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HABILITATION THESIS

Deterministic and Stochastic Variational Inequalities. A convex to nonconvex journey

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ABSTRACT

Domeniul fundamental: Matematică și științe ale naturii

Domeniul de abilitare: Matematică

Teză elaborată în vederea obţinerii atestatului de abilitare în scopul conducerii lucrărilor de doctorat în domeniul Matematică.

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Contents

1 A short history on the domain. Main personal achievements

Since early sixties, research has paid increasing attention to the study of reflected stochastic differential equations, the reflection process being interpreted in different ways. As the first approaches of the topic, Skorokhod considers the problem of reflection for diffusion processes into a bounded domain (see, e.g., [160]), while Tanaka focuses on the problem of reflecting boundary conditions into convex sets for SDEs (see [165]). This kind of problem became the interest of many other authors, who consider that the state process is reflected by one or two reflecting barriers, both for forward and backward stochastic differential equations. The field of applications of such problems also bloomed.

The aim of the present thesis consists in presenting the personal contribution in the area during the time from my Ph.D. thesis, defended in 2009. The research can be described as being focused on two mainstreams: forward stochastic variational inequalities and backward stochastic variational inequalities.

The intention is to extend, for the beginning, the notion of reflection for forward and backward stochastic differential equations by allowing to the reflecting directions to be perturbed by a Lipschitz term which destroys the maximal monotonicity of the multivalued operator which acts on the equation. In this manner, the classical Yosida approximating techniques must be reinterpreted. On the other hand, the author addresses some different kind of problems, by presenting the gap which arrises when we renounce at the convexity of the reflecting domains. This extension comes with two different kind of problems, apart from the ones given by the non-Lipschitz and the non-maximal-monotonicity of the term driving the involved equations. First, when we situate in the forward framework for stochastic differential equations, as we know, a deterministic (Skorokhod) problem must be analyzed. This is done by approximating it with Yosida penalizations. While, in the convex setup, we develop a technique which permits us to provide existence and uniqueness results, in the nonconvex setup, even for the approximating equations there are no known results for the existence of the solution. This is why we have to construct the tools needed for approximating the deterministic multivalued equations in this new framework. More precisely, we give some adapted and refined properties of the Yosida penalization, in a similar manner with the corresponding counterpart, found in the convex setup, as we can see, for example, in Barbu [6, Chapter 1] or Brézis [38, Chapter II]. The results are quite strong and permit to approach the second type of problem which appears when we leave the convex framework, as one can see in Chapter 3, Section 3.4. When dealing with backward stochastic differential equations, with oblique reflection and convex constraints, we do not have to consider first a deterministic problem and we provide directly some results concerning the existence and uniqueness of a solution. This solution is strong or weak, depending on the dependence on the state process of the perturbing term; the feedback term is absolutely continuous, not only a continuous and bounded variation one, as we can obtain in the forward case. However, the problem of the existence of a solution for backward stochastic variational inequalities with nonconvex constraints was - and partially remains - a very important open problem in stochastic analysis. With the tools developed in Chapter 2, Section 2.3 we are able to partially solve the problem by replacing the Brownian movement driving the multivalued equation with a different kind of stochastic process, PDMP (that is, Piecewise deterministic Markov process). Also, as a by-product, this is the first time when occupation measures were used to consider control problems for this type of backward variational inclusions. As application, we study some mathematical properties leading to the detection of infection time in a specific class of stochastic gene networks. The basic example one refers is a bistable (multistable) system consisting in a temperate virus and a host.

We now briefly present the structure of the thesis.

Chapter 1 presents a relatively detailed history of the domain, with precise informations regarding the author's contribution on the filed.

Chapter 2 is divided into five sections and deals with the analysis of some forward deterministic and stochastic variational inclusions, with generalized reflection. For the beginning, Section 2.1 presents some invariance criteria for a stochastic differential equation whose state evolution is constrained by time-dependent security tubes, by considering an equivalent problem where the square of distance function represents a viscosity solution to an adequately defined partial differential equation. We also derive a sufficient condition for the case of the drift coefficient defined by a linear expression; this sufficient condition has a significantly simpler form that facilitates the concrete use in applications. We then present a broader context when solutions are constrