; X', \Delta') = 1$. From $a(E; X, \Delta) = 0$, we deduce that $\mu(E)$ is the transverse intersection of some components of Δ with coefficient 1. That is $\mu(E)$ is the transverse intersection of some components of B with coefficients in $\mathbb{Z}_{\geq 1}$. In particular, $B \geq \Delta$ near the generic point of $\mu(E)$. We deduce

$$0 = a(E; X, \Delta) \geq a(E; X, B) = a(E; X', B')$$

That is $\operatorname{mult}_E B' \geq 1$. Then $\operatorname{mult}_E \Delta' > 0$, so $a(E; X', \Delta') = 1 - \operatorname{mult}_E \Delta' < 1$. Contradiction. □

Definition 4.4.3. Let (X, B) be a log variety. Let $\mu\colon (X', B_{X'}) \to (X, B)$ be a log crepant log resolution. Define

$$\mathcal{I} = \mu_* \mathcal{O}_{X'}(-N(B_{X'})).$$

The coherent $\mathcal{O}_X$-module $\mathcal{I}$ is independent of the choice of μ, by Lemma 4.4.2. Since B is effective, the divisor $N(B_{X'})^- = -\lfloor B_{X'}^{<0} \rfloor$ is μ-exceptional. Therefore

$$\mathcal{I} \subseteq \mu_* \mathcal{O}_{X'}(N(B_{X'})^-) = \mathcal{O}_X.$$

We call $\mathcal{I}$ the *ideal sheaf of the non-log canonical locus* of (X, B). It defines a closed subscheme $(X, B)_{-\infty}$ of X by the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{(X,B)_{-\infty}} \to 0.$$

We call $(X, B)_{-\infty}$ the *locus of non-log canonical singularities* of (X, B). It is empty if and only if (X, B) has log canonical singularities. The complement $X \setminus (X, B)_{-\infty}$ is the largest open subset on which (X, B) has log canonical singularities.

Remark 4.4.4. We introduced in [3] another scheme structure on the locus of non-log canonical singularities of a log variety (X, B). The two schemes have the same support, but their structure sheaves usually differ. To compare them, consider a log crepant log resolution $\mu\colon (X', B_{X'}) \to (X, B)$. Define

$$N^s = \lfloor B_{X'}^{\neq 1} \rfloor = N(B_{X'}) + \sum_{\operatorname{mult}_E(B_{X'}) \in \mathbb{Z}_{>1}} E.$$

Denote $B_{X'} = \sum_E b_E E$. Then $N^s - N(B_{X'}) = \sum_{b_E \in \mathbb{Z}_{>1}} E$ and $\lfloor B_{X'} \rfloor - N^s = \sum_{b_E = 1} E$. In particular

$$N \leq N^s \leq \lfloor B_{X'} \rfloor.$$

We obtain inclusions of ideal sheaves $\mu_*\mathcal{O}_{X'}(-N) \supseteq \mu_*\mathcal{O}_{X'}(-N^s) \supseteq \mu_*\mathcal{O}_{X'}(-\lfloor B_{X'}\rfloor)$. Equivalently, we have closed embeddings of subschemes of X

$$Y \hookrightarrow Y^s \hookrightarrow \mathrm{LCS}(X,B),$$

where Y^s is the scheme structure introduced in [3] and $\mathrm{LCS}(X,B)$ is the subscheme structure on the non-klt locus of (X,B).

Consider for example the log variety $(\mathbb{A}^2, 2H_1+H_2)$, where H_1, H_2 are the coordinate hyperplanes. The above inclusions are

$$H_1 \hookrightarrow 2H_1 \hookrightarrow 2H_1+H_2.$$

Lemma 4.4.5. *Let $\mu\colon (X',B') \to (X,B)$ be a log crepant birational contraction of log pairs with log smooth support. Then $\mu^*\lfloor B^{\neq 1}\rfloor - \lfloor B^{\neq 1}_{X'}\rfloor$ is an effective μ-exceptional divisor. In particular,*

$$\mathcal{O}_X(-\lfloor B^{\neq 1}\rfloor) = \mu_*\mathcal{O}_{X'}(-\lfloor B^{\neq 1}_{X'}\rfloor).$$

Proof. The operation $B \mapsto \lfloor B^{\neq 1}\rfloor$ is defined componentwise, so $\mu^*\lfloor B^{\neq 1}\rfloor - \lfloor B^{\neq 1}_{X'}\rfloor$ is clearly μ-exceptional. The equality $\mu^*(K+B) = K_{X'}+B_{X'}$ becomes

$$\mu^*\lfloor B^{\neq 1}\rfloor - \lfloor B^{\neq 1}_{X'}\rfloor = K_{X'} + B^{=1}_{X'} + \{B^{\neq 1}_{X'}\} - \mu^*(K + B^{=1} + \{B^{\neq 1}\}).$$

Consider the multiplicity of the left hand side at a prime on X'. It is an integer. The right hand side is ≥ -1. If > -1, it is ≥ 0. Suppose it equals -1. This implies $a(E; X, B^{=1} + \{B^{\neq 1}\}) = 0$. Then $a(E; X, B^{=1}) = 0$ and $B = B^{=1}$ near the generic point of $\mu(E)$. Then $a(E; X', B_{X'}) = 0$. Then the difference is zero. Contradiction. □

4.4.3 Lc centers

Lemma 4.4.6. *Let (X,B) be a log variety with log canonical singularities. Let D be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, let Z be the union of lc centers of (X,B) contained in* $\operatorname{Supp} D$*, with reduced structure. Then $(X, B+\epsilon D)_{-\infty} = Z$ for $0 < \epsilon \ll 1$.*

Proof. Let $\mu\colon X' \to X$ be a resolution of singularities such that $(X', \operatorname{Supp} B_{X'} \cup \operatorname{Supp} \mu^*D)$ is log smooth, where $\mu^*(K_X+B) = K_{X'}+B_{X'}$, and $\mu^{-1}(Z)$ has pure codimension one. We have $\mu^*(K_X+B+\epsilon D) = K_{X'}+B_{X'}+\epsilon\mu^*D$. Denote

$$\Sigma' = \sum_{\mathrm{mult}_E(B_{X'})=1, \mu(E)\subseteq Z} E.$$

Since the coefficients of $B_{X'}$ are at most 1, for $0 < \epsilon \ll 1$ we obtain the formula

$$\begin{aligned} N(B_{X'}+\epsilon\mu^*D) &= \lfloor (B_{X'})^{<0}\rfloor + \sum_{\mathrm{mult}_E(B_{X'})=1, \mu(E)\subseteq \operatorname{Supp} D} E \\ &= \lfloor (B_{X'})^{<0}\rfloor + \Sigma'. \end{aligned}$$

Denote $A = -\lfloor (B_{X'})^{<0} \rfloor$, an effective μ-exceptional divisor on X'. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mu_*\mathcal{O}_{X'}(A-\Sigma') \arrow[r] & \mu_*\mathcal{O}_{X'}(A) \arrow[r, "r"] & \mu_*\mathcal{O}_{\Sigma'}(A|_{\Sigma'}) \arrow[r, "\partial"] & R^1\mu_*\mathcal{O}_{X'}(A-\Sigma') \\
0 \arrow[r] & \mathcal{I}_Z \arrow[r] \arrow[u, "\alpha"] & \mathcal{O}_X \arrow[r] \arrow[u, "\beta"] & \mathcal{O}_Z \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

We claim that $\partial = 0$. Indeed, denote $B' = \{B_{X'}^{<0}\} + B_{X'}^{>0} - \Sigma'$. Then $A - \Sigma' \sim_{\mathbb{R}} K_{X'} + B'$ over X, (X', B') has log canonical singularities, and $\mu(C) \not\subseteq Z$ for every lc center C of (X', B'). The sheaf $\mu_*\mathcal{O}_{\Sigma'}(A|_{\Sigma'})$ is supported by Z, so the image of ∂ is supported by Z. Suppose by contradiction that ∂ is non-zero. Let s be a non-zero local section of $\operatorname{Im} \partial$. By [3, Theorem 3.2.(i)], (X', B') admits an lc center C such that $\mu(C) \subseteq \operatorname{Supp}(s)$. Since $\operatorname{Supp}(s) \subseteq Z$, we obtain $\mu(C) \subseteq Z$, a contradiction.

Since A is effective and μ-exceptional, β is an isomorphism. The map γ is injective. Since r is surjective, γ is also surjective, hence an isomorphism. We conclude that α is an isomorphism. That is $\mathcal{I}_Z = \mu_*\mathcal{O}_{X'}(-N(B_{X'} + \epsilon\mu^* D)) = \mathcal{I}_{(X, B+\epsilon D)_{-\infty}}$. □

4.5 Injectivity for log varieties

Theorem 4.5.1. *Let (X, B) be a proper log variety with log canonical singularities. Let U be the totally canonical locus of (X, B). Let L be a Cartier divisor on X such that $L \sim_{\mathbb{R}} K + B$. Then the restriction homomorphism*

$$H^1(X, \mathcal{O}_X(L)) \to H^1(U, \mathcal{O}_U(L|_U))$$

is injective.

Proof. Let $\mu \colon X' \to X$ be a birational contraction such that X' is non-singular, the exceptional locus $\operatorname{Exc} \mu$ has codimension one, and $\operatorname{Exc} \mu \cup \operatorname{Supp}(\mu_*^{-1} B)$ has normal crossings. We can write

$$K_{X'} + \mu_*^{-1} B + \operatorname{Exc} \mu = \mu^*(K + B) + A,$$

with A supported by $\operatorname{Exc} \mu$. Since (X, B) has log canonical singularities, A is effective. Denote $B' = \mu_*^{-1} B + \operatorname{Exc} \mu - \{A\}$ and $L' = \mu^* L + \lfloor A \rfloor$. We obtain

$$L' \sim_{\mathbb{R}} K_{X'} + B'.$$

Denote $U' = X' \setminus B'$. We claim that $U' \subseteq \mu^{-1}(U)$. Indeed, this is equivalent to the inclusion

$$\operatorname{Supp}(B') \supseteq \mu^{-1}\mu(\operatorname{Supp} B_{X'}^{>0}).$$

By Zariski's Main Theorem, $\operatorname{Exc} \mu = \mu^{-1}(X \setminus V)$, where V is the largest open subset of X such that μ is an isomorphism over V. Over $X \setminus V$, the inclusion is clear since

$\operatorname{Exc}\mu \subseteq \operatorname{Supp} B'$. Over V, μ is an isomorphism and the inclusion becomes an equality. This proves the claim.

Since A is effective and $\mu_* A = 0$, we have $\mathcal{O}_X(L) \tilde{\to} \mu_* \mathcal{O}_{X'}(L')$. From $U' \subseteq \mu^{-1}(U)$ we obtain a commutative diagram

$$\begin{tikzcd} H^1(X', \mathcal{O}_{X'}(L')) \arrow[r, "\alpha'"] & H^1(U', \mathcal{O}_{U'}(L'|_{U'})) \\ H^1(X, \mathcal{O}_X(L)) \arrow[r, "\alpha"] \arrow[u, "\beta"] & H^1(U, \mathcal{O}_U(L|_U)) \arrow[u] \end{tikzcd}$$

By Theorem 4.2.3, α' is injective. Since $\mathcal{O}_X(L) = \mu_* \mathcal{O}_{X'}(L')$, Lemma 4.1.2 implies that β is injective. Then $\alpha' \circ \beta$ is injective. The diagram is commutative, so α is injective. □

Corollary 4.5.2. *In the assumptions of Theorem 4.5.1, let D be an effective Cartier divisor such that* $\operatorname{Supp}(D) \cap U = \emptyset$. *Then we have a short exact sequence*

$$0 \to \Gamma(X, \mathcal{O}_X(L)) \to \Gamma(X, \mathcal{O}_X(L+D)) \to \Gamma(D, \mathcal{O}_D(L+D)) \to 0.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} H^1(X, \mathcal{O}_X(L)) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H^1(X, \mathcal{O}_X(L+D)) \arrow[d] \\ H^1(U, \mathcal{O}_U(L|_U)) \arrow[r, "\gamma"] & H^1(U, \mathcal{O}_U((L+D)|_U)) \end{tikzcd}$$

Since D is disjoint from U, γ is an isomorphism. Since β is injective, we obtain that $\gamma \circ \beta$ is injective. Therefore α is injective. The long exact sequence induced in cohomology by the short exact sequence $0 \to \mathcal{O}_X(L) \to \mathcal{O}_X(L+D) \to \mathcal{O}_D(L+D) \to 0$ gives the claim. □

4.5.1 Applications

Let (X, B) be a proper log variety with log canonical singularities, let L, H be Cartier divisors on X.

Corollary 4.5.3. *Suppose $L \sim_{\mathbb{R}} K_X + B$. Suppose the totally canonical locus of (X, B) is contained in some affine open subset $U' \subseteq X$. Then $H^1(X, \mathcal{O}_X(L)) = 0$.*

Proof. Let U be the totally canonical locus of (X, B). The restriction homomorphism $H^1(X, \mathcal{O}_X(L)) \to H^1(U, \mathcal{O}_U(L|_U))$ is injective. It factors through $H^1(U', \mathcal{O}_{U'}(L|_{U'})) = 0$, hence it is zero. Therefore $H^1(X, \mathcal{O}_X(L)) = 0$. □

Corollary 4.5.4. *Let $L \sim_{\mathbb{R}} K_X + B$. Let H be a Cartier divisor on X such that the linear system $|nH|$ is base point free for some positive integer n. Let $m_0 \geq 1$ and $s \in \Gamma(X, \mathcal{O}_X(m_0 H))$ such that $s|_C \neq 0$ for every lc center of (X, B). Then the multiplication*

$$\otimes s \colon H^1(X, \mathcal{O}_X(L + mH)) \to H^1(X, \mathcal{O}_X(L + (m+m_0)H))$$

is injective for $m \geq 1$.

Proof. Let D be the zero locus of s. There exists a rational number $0 < \epsilon < \frac{1}{m_0}$ such that $(X, B + \epsilon D)$ has log canonical singularities. We have

$$L + mH \sim_{\mathbb{R}} K_X + B + \epsilon D + (m - \epsilon m_0)H.$$

There exists $n \geq 1$ such that the linear system $|n(m - \epsilon m_0)H|$ has no base points. Let Y be a general member, and denote $B' = B + \epsilon D + \frac{1}{n}Y$. Then (X, B') has log canonical singularities, $\operatorname{Supp} D \subseteq \operatorname{Supp} B'$ and

$$L + mH \sim_{\mathbb{R}} K_X + B'$$

Since $\operatorname{Supp}(D)$ is disjoint from the totally canonical locus of (X, B'), Corollary 4.5.2 gives the injectivity of $H^1(X, L + mH) \to H^1(X, L + mH + D)$. □

Corollary 4.5.5. *Let $V \subseteq \Gamma(X, \mathcal{O}_X(H))$ be a vector subspace such that $V \otimes_k \mathcal{O}_X \to \mathcal{O}_X(H)$ is surjective. If $L \sim_{\mathbb{R}} K + B + tH$ and $t > \dim_k V$, then the multiplication map*

$$V \otimes_k \Gamma(X, \mathcal{O}_X(L - H)) \to \Gamma(X, \mathcal{O}_X(L))$$

is surjective.

Proof. We use induction on $\dim V$. If $\dim V = 1$, then $V = k\varphi$, with $\varphi \colon \mathcal{O}_X \tilde{\to} \mathcal{O}_X(H)$. Then $\otimes\varphi \colon \mathcal{O}_X(L - H) \to \mathcal{O}_X(L)$ is an isomorphism, so the claim holds.

Let $\dim V > 1$. Let $\varphi \in V$ be a general element, let $Y = (\varphi) + H$. Then the claim is equivalent to the surjectivity of the homomorphism

$$V|_Y \otimes \Gamma(X, \mathcal{O}_X(L - H))|_Y \to \Gamma(X, \mathcal{O}_X(L))|_Y$$

where $\Gamma(X, \mathcal{F})|_Y$ denotes the image of the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(Y, \mathcal{F} \otimes \mathcal{O}_Y)$, and $V|_Y$ is the image of V under this restriction for $\mathcal{F} = \mathcal{O}_X(H)$.

Assuming $\Gamma(X, \mathcal{O}_X(L - H))|_Y = \Gamma(Y, \mathcal{O}_Y(L))$ and $\Gamma(X, \mathcal{O}_X(L - H))|_Y = \Gamma(Y, \mathcal{O}_Y(L - H))$, we prove the claim as follows: we have $L \sim_{\mathbb{R}} K_X + B + Y + (t-1)H$. By adjunction, using that Y is general, we have $L|_Y \sim_{\mathbb{R}} K_Y + B|_Y + (t-1)H|_Y$, $(Y, B|_Y)$ has log canonical singularities, and $t - 1 > \dim V - 1 = \dim V|_Y$. Therefore $V|_Y \otimes \Gamma(Y, \mathcal{O}_Y(L - H)) \to \Gamma(Y, \mathcal{O}_Y(L))$ is surjective by induction.

It remains to show that $\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L))$ and $\Gamma(X, \mathcal{O}_X(L-H)) \to \Gamma(Y, \mathcal{O}_Y(L-H))$ are surjective. Consider the second homomorphism. We have

$$L - Y \sim_{\mathbb{R}} K_X + B + (t-1)H = K_X + B + \epsilon Y + (t - 1 - \epsilon)H.$$

Since Y is general, $(X, B + \epsilon Y)$ has log canonical singularities for $0 < \epsilon \ll 1$. Since H is free, we deduce that $L - Y \sim_{\mathbb{R}} K_X + B'$ with (X, B') having log canonical singularities, and $Y \subseteq \operatorname{Supp} B'$. By Corollary 4.5.2, $\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L))$ is surjective. The surjectivity of the other homomorphism is proved in the same way. □

4.6 Restriction to the non-log canonical locus

Let (X,B) be a proper log variety, and L a Cartier divisor on X such that $L \sim_{\mathbb{R}} K_X + B$. Suppose the locus of non-log canonical singularities $Y = (X,B)_{-\infty}$ is non-empty.

Lemma 4.6.1. *Suppose* $(X, \operatorname{Supp} B)$ *is log smooth.*

1) The long exact sequence induced in cohomology by the short exact sequence

$$0 \to \mathcal{I}_Y(L) \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L) \to 0$$

breaks up into short exact sequences

$$0 \to H^q(X, \mathcal{I}_Y(L)) \to H^q(X, \mathcal{O}_X(L)) \to H^q(Y, \mathcal{O}_Y(L)) \to 0 \ (q \geq 0).$$

2) Let E be a prime divisor on X such that $\operatorname{mult}_E B = 1$. *The long exact sequence induced in cohomology by the short exact sequence*

$$0 \to \mathcal{I}_Y(L-E) \to \mathcal{O}_X(L-E) \to \mathcal{O}_Y(L-E) \to 0$$

breaks up into short exact sequences

$$0 \to H^q(X, \mathcal{I}_Y(L-E)) \to H^q(X, \mathcal{O}_X(L-E)) \to H^q(Y, \mathcal{O}_Y(L-E)) \to 0 \ (q \geq 0).$$

Proof. 1) Let $N = N(B)$, so that $\mathcal{I}_Y = \mathcal{O}_X(-N)$. We have $L - N \sim_{\mathbb{R}} K_X + \Delta$ and N is supported by Δ. By Remark 4.2.6, the natural map $H^q(X, \mathcal{O}_X(L-N)) \to H^q(X, \mathcal{O}_X(L))$ is injective for all q.

2) We have $L - E \sim_{\mathbb{R}} K_X + B - E$ and $(X, B-E)_{-\infty} = (X,B)_{-\infty} = Y$. Therefore 2) follows from 1). □

Theorem 4.6.2 (Extension from non-lc locus)**.** *We have a short exact sequence*

$$0 \to \Gamma(X, \mathcal{I}_Y(L)) \to \Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L)) \to 0.$$

Proof. Let $\mu \colon (X', B_{X'}) \to (X,B)$ be a log crepant log resolution. Let $N(B_{X'}) = N = N^+ - N^-$ and $\Delta = B_{X'} - N(B_{X'})$. We have

$$\mu^* L - N \sim_{\mathbb{R}} K_{X'} + \Delta$$

and N^+ is supported by Δ. By Remark 4.2.6, we obtain for all q short exact sequences

$$0 \to H^q(X', \mathcal{O}_{X'}(\mu^*L - N)) \to H^q(X', \mathcal{O}_{X'}(\mu^*L + N^-)) \to H^q(N', \mathcal{O}_{N^+}(\mu^*L + N^-)) \to 0$$

By definition, $\mathcal{I}_Y = \mu_* \mathcal{O}_{X'}(-N)$. Thus $\mathcal{I}_Y(L) = \mu_* \mathcal{O}_{X'}(\mu^*L - N)$, and we obtain a commutative diagram

$$\begin{tikzcd}
H^q(X', \mathcal{O}_{X'}(\mu^*L + N^- - N^+)) \arrow[r, "\gamma^q"] & H^q(X', \mathcal{O}_{X'}(\mu^*L + N^-)) \\
H^q(X, \mathcal{I}_Y(L)) \arrow[u, "\beta^q"] \arrow[r, "\alpha^q"] & H^q(X, \mathcal{O}_X(L)) \arrow[u]
\end{tikzcd}$$

From above, γ^q is injective. By Lemma 4.1.2, β^1 is injective. Therefore $\gamma^1 \circ \beta^1$ is injective. Therefore α^1 is injective, which is equivalent to our statement. □

4.6.1 Applications

The first application was first stated by Shokurov, who showed that it follows from the Log Minimal Model Program and Log Abundance in the same dimension (see the proof of [59, Lemma 10.15]).

Theorem 4.6.3 (Global inversion of adjunction)**.** *Let (X,B) be a proper connected log variety such that $K_X+B\sim_{\mathbb{R}} 0$. Suppose $Y=(X,B)_{-\infty}$ is non-empty. Then Y is connected, and intersects every lc center of (X,B).*

Proof. By Theorem 4.6.2, we have a short exact sequence

$$0\to\Gamma(X,\mathcal{I}_Y)\to\Gamma(X,\mathcal{O}_X)\to\Gamma(Y,\mathcal{O}_Y)\to 0.$$

We have $0=\Gamma(X,\mathcal{I}_Y), k\tilde{\to}\Gamma(X,\mathcal{O}_X)$. Therefore $k\tilde{\to}\Gamma(Y,\mathcal{O}_Y)$, so Y is connected.

Let C be a log canonical center of (X,B). Let $\mu\colon(X',B_{X'})\to(X,B)$ be a log resolution such that $\mu^{-1}(C)$ has codimension one. Let Σ be the part of $B^{=1}_{X'}$ contained in $\mu^{-1}(C)$. We have $\mu(\Sigma)=C$. Let $B'=B_{X'}-\Sigma$ and $N=N(B')=N(B_{X'})$. We have

$$-\Sigma-N\sim_{\mathbb{R}} K_{X'}+\Delta(B')$$

The boundary $\Delta(B')$ supports N^+. By Remark 4.2.6, we obtain a surjection

$$\Gamma(X',\mathcal{O}_{X'}(-\Sigma+N^-))\to\Gamma(N^+,\mathcal{O}_{N^+}(-\Sigma+N^-)).$$

We have $\Gamma(X',\mathcal{O}_{X'}(-\Sigma+N^-))\subseteq\Gamma(X,\mathcal{I}_C)=0$. Therefore $\Gamma(X',\mathcal{O}_{X'}(-\Sigma+N^-))=0$. We obtain $\Gamma(N^+,\mathcal{O}_{N^+}(-\Sigma+N^-))=0$. Since

$$0=\Gamma(N^+,\mathcal{O}_{N^+}(-\Sigma+N^-))\subseteq\Gamma(N^+,\mathcal{O}_{N^+}(N^-))\neq 0,$$

we infer $\Sigma\cap N^+\neq\emptyset$. This implies $C\cap Y\neq\emptyset$. □

The next application is a corollary of [3, Theorem 4.4.], if H is $\mathbb{Q}$-ample.

Theorem 4.6.4 (Extension from lc centers)**.** *Let (X,B) be a proper log variety with log canonical singularities. Let L be a Cartier divisor on X such that $H=L-(K_X+B)$ is a semiample $\mathbb{Q}$-divisor. Let $m_0\geq 1$, $D\in|m_0H|$, and denote by Z the union of lc centers of (X,B) contained in* $\operatorname{Supp} D$. *Then the restriction homomorphism*

$$\Gamma(X,\mathcal{O}_X(L))\to\Gamma(Z,\mathcal{O}_Z(L))$$

is surjective.

Proof. By Lemma 4.4.6, there exists $\epsilon\in(0,1)\cap\mathbb{Q}$ such that $(X,B+\epsilon D)_{-\infty}=Z$. Let $m_1\geq 1$ such that the linear system $|m_1H|$ has no base points. Let $D'\in|m_1H|$ be a general member. Then $(X,B+\epsilon D+(\frac{1}{m_1}-\frac{\epsilon}{m_0m_1})D')_{-\infty}=Z$ and

$$L\sim_{\mathbb{Q}} K_X+B+\epsilon D+(\frac{1}{m_1}-\frac{\epsilon}{m_0m_1})D'.$$

By Theorem 4.6.2, $\Gamma(X,\mathcal{O}_X(L))\to\Gamma(Z,\mathcal{O}_Z(L))$ is surjective. □

Corollary 4.6.5. *Let (X, B) be a proper log variety with log canonical singularities such that the linear system $|m_1(K_X + B)|$ has no base points for some $m_1 \geq 1$. Let $m_0 \geq 1$, $D \in |m_0(K_X+B)|$, and denote by Z the union of lc centers of (X, B) contained in* $\operatorname{Supp} D$*. Then*

$$\Gamma(X, \mathcal{O}_X(mK_X + mB)) \to \Gamma(Z, \mathcal{O}_Z(mK_X + mB))$$

is surjective for every $m \geq 2$ such that $mK_X + mB$ is Cartier.

Proof. Apply Theorem 4.6.4 to $m(K_X + B) = K_X + B + (m-1)(K_X + B)$. $\square$

Chapter 5

Vanishing theorems for GNC log varieties

The birational classification of complex manifolds rests on vanishing theorems for Cartier divisors of the form $L \sim_{\mathbb{Q}} K_X + B$, where (X, B) is a *log smooth variety* (i.e. X is a smooth complex variety and $B = \sum_i b_i E_i$ is a boundary with coefficients $b_i \in \mathbb{Q} \cap [0, 1]$, such that $\sum_i E_i$ is a normal crossings divisor on X). In the order in which one may prove these vanishing theorems, they are Esnault-Viehweg injectivity, Tankeev-Kollár injectivity, Kollár's torsion freeness, Ohsawa-Kollár vanishing, Kawamata-Viehweg vanishing. The injectivity theorems imply the rest. Modulo cyclic covering tricks and Hironaka's desingularization, the injectivity theorems are a direct consequence of the E_1-degeneration of the Hodge to de Rham spectral sequence associated to an open manifold.

To study the category of log smooth varieties, it is necessary to enlarge it to allow certain non-normal, even reducible objects, which appear in inductive arguments in the study of linear systems, or in compactification problems for moduli spaces of manifolds. The smallest such enlargement is the category of *normal crossings log varieties* (X, B), which may be thought as glueings of log smooth varieties, in the simplest possible way. By definition, they are locally analytically isomorphic to the local model $0 \in X = \cup_{i \in I}\{z_i = 0\} \subset \mathbb{A}^N$, endowed with the boundary $B = \sum_{j \in J} b_j \{z_j = 0\}|_X$, where I, J are disjoint subsets of $\{1, \ldots, N\}$ and $b_j \in \mathbb{Q} \cap [0, 1]$. Since X has Gorenstein singularities, the dualizing sheaf ω_X is an invertible $\mathcal{O}_X$-module. We denote by K_X a Cartier divisor on X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. By definition, B is $\mathbb{Q}$-Cartier. Normal crossings varieties are build up of their lc centers, closed irreducible subvarieties, which on the local analytic model correspond to (unions of) affine subspaces $\cap_{i \in I'}\{z_i = 0\} \cap \cap_{j \in J'}\{z_j = 0\} \subset \mathbb{A}^N$, where $I' \subset I$ is a non-empty subset, and $J' \subset \{j \in J; b_j = 1\}$ is a possibly empty subset. For example, the irreducible components of X are lc centers of (X, B). Inside the category of normal crossings log varieties, log smooth varieties are exactly those with normal ambient space. The aim of this chapter is to show that the above mentioned vanishing theorems remain true in the category of normal crossings varieties.

Theorem 5.0.6. *Let (X, B) be a normal crossings log variety, L a Cartier divisor on X,* *and $f\colon X \to Y$ a proper morphism.*

1) *(Esnault-Viehweg injectivity) Suppose $L \sim_{\mathbb{Q}} K_X+B$. Let D be an effective Cartier divisor supported by B. Then the natural homomorphisms $R^q f_*\mathcal{O}_X(L) \to R^q f_*\mathcal{O}_X(L+D)$ are injective.*

2) *(Tankeev-Kollár injectivity) Suppose $L \sim_{\mathbb{Q}} K_X+B+H$, where H is an f-semiample $\mathbb{Q}$-divisor. Let D be an effective Cartier divisor which contains no lc center of (X,B), and such that $D \sim_{\mathbb{Q}} uH$ for some $u > 0$. Then the natural homomorphisms $R^q f_*\mathcal{O}_X(L) \to R^q f_*\mathcal{O}_X(L+D)$ are injective.*

3) *(Kollár's torsion freeness) Suppose $L \sim_{\mathbb{Q}} K_X+B$. Let s be a local section of $R^q f_*\mathcal{O}_X(L)$ whose support does not contain $f(C)$, for every lc center C of (X,B). Then $s=0$.*

4) *(Ohsawa-Kollár vanishing) Let $g\colon Y \to Z$ be a projective morphism. Suppose $L \sim_{\mathbb{Q}} K_X+B+f^*A$, where A is a g-ample $\mathbb{Q}$-Cartier divisor on Y. Then $R^p g_* R^q f_*\mathcal{O}_X(L) = 0$ for $p \neq 0$.*

The notation $L \sim_{\mathbb{Q}} M$ means that there exists a positive integer r such that both rL and rM are Cartier divisors, and $\mathcal{O}_X(rL) \simeq \mathcal{O}_X(rM)$. Kawamata-Viehweg vanishing is the case $\dim Z = 0$ of the Ohsawa-Kollár vanishing.

Theorem 5.0.6.2)-4) was proved by Kawamata [38] if B has coefficients strictly less than 1, and it was proved for *embedded* normal crossings varieties (X,B) in [3, Section 3]. We remove the global embedded assumption in this chapter, as expected in [3, Remark 2.9]. Theorem 5.0.6.1) is implicit in the proof of [3, Theorem 3.1], in the case when (X,B) is embedded normal crossings and D is supported by the part of B with coefficients strictly less than 1, which is the original setting of Esnault and Viehweg. We observed in [5] that the same results holds if (X,B) is log smooth and D is supported by B, and Theorem 5.0.6.1) extends [5] to the normal crossings case.

Theorem 5.0.6 is proved by reduction to the log smooth case. There are two known methods of proof. Let $\bar{X} \to X$ be the normalization, let $X_n = (\bar{X}/X)^{n+1}$ for $n \geq 0$. With the natural projections and diagonals, we obtain a simplicial algebraic variety $X_\bullet$, together with a natural augmentation $\epsilon\colon X_\bullet \to X$. The key point is that each X_n is smooth, so we may really think of ϵ as a resolution of singularities. The method in [38] is to use the descent spectral sequence to deduce a statement on X from the same statement on each X_n. The method in [3] is to lift the statement from X to a statement on $X_\bullet$, and imitate the proof used in the log smooth case in this simplicial setting. In this chapter we use the method in [38]. The new idea is an adjunction formula

$$(K_X+B)|_{X_n} \sim_{\mathbb{Q}} K_{X_n}+B_n,$$

for a suitable log smooth structure (X_n,B_n), for each n. Moreover, (X_n,B_n) glue to a log smooth simplicial variety. To achieve this, we observe that each irreducible component of X_n is the normalization of some lc center of (X,B). Then the adjunction formula follows from the theory of residues for normal crossings varieties developed in Chapter 3. To construct residues for normal crossings varieties we have to deal with slightly more general singularities, namely*generalized normal crossings log varieties*. The motivation for this enlargement, is that if X has normal crossings singularities, then $\operatorname{Sing} X$ may not have normal crossings singularities. But if X has generalized normal crossings singularities, so does $\operatorname{Sing} X$. We actually prove Theorem 5.0.6 in the category of generalized normal crossings singularities (Theorems 5.3.2, 5.3.4, 5.3.5, 5.3.6). The same proof works in the category of normal crossings log varieties, provided their residues to lc centers are taken for granted. Note that generalized normal crossings singularities in our sense are more general than those defined by Kawamata [38]. For example, every seminormal curve is generalized normal crossings.

To illustrate how generalized normal crossings appear, let us consider two examples of residues. First, consider the log smooth variety $(\mathbb{A}^2, H_1 + H_2)$, where H_1, H_2 are the standard hyperplanes, intersecting at the origin 0. We want to perform adjunction from $(\mathbb{A}^2, H_1 + H_2)$ to its lc center 0. We may first take residue onto H_1, and end up with the log structure $(H_1, 0)$, and then take residue from $(H_1, 0)$ to 0. But we may also restrict to $(H_2, 0)$, and then to 0. The two chains of residues do not coincide; they differ by -1. Since an analytic isomorphism interchanges the two hyperplanes, none of the above compositions of residues is canonical. But they become canonical if raised to even powers. We obtain a canonical residue isomorphism

$$\operatorname{Res}^{[2]}_{\mathbb{A}^2\to 0}\colon \omega_{\mathbb{A}^2}(\log H_1 + H_2)^{\otimes 2}|_0 \xrightarrow{\sim} \omega_0^{\otimes 2}$$

Now we construct the same residue isomorphism, without coordinates. Denote $C = H_1 + H_2$. Let ω_C be the sheaf whose sections are rational differential forms which are regular outside 0, and on the normalization $H_1 \sqcup H_2$ of C induce forms with logarithmic poles along the two points O_1, O_2 above the origin, and have the same residues at O_1, O_2. One checks that ω_C is an invertible $\mathcal{O}_C$-module. The residues from $\mathbb{A}^2$ to the irreducible components of normalization of C glue to a residue isomorphism

$$\operatorname{Res}^{[2]}_{\mathbb{A}^2\to C}\colon \omega_{\mathbb{A}^2}(\log C)^{\otimes 2}|_C \xrightarrow{\sim} \omega_C^{\otimes 2}.$$

Since the forms of ω_C have the same residues above the origin, we also obtain a residue isomorphism

$$\operatorname{Res}^{[2]}_{C\to 0}\colon \omega_C^{\otimes 2}|_0 \xrightarrow{\sim} \omega_0^{\otimes 2}.$$

The composition $\operatorname{Res}^{[2]}_{C\to 0} \circ \operatorname{Res}^{[2]}_{\mathbb{A}^2\to C}$ is exactly $\operatorname{Res}^{[2]}_{\mathbb{A}^2\to 0}$. It is intrinsic, independent of the choice of coordinates, or analytic isomorphisms. Note that ω_C differs from the Rosenlicht dualizing sheaf Ω_C, but $\omega_C^{\otimes m} = \Omega_C^{\otimes m}$ for $m \in 2\mathbb{Z}$ (at the origin, the local generator for ω_C is $(\frac{dz_1}{z_1}, \frac{dz_2}{z_2})$, and for Ω_C is $(\frac{dz_1}{z_1}, -\frac{dz_2}{z_2})$).

Second, let S be the normal crossings surface $(xyz = 0) \subset \mathbb{A}^3$, set $B = 0$. We want to perform adjunction from S to its lc center the origin. As above, we may first restrict to a plane, then to a line, and then to the origin. There are several choices of chains, which coincide up to a sign. If we raise to an even power, we obtain residue isomorphisms from

S to 0. These are invariant under analytic isomorphisms, since we can also define them in the following invariant way. Let $C = \operatorname{Sing} S$. Then C is the union of coordinate axis in $\mathbb{A}^3$, a seminormal curve which is not Gorenstein. The usual dualizing sheaf is useless in this situation. We may define ω_C as above (requiring same residues over the origin), and then ω_C is an invertible $\mathcal{O}_C$-module (at the origin, the local generator is $(\frac{dz_1}{z_1}, \frac{dz_2}{z_2}, \frac{dz_3}{z_3})$), and residues from S to the irreducible components of the normalization of C glue to a residue isomorphism

$$\operatorname{Res}^{[2]}_{S \to C} \colon \omega_S^{\otimes 2}|_C \xrightarrow{\sim} \omega_C^{\otimes 2}.$$

The singular locus of C is 0, and we again obtain a residue isomorphism

$$\operatorname{Res}^{[2]}_{C \to 0} \colon \omega_C^{\otimes 2}|_0 \xrightarrow{\sim} \omega_0^{\otimes 2}.$$

The composition $\operatorname{Res}^{[2]}_{C \to 0} \circ \operatorname{Res}^{[2]}_{S \to C}$ is exactly $\operatorname{Res}^{[2]}_{S \to 0}$, defined from coordinates.

The conclusion we draw from these two examples is that we must redefine the powers of log canonical sheaf $\omega^{[n]}_{(X,B)}$ ($n \in \mathbb{Z}$) (without dualizing property), and we must allow singularities which are not normal crossings, but very close. In [8], we constructed residues for so called n-wlc varieties. Generalized normal crossings varieties are a special case of n-wlc varieties.

We outline the structure of this chapter. In Section 1, we construct the simplicial log variety induced by a n-wlc log variety. In Section 2, we define generalized normal crossings log varieties, and analyze the induced simplicial log variety. In Section 3, we prove the vanishing theorems. The injectivity theorems are reduced to the smooth case, using the simplicial log structure induced. The torsion freeness and vanishing theorems are deduced then by standard arguments. In Section 4, we collect some inductive properties of generalized normal crossings varieties. The key inductive property is that the LCS-locus of a generalized normal crossings log variety is again a generalized normal crossings log variety, for a suitable boundary, and we can perform adjunction onto the LCS-locus. We hope that in the future one may be able to use these inductive properties to reprove the vanishing theorems in Section 3.

5.1 Preliminary

All varieties are defined over an algebraically closed field k, of characteristic zero.

A *log smooth variety* is a pair (X, B), where X is a smooth k-variety and $B = \sum_i b_i E_i$ is a boundary such that $b_i \in \mathbb{Q} \cap [0, 1]$ and $\sum_i E_i$ is a NC divisor.

We refer the reader to [8] for the definition and basic properties of wlc varieties $(X/k, B)$, and some special cases: toric and n-wlc. We will remove the fixed ground field k from notation; for example we denote $\omega^{[n]}_{(X/k,B)}$ by $\omega^{[n]}_{(X,B)}$.

Lemma 5.1.1. *Let $(X', B_{X'})$ and (X, B) be normal log pairs, let $f \colon (X', B_{X'}) \to (X, B)$ be étale and log crepant. Let $Z' \subset X'$ be a closed irreducible subset. Then Z' is an lc center of $(X', B_{X'})$ if and only if $f(Z')$ is an lc center of (X, B_X).*

Proof. Cutting $f(Z')$ with general hyperplane sections, we may suppose Z' is a closed point P'. Since f is open, we may replace X by the image of f and suppose f is surjective. After removing from X' the finite set $f^{-1}f(P') \setminus P'$, we may also suppose $f^{-1}f(P') = P'$. Then the claim follows from [44, page 46, 2.14.(2)]. $\square$

5.1.1 Simplicial log structure induced by a n-wlc log variety

Let (X,B) be a n-wlc log variety. Let $r \in (2\mathbb{Z})_{>0}$ such that rB has integer coefficients and $\omega^{[r]}_{(X,B)}$ is an invertible $\mathcal{O}_X$-module. Let $\pi\colon \bar{X} \to X$ be the normalization. Then $X_n = (\bar{X}/X)^{n+1}$ $(n \geq 0)$ are the components of a simplicial k-algebraic variety $X_\bullet$, endowed with a natural augmentation $\epsilon\colon X_\bullet \to X$.

Proposition 5.1.2. *The following properties hold:*

a) Each X_n is normal. Let Z_n be an irreducible component of X_n. Then $\epsilon_n\colon Z_n \to X$ is the normalization of an lc center of (X,B). Let (Z_n, B_{Z_n}) be the n-wlc log variety structure induced by the residue isomorphism $\operatorname{Res}^{[r]}\colon \omega^{[r]}_{(X,B)}|_{Z_n} \xrightarrow{\sim} \omega^{[r]}_{(Z_n,B_{Z_n})}$. Let $(X_n,B_n) = \sqcup_{Z_n}(Z_n,B_{Z_n})$ be the induced structure of normal log variety, with n-wlc singularities (independent of the choice of r). We obtain isomorphisms

$$\operatorname{Res}^{[r]}_{X\to X_n} : \epsilon_n^*\omega^{[r]}_{(X,B)} \xrightarrow{\sim} \omega^{[r]}_{(X_n,B_n)}.$$

Moreover, each lc center of (X,B) is the image of some lc center of (X_n,B_n).

b) Let $\varphi\colon X_m \to X_n$ be the simplicial morphism induced by an order preserving morphism $\Delta_n \to \Delta_m$, for some $m,n \geq 0$. It induces a commutative diagram

$$\begin{tikzcd} X_m \arrow[rr, "\varphi"] \arrow[dr, "\epsilon_m"'] & & X_n \arrow[dl, "\epsilon_n"] \\ & X & \end{tikzcd}$$

Let Z_m be an irreducible component of X_m. Then $\varphi\colon Z_m \to X_n$ is the normalization of an lc center of (X_n,B_n). Let $\operatorname{Res}^{[r]}\colon \omega^{[r]}_{(X_n,B_n)}|_{Z_m} \xrightarrow{\sim} \omega^{[r]}_{(Z_m,B_{Z_m})}$ be the induced residue isomorphism. Let $\operatorname{Res}^{[r]}_\varphi\colon \varphi^*\omega^{[r]}_{(X_n,B_n)} \xrightarrow{\sim} \omega^{[r]}_{(X_m,B_m)}$ be the induced isomorphism. Then

$$\operatorname{Res}^{[r]}_\varphi \circ \varphi^* \operatorname{Res}^{[r]}_{X\to X_n} = \operatorname{Res}^{[r]}_{X\to X_m}.$$

In particular, $\omega^{[r]}_{(X_n,B_n)}$ and $\operatorname{Res}^{[r]}_\varphi$ form an $\mathcal{O}_{X_\bullet}$-module $\omega^{[r]}_{(X_\bullet,B_\bullet)}$, endowed with an isomorphism $\epsilon^*\omega^{[r]}_{(X,B)} \xrightarrow{\sim} \omega^{[r]}_{(X_\bullet,B_\bullet)}$.

Proof. By [10], we may suppose (X,B) coincides with a local model. That is $X = \operatorname{Spec} k[\mathcal{M}]$ is the toric variety associated with a monoidal complex $\mathcal{M} = (M,\Delta,(S_\sigma)_{\sigma\in\Delta})$,

X has normal irreducible components, B is an effective boundary supported by invariant prime divisors at which X is smooth, and (X, B) has wlc singularities.

Let $X = \cup_F X_F$ be the decomposition into irreducible components, where the union runs after all facets F of Δ. Let $\psi \in \cap_F \frac{1}{r} S_F$ be the log discrepancy function of (X, B). By assumption, each irreducible component X_F is normal. Therefore $\bar{X} = \sqcup_F X_F$. We obtain

$$X_n = \sqcup_{F_0,\ldots,F_n} X_{F_0 \cap \cdots \cap F_n}.$$

Since $\psi \in F_0 \cap \cdots \cap F_n$, each $X_{F_0 \cap \cdots \cap F_n}$ is an lc center of (X, B). The toric log structure induced via residues on $X_{F_0 \cap \cdots \cap F_n}$ is that induced by the log discrepancy function $\psi \in F_0 \cap \cdots \cap F_n$.

An lc center of (X, B) is of the form X_γ, with $\psi \in \gamma \in \Delta$. If F is a facet of Δ which contains γ, then X_γ is also an lc center of the irreducible component $(X_{F \cap \cdots \cap F}, B_n)$ of (X_n, B_n). This proves a).

For b), recall that any simplicial morphism is a composition of face morphisms $\delta_i \colon X_{n+1} \to X_n$ and degeneracy morphisms $s_i \colon X_n \to X_{n+1}$. Hence suffices to verify b) for face and degeneracy morphisms. For our local model, δ_i embeds $X_{F_0 \cap \cdots \cap F_{n+1}}$ into $X_{F_0 \cap \cdots \widehat{F_i} \cdots \cap F_{n+1}}$, and s_i maps $X_{F_0 \cap \cdots \cap F_n}$ isomorphically onto $X_{F_0 \cap \cdots \cap F_i \cap F_i \cap \cdots \cap F_n}$. Then b) holds in our case, since all log structures involved have the same log discrepancy function ψ. $\square$

5.2 GNC log varieties

Recall first some standard notation. The set $\{1, 2, \ldots, N\}$ is denoted by $[N]$, the k-affine space $\mathbb{A}^N_k$ has coordinates $(z_i)_{i \in [N]}$, and $H_i = \{z \in \mathbb{A}^N; z_i = 0\}$ is the standard i-th hyperplane. For a subset $F \subseteq [N]$, denote $\mathbb{A}_F = \cap_{i \in [N] \setminus F} \{z \in \mathbb{A}^N; z_i = 0\}$. It is an affine space with coordinates $(z_i)_{i \in F}$.

Definition 5.2.1. A *GNC (generalized normal crossings) local model* is a pair (X, B), of the following form:

a) $X = \cup_F \mathbb{A}_F \subset \mathbb{A}^N_k$, where the union is indexed after finitely many subsets $F \subseteq [N]$ (called facets), not contained in one another. We assume X satisfies Serre's property S_2, that is for any two facets $F \neq F'$, there exists a chain of facets $F = F_0, F_1, \ldots, F_l = F'$ such that for every $0 \le i < l$, $F_i \cap F_{i+1}$ contains $F \cap F'$ and it has codimension one in both F_i and F_{i+1}.

b) Denote $\sigma = \cap_F F$. If $\sigma \prec \tau \prec F$ and τ has codimension one in F, then there exists a facet F' such that $\tau = F \cap F'$.

c) $B = (\sum_{i \in \sigma} b_i H_i)|_X$, where $b_i \in \mathbb{Q} \cap [0, 1]$ and $H_i = \{z \in \mathbb{A}^N; z_i = 0\}$. We may rewrite $B = \sum_F \sum_{i \in \sigma} b_i \mathbb{A}_{F \setminus i}$.

We claim that (X, B) is a toric wlc log variety. Note first that X is the toric variety $\operatorname{Spec} k[\mathcal{M}]$ associated to the monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$, where $M = \mathbb{Z}^N$,

Δ is the fan consisting of the cones $\sum_{i\in F}\mathbb{R}_{\geq 0}m_i$ and all their faces, and $S_\sigma = \mathbb{Z}^N \cap \sigma$ for $\sigma \in \Delta$. Here $m_1, \ldots, m_N$ denotes the standard basis of the semigroup $\mathbb{N}^N$. Each irreducible component of X is smooth. The normalization of X is $\bar{X} = \sqcup_F \mathbb{A}_F$. Denote $\psi = \sum_{i\in\sigma}(1-b_i)m_i$. On $\mathbb{A}_F$, ψ induces the log structure with boundary

$$B_{\mathbb{A}_F} = \sum_{i\in F\setminus\sigma}\mathbb{A}_{F\setminus i} + \sum_{i\in\sigma} b_i\mathbb{A}_{F\setminus i}.$$

Let $\bar{C} \subset \bar{X}$ be the conductor subscheme. By a), $\bar{C}|_{\mathbb{A}_F} \leq \sum_{i\in F\setminus\sigma}\mathbb{A}_{F\setminus i}$. Equality holds if and only if b) holds. Therefore

$$B_{\mathbb{A}_F} = \bar{C}|_{\mathbb{A}_F} + \sum_{i\in\sigma} b_i\mathbb{A}_{F\setminus i} = (\bar{C}+\bar{B})|_{\mathbb{A}_F}.$$

We conclude that the irreducible components of $(\bar{X}, \bar{C}+\bar{B})$ have the same log discrepancy function ψ, and therefore (X,B) is a toric wlc log variety, by [8, Proposition 4.10]. Note that X is $\mathbb{Q}$-orientable by [8, Lemma 4.7 and Example 4.8.(2)]. If $2 \mid r$ and $rb_i \in \mathbb{Z}$ for all $i \in \sigma$, then $\omega^{[r]}_{(X,B)} \simeq \mathcal{O}_X$. Given a), properties b) and c) are equivalent to

- b') $(X,0)$ is a toric wlc log variety.
- c') B is a torus-invariant boundary whose support contains no lc center of $(X,0)$.

The $\mathbb{Q}$-divisors B, $B^{=1}$, $B^{<1}$ are $\mathbb{Q}$-Cartier (so is the part of B with coefficients in a given interval in $\mathbb{R}$).

Example 5.2.2. A *NC (normal crossings) local model* is a pair (X,B), where $X = \cup_{i\in I}H_i \subset \mathbb{A}^N_k$ and $B = (\sum_{i\notin I} b_iH_i)|_X$, where I is a non-empty subset of $[N]$ and $b_i \in \mathbb{Q}\cap[0,1]$. If we set $F = [N]\setminus i$ $(i \in I)$, we see that (X,B) is a GNC local model. Here we have $\sigma = [N]\setminus I$.

Example 5.2.3. Let $\sigma \subsetneq [N]$, let $|\sigma| \leq p < N$. Let $\{F\}$ consist of all subsets of $[N]$ which have cardinality p, and contain σ. Let $b_i \in \mathbb{Q}\cap[0,1]$, for $i \in \sigma$. Then $(X = \cup_F\mathbb{A}_F \subset \mathbb{A}^N_k, (\sum_{i\in\sigma} b_iH_i)|_X)$ is a GNC local model.

Example 5.2.4. Let $X = \mathbb{A}_{12}\cup\mathbb{A}_{23}\cup\mathbb{A}_{34} \subset \mathbb{A}^4_k$ and $B = \mathbb{A}_1 + \mathbb{A}_4$. Then (X,B) is a toric wlc log variety (with log discrepancy function $\psi = 0$), but not a GNC local model.

Definition 5.2.5. A *GNC (NC) log variety* (X,B) is a wlc log variety such that for every closed point $x \in X$, there exists a GNC (NC) local model (X',B') and an isomorphism of complete local k-algebras $\mathcal{O}^\wedge_{X,x} \simeq \mathcal{O}^\wedge_{X',0}$, such that $(\omega^{[r]}_{(X,B)})^\wedge_x$ corresponds to $(\omega^{[r]}_{(X',B')})^\wedge_0$ for r sufficiently divisible.

By [10], there exists a common étale neighborhood

$$\begin{tikzcd} & (U,y) \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ (X,x) & & (X',0) \end{tikzcd}$$

and a wlc log variety structure (U, B_U) on U such that $i^*\omega^{[n]}_{(X,B)} = \omega^{[n]}_{(U,B_U)} = i'^*\omega^{[n]}_{(X',B')}$ for all $n \in \mathbb{Z}$.

It follows that $(X, 0)$ is a GNC (NC) log variety, and B, $B^{=1}$, $B^{<1}$ are $\mathbb{Q}$-Cartier divisors.

Remark 5.2.6. Let (X, B) be a NC log variety. Let ω_X be the canonical choice of dualizing sheaf, defined by Rosenlicht. It is an invertible $\mathcal{O}_X$-module, since X is locally complete intersection. If rB has integer coefficients and r is divisible by 2, then $\omega_X^{\otimes r} \otimes \mathcal{O}_X(rB) = \omega^{[r]}_{(X,B)}$ (see [8]).

5.2.1 Simplicial log structure induced by a GNC log variety

Let (X, B) be a GNC log variety. Let $\epsilon \colon X_\bullet \to X$ be the simplicial resolution induced by the normalization of X. A GNC log variety is n-wlc. By Proposition 5.1.2, residues induce a natural simplicial log variety structure $(X_\bullet, B_\bullet)$. In this case (X_n, B_n) is a disjoint union of log smooth log varieties, and we have residue isomorphisms

$$\operatorname{Res}^{[r]}_{X \to X_n} : \epsilon_n^*\omega^{[r]}_{(X,B)} \xrightarrow{\sim} \omega^{[r]}_{(X_n,B_n)}$$

for $r \in (2\mathbb{Z})_{>0}$ such that rB has integer coefficients.

Lemma 5.2.7. *The following properties hold:*

1) *$\epsilon \colon X_\bullet \to X$ is a smooth simplicial resolution, and $\mathcal{O}_X \to R\epsilon_*\mathcal{O}_{X_\bullet}$ is a quasi-isomorphism.*

2) *The lc centers of $(X, 0)$ are the images of the irreducible components of X_n ($n \geq 0$).*

3) *(X_n, B_n) is a log smooth variety, for all n.*

4) *The support of B contains no lc center of $(X, 0)$, and each ϵ_n^*B is supported by B_n.*

Proof. We may suppose (X, B) is a GNC local model. Then

$$(X_n, B_n) = \sqcup_{F_0,\ldots,F_n}(\mathbb{A}_{F_0\cap\cdots\cap F_n}, \sum_{i\in F_0\cap\cdots\cap F_n} \mathbb{A}_{F_0\cap\cdots\cap F_n\setminus i} + \sum_{i\in\sigma} b_i \mathbb{A}_{F_0\cap\cdots\cap F_n\setminus i}).$$

1) Each X_n is smooth, so $\epsilon \colon X_\bullet \to X$ is a smooth simplicial resolution. By [7, Theorem 0.1.b)], $\mathcal{O}_X \to R\epsilon_*\mathcal{O}_{X_\bullet}$ is a quasi-isomorphism.

2) The log variety $(X, 0)$ has log discrepancy function $\psi = \sum_{i\in\sigma} m_i \in \operatorname{relint}\sigma$. Therefore its lc centers are X_γ, where $\sigma \prec \gamma \in \Delta$. We claim that each such γ is an intersection of facets of Δ. Indeed, if γ is a facet, the claim holds. Else, choose a facet F which contains γ. Since $\gamma \subsetneq F$, γ is the intersection after all codimension one faces $\tau \prec F$ which contain γ. Each τ contains the core σ. Therefore $\tau = F \cap F'$ for some facet F', by axiom b) in the definition of GNC local models. We conclude that $\gamma = F_0 \cap \cdots \cap F_n$ for some $n \geq 0$. Therefore X_γ appears as an irreducible component of X_n.

3) This is clear from the explicit formula for (X_n, B_n).

4) The support of B does not contain the core X_σ. Since the image on X of an irreducible component of X_n does contain X_σ, we obtain that $\epsilon_n^* B$ is well $\mathbb{Q}$-Cartier defined for all n. Moreover,

$$(B_n - \epsilon_n^* B)|_{\mathbb{A}_{F_0 \cap \cdots \cap F_n}} = \sum_{i \in F_0 \cap \cdots \cap F_n} \mathbb{A}_{F_0 \cap \cdots \cap F_n \setminus i}.$$

$\square$

5.3 Vanishing theorems

Lemma 5.3.1. *Let (X, B) be a log smooth variety. Let L be a Cartier divisor on X such that $L \sim_\mathbb{Q} K_X + B$. Let D be an effective Cartier divisor supported by B. Let $f\colon X \to Z$ be a proper morphism. Then the natural homomorphisms $R^q f_* \mathcal{O}_X(L) \to R^q f_* \mathcal{O}_X(L + D)$ are injective.*

Proof. We may suppose X is irreducible, f is surjective, and Z is affine. Let $Z \hookrightarrow \mathbb{A}^N$ be a closed embedding into an affine space. Compactify $\mathbb{A}^N \subset \mathbb{P}^N$ by adding the hyperplane at infinity H_0. Let $Z' \subset \mathbb{P}^N$ be the closure of Z. Let $H = H_0|_{Z'}$. Then $Z \subset Z'$ is an open dense embedding, whose complement H is a hyperplane section.

By Nagata, there exists an open dense embedding $X \subset X''$ such that X'' is proper. The induced rational map $f\colon X'' \dashrightarrow Z'$ is regular on X. By Hironaka's desingularization, there exists a birational contraction $X' \to X''$, which is an isomorphism over X, such that X' is smooth and f induces a regular map $f'\colon X' \to Z'$. We may also suppose $\Sigma = X' \setminus X$ is a NC divisor, and $(X', B' + \Sigma)$ is log smooth, where $B' = \sum_i b_i (E_i)'$ is the closure of B in X' (defined componentwise). We obtained a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Z \arrow[r] & Z' \end{tikzcd}$$

where the vertical arrows are open dense embeddings, Z' is projective and X' is proper. The properness of f is equivalent to $X = f'^{-1}(Z)$, so the diagram is also cartesian.

We represent L by a Weil divisor on X. Let L' be its closure in X'. Then $L' \sim_\mathbb{Q} K_{X'} + B' + N$, where N is a $\mathbb{Q}$-divisor supported by Σ. Denote $P = L' - \lfloor N \rfloor$ and $\Delta = B' + \{N\}$. Then $P \sim_\mathbb{Q} K_{X'} + \Delta$ and (X', Δ) is log smooth. The closure D' of D in X' is supported by B', hence it is supported by Δ.

Let m be a positive integer. Let S be a general member of the free linear system $|f'^*(mH)|$. Then $P + f'^*(mH) \sim_\mathbb{Q} K_{X'} + \Delta + S$, $(X', \Delta + S)$ is log smooth, and D' is supported by $\Delta + S$. Denote $\mathcal{F} = \mathcal{O}_{X'}(P)$. By Theorem 4.2.3, the natural homomorphism

$$H^n(X', \mathcal{F}(f'^*(mH))) \to H^n(X', \mathcal{F}(f'^*(mH) + D')) \ (n \geq 0)$$

is injective. We have the Leray spectral sequence

$$E_2^{pq} = H^p(Z', R^q f'_* \mathcal{F}(m)) \Longrightarrow H^{p+q}(X', \mathcal{F}(f'^*(mH))).$$

Suppose m is sufficiently large. Serre vanishing gives $E_2^{pq} = 0$ if $p \neq 0$. Therefore we obtain a natural isomorphism $H^0(Z', R^n {f'}_* \mathcal{F}(m)) \tilde{\to} H^n(X', \mathcal{F}(f'^*(mH)))$. By the same argument, we have a natural isomorphism $H^0(Z', R^n f'_* \mathcal{F}(D')(m)) \tilde{\to} H^n(X', \mathcal{F}(D' + f'^*(mH)))$. The injective homomorphism above becomes the injective homomorphism

$$H^0(Z', R^n f'_* \mathcal{F}(m)) \to H^0(Z', R^n f'_* \mathcal{F}(D')(m)).$$

Since $\mathcal{O}_{Z'}(m)$ is very ample, this means that $R^n f'_* \mathcal{F} \to R^n f'_* \mathcal{F}(D')$ is injective. But $X = f'^{-1}(Z)$, $P|_X = L$, $\mathcal{F}|_X = \mathcal{O}_X(L)$ and $D'|_X = D$, so the restriction of this injective homomorphism to Z is just the injective homomorphism $R^n f_* \mathcal{O}_X(L) \to R^n f_* \mathcal{O}_X(L + D)$. □

Theorem 5.3.2 (Esnault-Viehweg injectivity). *Let (X, B) be a GNC log variety. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $\mathcal{L}^{\otimes r} \simeq \omega_{(X,B)}^{[r]}$ for some $r \geq 1$ such that rB has integer coefficients. Let D be an effective Cartier divisor supported by B. Let $f \colon X \to Z$ be a proper morphism. Then the natural homomorphism $R^i f_* \mathcal{L} \to R^i f_* \mathcal{L}(D)$ is injective, for every i.*

Proof. We may suppose Z is affine. Denote $\Sigma = \operatorname{Supp} B$ and $U = X \setminus \Sigma$. Since rB is Cartier, we have an isomorphism $\varinjlim_{m \in \mathbb{N}} H^i(X, \mathcal{O}_X(mrB)) \tilde{\to} H^i(U, \mathcal{L}|_U)$. The claim for all D is thus equivalent to the injectivity of the restriction homomorphisms

$$H^i(X, \mathcal{L}) \to H^i(U, \mathcal{L}|_U).$$

Let $\epsilon \colon X_\bullet \to X$ be the smooth simplicial resolution induced by the normalization of X. Let $\Sigma_n = \epsilon_n^{-1}(\Sigma)$ and $U_n = X_n \setminus \Sigma_n$. The restriction $\epsilon \colon U_\bullet \to U$ is also a smooth simplicial resolution. By Lemma 5.2.7, $\mathcal{L} \to R\epsilon_* \mathcal{L}_\bullet$ and $\mathcal{L}|_U \to R\epsilon_* \mathcal{L}_\bullet|_{U_\bullet}$ are quasi-isomorphisms. Therefore the claim is equivalent to the injectivity of the restriction homomorphisms

$$\alpha \colon H^i(X_\bullet, \mathcal{L}_\bullet) \to H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}).$$

Both spaces are endowed with simplicial filtrations S. The Godement resolutions $\mathcal{L}_p \to \mathcal{K}_p^*$ $(p \geq 0)$ glue to a simplicial resolution $\mathcal{L}_\bullet \to \mathcal{K}_\bullet^*$. Denote $A_p^q = \Gamma_{\Sigma_p}(X_p, \mathcal{K}_p^q)$, $B_p^q = \Gamma(X_p, \mathcal{K}_p^q)$ and $C_p^q = \Gamma(U_p, \mathcal{K}_p^q)$. The associated simple complexes fit into a short exact sequence

$$0 \to A \to B \to C \to 0$$

which induces in homology the long exact sequence

$$\cdots \to H^i_{\Sigma_\bullet}(X_\bullet, \mathcal{L}_\bullet) \to H^i(X_\bullet, \mathcal{L}_\bullet) \to H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}) \to \cdots .$$

Let S be the simplicial filtration (naive with respect to p) on A, B, C. For each p, the short exact sequence

$$0 \to \Gamma_{\Sigma_p}(X_p, \mathcal{K}_p^*) \to \Gamma(X_p, \mathcal{K}_p^*) \to \Gamma(U_p, \mathcal{K}_p^*) \to 0$$

is split. That is $0 \to E_0A \to E_0B \to E_0C \to 0$ is a split short exact sequence. Passing to homology, we obtain that $0 \to E_1A \to E_1B \to E_1C \to 0$ is a split short exact sequence. Iterating this argument, we conclude that $0 \to E_rA \to E_rB \to E_rC \to 0$ is a split short exact sequence, for every r. Therefore $0 \to E_\infty A \to E_\infty B \to E_\infty C \to 0$ is a short exact sequence, which induces in homology the long exact sequence

$$\cdots \to \operatorname{Gr}_S H^i_{\Sigma_\bullet}(X_\bullet, \mathcal{L}_\bullet) \to \operatorname{Gr}_S H^i(X_\bullet, \mathcal{L}_\bullet) \to \operatorname{Gr}_S H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}) \to \cdots .$$

Step 1: $H^q_{\Sigma_p}(X_p, \mathcal{L}_p) \to H^q(X_p, \mathcal{L}_p)$ is zero for all p, q. Indeed,

$$\mathcal{L}_p^{\otimes 2r} = \mathcal{L}^{\otimes 2r}|_{X_p} \simeq \omega^{[2r]}_{(X,B)}|_{X_p} \xrightarrow{\sim} \omega^{[2r]}_{(X_p,B_p)},$$

(X_p, B_p) is a log smooth variety, $U_p \supseteq X_p \setminus B_p$ by Lemma 5.2.7.4), and $X_p \to Z$ is proper. By Lemma 5.3.1, $H^q(X_p, \mathcal{L}_p) \to H^q(U_p, \mathcal{L}_p|_{U_p})$ is injective for all p, q. Equivalently,

$$H^q_{\Sigma_p}(X_p, \mathcal{L}_p) \to H^q(X_p, \mathcal{L}_p)$$

is zero for all p, q.

Step 2: $\operatorname{Gr}_S \alpha$ is injective. Indeed, $E_1A \to E_1B$ is the direct sum of $H^q_{\Sigma_p}(X_p, \mathcal{L}_p) \to H^q(X_p, \mathcal{L}_p)$. By Step 1, $E_1A \to E_1B$ is zero. Step by step, we deduce that $E_rA \to E_rB$ is zero for every $r \geq 1$. Then $E_\infty A \to E_\infty B$ is zero, that is $Gr_S H^i_{\Sigma_\bullet}(X_\bullet, \mathcal{L}_\bullet) \to \operatorname{Gr}_S H^i(X_\bullet, \mathcal{L}_\bullet)$ is zero. Therefore the last long exact sequence breaks up into short exact sequences

$$0 \to \operatorname{Gr}_S H^i(X_\bullet, \mathcal{L}_\bullet) \to \operatorname{Gr}_S H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}) \to \operatorname{Gr}_S H^{i+1}_{\Sigma_\bullet}(X_\bullet, \mathcal{L}_\bullet) \to 0.$$

Step 3: Since $S_{i+1} H^i(X_\bullet, L_\bullet) = 0$, the filtration S on $H^i(X_\bullet, L_\bullet)$ is finite. Therefore the injectivity of $\operatorname{Gr}_S \alpha$ means that α is injective and strict with respect to the filtration S. □

Lemma 5.3.3. *Let (X, B) be a log smooth variety, let $f\colon X \to Z$ be a proper morphism. Let L be a Cartier divisor such that the $\mathbb{Q}$-divisor $A = L - (K_X + B)$ is f-semiample. Let D be an effective Cartier divisor on X such that $D \sim_\mathbb{Q} uA$ for some $u > 0$, and D contains no lc center of (X, B). Then the natural homomorphism $R^q f_* \mathcal{O}_X(L) \to R^q f_* \mathcal{O}_X(L + D)$ is injective, for all q.*

Proof. We may suppose Z is affine, and A is f-semiample.

Step 1: Suppose $(X, B + \epsilon D)$ is log smooth, for some $0 < \epsilon < \frac{1}{u}$. We have

$$L = K_X + B + \epsilon D + (A - \epsilon D) \sim_\mathbb{Q} K_X + B + \epsilon D + (1 - \epsilon u)A.$$

Let $n \geq 1$ such that $\mathcal{O}_X(nA)$ is generated by global sections. Let S be the zero locus of a generic global section. Then

$$L \sim_\mathbb{Q} K_X + B + \epsilon D + \frac{1 - \epsilon u}{n} S,$$

the log variety $(X, B + \epsilon D + \frac{1-\epsilon u}{n} S)$ is log smooth, and its boundary supports D. By Lemma 5.3.1, $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ is injective, for all q.

Step 2: By Hironaka, there exists a desingularization $\mu\colon X' \to X$ such that the exceptional locus of μ and the proper transforms of B and D are supported by a NC divisor on X'. Let $\mu^*(K_X + B) = K_{X'} + B_{X'}$, let $E = \lceil -B_{X'}^{<0} \rceil$. Then

$$\mu^* L + E = K_{X'} + B_{X'}^{\geq 0} + \{B_{X'}^{<0}\} + \mu^* A.$$

The log variety $(X', B_{X'}^{\geq 0} + \{B_{X'}^{<0}\} + \epsilon \mu^* D)$ is log smooth for $0 < \epsilon \ll 1$, by the choice of the resolution, and since D contains no lc centers of (X, B). We also have $\mu^* D \sim_{\mathbb{Q}} u \mu^* A$. By Step 1, the natural homomorphisms

$$H^q(X', \mathcal{O}_{X'}(\mu^* L + E)) \to H^q(X', \mathcal{O}_{X'}(\mu^* L + E + \mu^* D))$$

are injective. Consider now the commutative diagram

$$\begin{tikzcd}
H^q(X', \mathcal{O}_{X'}(\mu^* L + E)) \arrow[r, "\alpha'"] & H^q(X', \mathcal{O}_{X'}(\mu^* L + E + \mu^* D)) \\
H^q(X, \mathcal{O}_X(L)) \arrow[u, "\beta"] \arrow[r, "\alpha"] & H^q(X, \mathcal{O}_X(L+D)) \arrow[u]
\end{tikzcd}$$

From above, α' is injective. If β is injective, it follows that α is injective. To show that β is injective, suffices to show that $\mathcal{O}_X \to R\mu_* \mathcal{O}_{X'}(E)$ has a left inverse. The Cartier divisor $E' = K_{X'} - \mu^* K_X$ is effective, and $-B_{X'} \leq E'$. Therefore $E \leq E'$. We obtain homomorphisms

$$\mathcal{O}_X \to R\mu_* \mathcal{O}_{X'}(E) \to R\mu_* \mathcal{O}_{X'}(E').$$

Suffices to show that the composition has a left inverse. Tensoring with ω_X, this is just the homomorphism $\omega_X \to R\mu_* \omega_{X'}$, which admits a left inverse defined by trace (see the proof of [18, Proposition 4.3]). $\square$

Theorem 5.3.4 (Tankeev-Kollár injectivity)**.** *Let (X, B) be a GNC log variety, let $f\colon X \to Z$ be a proper morphism. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $\mathcal{L}^{\otimes r} \simeq \omega_{(X,B)}^{[r]} \otimes \mathcal{H}$, where $r \geq 1$ and rB has integer coefficients, and $\mathcal{H}$ is an invertible $\mathcal{O}_X$-module such that $f^* f_* \mathcal{H} \to \mathcal{H}$ is surjective. Let $s \in \Gamma(X, \mathcal{H})$ be a global section which is invertible at the generic point of each lc center of (X, B), let D be the effective Cartier divisor defined by s. In particular, D contains no lc center of (X, B). Then the natural homomorphism $R^q f_* \mathcal{O}_X(L) \to R^q f_* \mathcal{O}_X(L+D)$ is injective, for all q.*

Proof. We may suppose Z is affine. In particular, $\mathcal{H}$ is generated by global sections. Let $U = X \setminus \operatorname{Supp} D$. The claim for D and all its multiples is equivalent to the injectivity of the restriction homomorphisms $H^i(X, \mathcal{L}) \to H^i(U, \mathcal{L}|_U)$.

The proof is the same as that of Theorem 5.3.2, except that in Step 1 we use Lemma 5.3.3 instead of Lemma 5.3.1. Indeed, $\mathcal{L}_p^{\otimes 2r} \xrightarrow{\sim} \omega_{(X_p, B_p)}^{[2r]} \otimes \mathcal{H}_p^{\otimes 2}$, (X_p, B_p) is log smooth, $\mathcal{H}_p$ is generated by global sections, and $\epsilon_p^* D \in |\mathcal{H}_p|$ contains no lc center of (X_p, B_p). Therefore $H^q(X_p, \mathcal{L}_p) \to H^q(U_p, \mathcal{L}_p|_{U_p})$ is injective, where $U_p = \epsilon_p^{-1}(U) = X_p \setminus \operatorname{Supp} \epsilon_p^* D$. $\square$

Theorem 5.3.5 (Kollár's torsion freeness). *Let (X,B) be a GNC log variety. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $\mathcal{L}^{\otimes r} \simeq \omega^{[r]}_{(X,B)}$ for some $r \geq 1$ such that rB has integer coefficients. Let $f\colon X \to Z$ be a proper morphism. Let s be a local section of $R^q f_*\mathcal{L}$ whose support does not contain $f(C)$, for every lc center C of (X,B). Then $s = 0$.*

Proof. Suppose by contradiction that $s \neq 0$. Choose a closed point $P \in \operatorname{Supp}(s)$. We shrink Z to an affine neighborhood of P. There exists a non-zero divisor $h \in \mathcal{O}_{Z,P}$ which vanishes on $\operatorname{Supp}(s)$, but does not vanish identically on $f(C)$, for every lc center C of (X,B). There exists $n \geq 1$ such that $h^n s = 0$ in $(R^q f_*\mathcal{L})_P$.

After shrinking Z near P, we may suppose that $0 \neq s \in \Gamma(Z, R^q f_*\mathcal{L})$, $h \in \Gamma(Z, \mathcal{O}_Z)$ is a non-zero divisor, $h^n s = 0$, and h is invertible at the generic point of $f(C)$, for every lc center C of (X,B). Since Z is affine, we have an isomorphism $\Gamma(Z, R^q f_*\mathcal{L}) \simeq H^q(X,\mathcal{L})$. Therefore the multiplication $\otimes f^*h^n\colon H^q(X,\mathcal{L}) \to H^q(X,\mathcal{L})$ is not injective. But $f^*h \in \Gamma(X,\mathcal{O}_X)$ is invertible at the generic point of each lc center of (X,B). By Theorem 5.3.4 with $\mathcal{H} = \mathcal{O}_X$, the multiplication $\otimes f^*h\colon H^q(X,\mathcal{L}) \to H^q(X,\mathcal{L})$ is injective. Contradiction! □

Theorem 5.3.6 (Ohsawa-Kollár vanishing). *Let (X,B) be a GNC log variety, let $f\colon X \to Y$ be a proper morphism and $g\colon Y \to Z$ a projective morphism. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $\mathcal{L}^{\otimes r} \simeq \omega^{[r]}_{(X,B)} \otimes f^*\mathcal{A}$, where $r \geq 1$ and rB has integer coefficients, and $\mathcal{A}$ is a g-ample invertible $\mathcal{O}_Y$-module. Then $R^p g_* R^q f_*\mathcal{L} = 0$ for all $p > 0, q \geq 0$.*

Proof. We use induction on the dimension of X. We may suppose Z is affine. Replacing r by a multiple, we may suppose $\mathcal{A}$ is g-generated. Let m be a sufficiently large integer, to be chosen later. Let S be the zero locus of a general global section of $\mathcal{A}^{\otimes m}$. Denote $T = f^*S$.

Consider the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(T) \to \mathcal{L}(T)|_T \to 0.$$

The connecting homomorphism $\partial\colon R^q f_*\mathcal{L}(T)|_T \to R^{q+1} f_*\mathcal{L}$ is zero by Theorem 5.3.5, since the image is supported by T, which contains no lc center of (X,B), and $\mathcal{L}^{\otimes r} \simeq \omega^{[r]}_{(X,B)}$ locally over Y. Therefore the long exact sequence in cohomology breaks up into short exact sequences

$$0 \to R^q f_*\mathcal{L} \to R^q f_*\mathcal{L}(T) \to R^q f_*\mathcal{L}(T)|_T \to 0.$$

We have $R^p g_* R^q f_*\mathcal{L}(T) \simeq R^p g_*(R^q f_*\mathcal{L}(S)) \simeq R^p g_*(R^q f_*\mathcal{L} \otimes \mathcal{A}^m)$. If m is sufficiently large, Serre vanishing gives $R^p g_* R^q f_*\mathcal{L}(T) = 0$ for $p \neq 0$. By [8], $(X, B+T)$ is a GNC log variety, T is S_2 and there exists a natural boundary $B_T = B|_T$ such that (T, B_T) is a GNC log variety, and codimension one residues glue to residue isomorphisms

$$\operatorname{Res}^{[2r]}_{X\to T}\colon \omega^{[2r]}_{(X,B+T)}|_T \xrightarrow{\sim} \omega^{[2r]}_{(T,B_T)}.$$

From $\mathcal{L}(T)^{\otimes r} \simeq \omega^{[r]}_{(X,B+T)} \otimes f^*\mathcal{A}$ we obtain $\mathcal{L}(T)|_T^{\otimes 2r} \simeq \omega^{[2r]}_{(T,B_T)} \otimes (f|_T)^*(\mathcal{A}|_S^{\otimes 2})$. Since $\dim T < \dim X$, we obtain by induction $R^p g_* R^q f_*\mathcal{L}(T)|_T = 0$ for $p \neq 0$.

From the short exact sequence above, we deduce $R^p g_* R^q f_* \mathcal{L} = 0$ for $p \geq 2$. For $p = 1$, consider the commutative diagram

$$\begin{array}{ccc} R^{1+q}(g\circ f)_*\mathcal{L} & \xrightarrow{\beta} & R^{1+q}(g\circ f)_*\mathcal{L}(T) \\ \uparrow & & \uparrow \\ R^1 g_* R^q f_* \mathcal{L} & \longrightarrow & R^1 g_* R^q f_* \mathcal{L}(T) = 0 \end{array}$$

The vertical arrows are injective, from the Leray spectral sequence. The homomorphism β is injective by Theorem 5.3.4, since $T \in |f^*\mathcal{A}^{\otimes m}|$ contains no lc centers of (X, B). A diagram chase gives $R^1 g_* R^q f_* \mathcal{L} = 0$. $\square$

5.4 Inductive properties of GNC log varieties

Proposition 5.4.1. *Let (X, B) be a GNC log variety. Let $Y = \mathrm{LCS}(X, B)$ and (Y, B_Y) the n-wlc structure induced by glueing of codimension one residues. Then $B_Y = (B - B^{=1})|_Y$ and (Y, B_Y) is a GNC log variety. If $2 \mid r$ and rB has integer coefficients, then $\mathrm{Res}^{[r]} \colon \omega^{[r]}_{(X,B)}|_Y \xrightarrow{\sim} \omega^{[r]}_{(Y,B_Y)}$ is an isomorphism. Moreover,*

1) *Let $\pi \colon (\bar{X}, B_{\bar{X}}) \to (X, B)$ be the normalization of X, with induced log variety structure (with log smooth support). Let $\bar{Y} = \mathrm{LCS}(\bar{X}, B_{\bar{X}})$. Let $n \colon Y^n \to Y$ and $\bar{n} \colon \bar{Y}^n \to \bar{Y}$ be the normalizations. In the commutative diagram*

$$\begin{array}{ccccc} \bar{X} & \longleftarrow & \bar{Y} & \xleftarrow{\ \bar{n}\ } & \bar{Y}^n \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle g} \\ X & \longleftarrow & Y & \xleftarrow{\ n\ } & Y^n \end{array}$$

each square is both cartesian and a push-out, and g is an étale covering. With the log structures induced by glueing of codimension one residues, we obtain a commutative diagram of GNC log varieties and log crepant morphisms

$$\begin{array}{ccccc} (\bar{X}, B_{\bar{X}}) & \longleftarrow & (\bar{Y}, B_{\bar{Y}}) & \xleftarrow{\ \bar{n}\ } & (\bar{Y}^n, B_{\bar{Y}^n}) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle g} \\ (X, B) & \longleftarrow & (Y, B_Y) & \xleftarrow{\ n\ } & (Y^n, B_{Y^n}) \end{array}$$

2) *The lc centers of (X, B) are the irreducible components of X and the lc centers of (Y, B_Y).*

Proof. 1) We may suppose (X, B) is a GNC local model. Let $X = \cup_F \mathbb{A}_F \hookrightarrow \mathbb{A}^N$ and $B = \sum_{i \in \sigma} b_i H_i|_X$, with core $\sigma = \cap_F F$. Denote $\sigma' = \{i \in \sigma; b_i < 1\}$. Then $\psi = \sum_{i\in\sigma}(1 - b_i)m_i = \sum_{i \in \sigma'}(1 - b_i)m_i$, which belongs to the relative interior of σ'. We deduce that

X_γ is an lc center of (X,B) if and only if $\sigma' \prec \gamma \in \Delta$. Therefore $Y = \cup_\tau \mathbb{A}_\tau \hookrightarrow \mathbb{A}^N$ is an irreducible decomposition, where the union is taken after all codimension one faces $\tau \in \Delta$ which contain σ'. In particular, the core of Y is σ'. One checks that $(Y,0)$ satisfies properties a) and b) of the GNC local model. The boundary induced by codimension one residues is $B_Y = \sum_{i\in\sigma'} b_i H_i|_Y = (B - B^{=1})|_Y$, which satisfies c). The commutative diagram becomes

$$\begin{tikzcd}
\sqcup_F \mathbb{A}_F \arrow[d, "\pi"'] & \sqcup_F \cup_{\tau\prec F} \mathbb{A}_\tau \arrow[l] \arrow[d, "\pi"'] & & \sqcup_F \sqcup_{\tau\prec F} \mathbb{A}_\tau \arrow[ll, "\bar{n}"'] \arrow[d, "g"] \\
\cup_F \mathbb{A}_F & \cup_\tau \mathbb{A}_\tau \arrow[l] & & \sqcup_\tau \mathbb{A}_\tau \arrow[ll, "n"']
\end{tikzcd}$$

and one checks that both squares are push-outs and cartesian, using axioms a) and b) of the GNC local models. Over $\mathbb{A}_\tau$, g consists of several identical copies of $\mathbb{A}_\tau$, one for each facet F which contains τ. Therefore g is an étale covering. All log structures have the same log discrepancy function ψ, hence the morphisms of the diagram are log crepant.

2) *Step 1*: The claim holds if (X,B) is a GNC local model. Indeed, the lc centers of (X,B) are the invariant cycles X_γ such that $\psi \in \gamma$ and $\gamma \in \Delta$, and the lc centers of (Y, B_Y) are the invariant cycles X_γ such that $\psi \in \gamma$ and $\gamma \in \Delta$ is a face of positive codimension.

Step 2: We reduce the claim to the case when (X,B) has log smooth support. Indeed, consider the commutative diagram of log structures in 1). The log structure on the normalization $(\bar{X}, B_{\bar{X}})$ has log smooth support. By Lemma 5.1.1 for g and a diagram chase, the claim for (X,B) and its LCS-locus is equivalent to the claim for $(\bar{X}, B_{\bar{X}})$ and its LCS-locus.

Step 3: Let (X,B) have log smooth support. Then $Y = B^{=1}$ and the induced boundary is $B_Y = (B-Y)|_Y$. We have to show that for a closed subset $Z \subseteq Y$, Z is an lc center of (X,B) if and only if Z is an lc center of (Y,B_Y), i.e. the image of an lc center of the normalization (Y^n, B_{Y^n}). We may cut with general hyperplane sections, and suppose Z is a closed point P. Note that if $f\colon (X', B_{X'}) \to (X,B)$ is étale log crepant, then $Y' = f^*Y$, and since normalization commutes with étale base change, we obtain a cartesian diagram

$$\begin{tikzcd}
(Y'^n, B_{Y'^n}) \arrow[r, "g"] \arrow[d, "n'"'] & (Y^n, B_{Y^n}) \arrow[d, "n"'] \\
(X', B_{X'}) \arrow[r, "f"] & (X,B)
\end{tikzcd}$$

with f, g étale log crepant. By Lemma 5.1.1 for f and g, the claim holds for n if and only if it holds for n'. By the existence of a common étale neighborhood [10] and Step 1, we are done. $\square$

Corollary 5.4.2. *Let X be a GNC log variety. Then $S = \operatorname{Sing} X$ coincides with the non-normal locus of X, and with $\mathrm{LCS}(X,0)$. The n-wlc structure induced by glueing of codimension one residues is $(S,0)$, a GNC log variety, and $\mathrm{Res}^{[2]}\colon \omega_X^{[2]}|_S \xrightarrow{\sim} \omega_S^{[2]}$ is an isomorphism. Moreover,*

1) *Let $\pi\colon(\bar{X},\bar{C})\to(X,0)$ be the normalization of X, with induced log variety structure (with log smooth support). Note that $\bar{C}=\mathrm{LCS}(\bar{X},\bar{C})$. Let $n\colon S^n\to S$ and $\bar{n}\colon\bar{C}^n\to\bar{C}$ be the normalizations. In the commutative diagram*

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"'] & \bar{C} \arrow[l] \arrow[d, "\pi"'] & & \bar{C}^n \arrow[ll, "\bar{n}"'] \arrow[d, "g"] \\ X & S \arrow[l] & & S^n \arrow[ll, "n"'] \end{tikzcd}$$

each square is both cartesian and a push-out, and g is an étale covering. With the log structures induced by glueing of codimension one residues, we obtain a commutative diagram of GNC log varietis and log crepant morphisms

$$\begin{tikzcd} (\bar{X},\bar{C}) \arrow[d, "\pi"'] & (\bar{C},0) \arrow[l] \arrow[d, "\pi"'] & & (\bar{C}^n,\mathrm{Cond}\,\bar{n}) \arrow[ll, "\bar{n}"'] \arrow[d, "g"] \\ (X,0) & (S,0) \arrow[l] & & (S^n,\mathrm{Cond}\,n) \arrow[ll, "n"'] \end{tikzcd}$$

2) *The lc centers of X are the irreducible components of X and the lc centers of S.*

Proof. It remains to check that $S=C=\mathrm{LCS}(X,0)$. First of all, we claim that $S=C$. Indeed, let $x\in X$. We show that $\mathcal{O}_{X,x}$ is normal if and only if $\mathcal{O}_{X,x}$ is nonsingular. We may suppose $x\in X$ is a local model $X=\cup_F X_F$ and x belongs to the closed orbit of X. Then $\mathcal{O}_{X,x}$ is normal if and only if there is only one facet F. As X_F is smooth, the latter is equivalent to $\mathcal{O}_{X,x}$ being smooth.

Since $X\setminus S$ is smooth, $\mathrm{LCS}(X,0)\subseteq S$. On the other hand, each irreducible component Q of S is an irreducible component of C. Therefore Q is an lc center. We conclude that $\mathrm{LCS}(X,0)=S$. $\square$

Remark 5.4.3. Let (X,B) be a GNC log variety. Let $S=\mathrm{Sing}\,X$ and $B_S=B|_S$. One can also show that (S,B_S) is a GNC log variety, induced by codimension one residues. If $2\mid r$ and rB has integer coefficients, the glueing of codimension one residues induces an isomorphism $\mathrm{Res}^{[r]}\colon\omega^{[r]}_{(X,B)}|_S\xrightarrow{\sim}\omega^{[r]}_{(S,B_S)}$.

Lemma 5.4.4. *Let (X,B) be a GNC log variety. Let $\pi\colon(\bar{X},B_{\bar{X}})\to(X,B)$ be the normalization of X, with the induced log variety structure. Let $Y=\mathrm{LCS}(X,B)$. Let Z be a union of lc centers of (X,B).*

1) *$Z\cap Y$ is a union of lc centers of (Y,B_Y).*

2) *$\pi^{-1}(Z)$ is a union of lc centers of $(\bar{X},B_{\bar{X}})$.*

3) *We have a short exact sequence $0\to\mathcal{I}_{Z\cup Y\subset X}\to\mathcal{I}_{Z\subset X}\xrightarrow{|_Y}\mathcal{I}_{Z\cap Y\subset Y}\to 0$.*

Proof. 1) We may suppose Z is an lc center. If $Z \subseteq Y$, the claim is clear. Therefore we may suppose Z is an irreducible component of X. Then the normalization $\bar{Z}$ of Z is an irreducible component of the normalization $\bar{X}$ of X. We have $\pi^{-1}(Y) = \bar{Y} = \mathrm{LCS}(\bar{X}, B_{\bar{X}})$. Therefore $Z \cap Y = \pi(\bar{Z} \cap \bar{Y})$. We have $\bar{Z} \cap \bar{Y} = \mathrm{LCS}(\bar{X}, B_{\bar{X}})|_{\bar{Z}}$, we deduce that $\bar{Z} \cap \bar{Y}$ is a union of lc centers of $(\bar{X}, B_{\bar{X}})$ contained in $\bar{Y}$. Therefore $Z \cap Y$ is a union of lc centers of (X, B) contained in Y, hence lc centers of (Y, B_Y), by Proposition 5.4.1.

2) We use induction on $\dim X$. We may suppose Z is an lc center. If Z is an irreducible component of X, then its normalization $\bar{Z}$ is an irreducible component of $\bar{X}$, and $\pi^{-1}(Z) = \bar{Z} \cup \pi^{-1}(Z \cap Y)$, since Y contains the non-normal locus of X. By induction, the claim holds.

Suppose Z is not an irreducible component of X. Then $Z \subseteq Y$, by Proposition 5.4.1. By induction, $n^{-1}(Z)$ is a union of lc centers of (Y^n, B_{Y^n}). Let W be such an lc center. Since g is finite flat, each irreducible component of $g^{-1}(W)$ dominates W. Therefore $g^{-1}n^{-1}(Z)$ is a union of lc centers of $(\bar{Y}^n, B_{\bar{Y}^n})$, by Lemma 5.1.1. Equivalently, $\bar{n}^{-1}\pi^{-1}(Z)$ is a union of lc centers of $(\bar{Y}^n, B_{\bar{Y}^n})$. Therefore $\pi^{-1}(Z)$ is a union of lc centers of $(\bar{Y}, B_{\bar{Y}})$. The latter lc centers are also lc centers of $(\bar{X}, B_{\bar{X}})$.

3) The sequence is exact if and only if $|_Y \colon \mathcal{I}_{Z \subset X} \to \mathcal{I}_{Z \cap Y \subset Y}$ is surjective, if and only if $|_Y \colon \mathcal{I}_{Z \subset Z \cup Y} \to \mathcal{I}_{Z \cap Y \subset Y}$ is surjective, if and only if the diagram

$$\begin{tikzcd} Y \arrow[d] & Y \cap Z \arrow[l] \arrow[d] \\ Y \cup Z & Z \arrow[l] \end{tikzcd}$$

is a push-out. By [47], this diagram is a push-out if $Y \cup Z$ is weakly normal. To show this, consider the normalization $\pi \colon \bar{X} \to X$. Denote $W = \pi^{-1}(Y \cup Z)$. Since

$$\begin{tikzcd} \bar{X} \arrow[d] & \bar{Y} \arrow[l] \arrow[d] \\ X & Y \arrow[l] \end{tikzcd}$$

is a push-out and $Y \cup Z$ contains Y, the diagram

$$\begin{tikzcd} \bar{X} \arrow[d] & W \arrow[l] \arrow[d] \\ X & Y \cup Z \arrow[l] \end{tikzcd}$$

is also a push-out. But $\bar{X}$ is smooth, and W is the union of $\bar{Y}$ with the irreducible components of $\bar{X}$ which are mapped into Z. Therefore the singularities of W are at most normal crossings. We conclude that $X, \bar{X}, W$ are weakly normal. From the last push-out diagram, we deduce that $Y \cup Z$ is weakly normal as well. $\square$

The results of this section can be used to reduce Kollár's torsion freeness theorem and Ohsawa-Kollár vanishing theorem from the GNC varieties to log smooth varieties. This is done by a using the push-out and cartesian diagram obtained from normalization and restriction to the LCS-locus. We were unable to use the same argument to reduce the injectivity theorems from GNC varieties to log smooth varieties, but we expect this is possible.

Chapter 6

Future developments

My future research is related to the central problems of the Classification Theory of Algebraic Varieties, such as the Abundance and Termination Conjecture, the study of singularities appearing on minimal models, or the study of moduli spaces of log canonically polarized algebraic varieties.

In this chapter we will restrict ourselves to future research directions related to the content of this thesis. First of all, we pose some questions arising naturally from Chapter 3 and 4.

Question 6.0.5. *Let $(X/k, B)$ be a wlc log pair which is locally analytically isomorphic to a toric wlc log pair (the toric local model may have non-normal irreducible components). Let Z be an lc center, let $Z^n \to Z$ be the normalization. Is there a residue isomorphism from X to Z^n? Is it torsion the moduli part in the higher codimension adjunction formula from $(X/k, B)$ to Z^n?*

Question 6.0.6. *Let (X, Σ) be a log smooth pair, with X proper. Denote $U = X \setminus \Sigma$. Is the restriction $H^q(X, \Omega^p_X(\log \Sigma)) \to H^q(U, \Omega^p_U)$ injective for $p + q > \dim X$?*

Example 6.0.7. Let $P \in S$ be the germ of non-singular point, of dimension $d \geq 2$. Let $\mu \colon X \to S$ be the blow-up at P, with exceptional locus $E \simeq \mathbb{P}^{d-1}$. Denote $U = X \setminus E$. The residue map

$$R^{d-1}\mu_*\mathcal{O}_X(K_X + E) \to R^{d-1}\mu_*\mathcal{O}_E(K_E)$$

is an isomorphism, so $R^{d-1}\mu_*\mathcal{O}_X(K_X+E)$ is a skyscraper sheaf on X centered at P. Since μ is an isomorphism on U, $R^{d-1}(\mu|_U)_*\mathcal{O}_E(K_E) = 0$. Therefore the restriction homomorphism

$$R^{d-1}\mu_*\mathcal{O}_X(K_X + E) \to R^{d-1}(\mu|_U)_*\mathcal{O}_U(K_U)$$

is not injective.

Question 6.0.8. *Let (X, Σ) be a log smooth pair. Denote $U = X \setminus \Sigma$. Let $\pi \colon X \to S$ be a proper morphism, let $\pi|_U \colon U \to S$ be its restriction to U. Suppose that $\pi(C) = \pi(X)$ for every strata C of (X, Σ). Is the restriction $R^q\pi_*\mathcal{O}_X(K_X+\Sigma) \to R^q(\pi|_U)_*\mathcal{O}_U(K_U)$ injective for all q?*

Question 6.0.9. *Let (X, B) be a proper log variety with log canonical singularities. Let U be the totally canonical locus of (X, B). Let L be a Cartier divisor on X such that $L \sim_{\mathbb{R}} K_X + B$. Is the restriction $H^q(X, \mathcal{O}_X(L)) \to H^q(U, \mathcal{O}_U(L|_U))$ injective for all q?*

Question 6.0.10. *Let (X, B) be a proper log variety. Suppose the locus of non-log canonical singularities $Y = (X, B)_{-\infty}$ is non-empty. Let L be a Cartier divisor on X such that $L \sim_{\mathbb{R}} K_X + B$. Does the long exact sequence induced in cohomology by the short exact sequence $0 \to \mathcal{I}_Y(L) \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L) \to 0$ break up into short exact sequences*

$$0 \to H^q(X, \mathcal{I}_Y(L)) \to H^q(X, \mathcal{O}_X(L)) \to H^q(Y, \mathcal{O}_Y(L)) \to 0 \ (q \geq 0)?$$

Second, an interesting research direction is to better understand weakly log canonical singularities. The toric case is classified in this thesis. The next case to understand would be that of curves.

Problem 6.0.11. A curve C is weakly log canonical if and only if it has seminormal singularities. The sheaf $\omega_C^{[1]}$ defined in this thesis is then invertible.

- Can we imitate the classification of smooth projective curves, to classify C according to the positivity properties of $\omega_C^{[1]}$?
- There exists a coarse moduli space for projective seminormal curves C with $\omega_C^{[1]}$ ample, of fixed degree?

Problem 6.0.12. Are the Base Point Free and Cone Theorems valid in the category of weakly normal log varieties? Are weakly normal singularities Du Bois? Is the usual calculus with lc centers valid for weakly log canonical log varieties? What is the structure of the unions of lc centers?

Third, we are interested in understanding the finite generation of the log canonical ring and the log minimal model program in the context of normal crossings varieties.

Problem 6.0.13. Let (X, B) be a projective normal crossings log variety defined over $\mathbb{C}$, with associated graded log canonical ring

$$R = \oplus_{n \geq 0} \Gamma(X, \omega_{(X,B)}^{[n]}).$$

There are simple examples where R is not finitely generated. But we expect that R is finitely generated under the following extra hypothesis: there exists $n > 0$ such that $\omega_{(X,B)}^{[n]}$ is generated by global sections at the generic point of each lc center of (X, B). This problem appears when trying to prove its special case when (X, B) is log smooth, a well known conjecture.

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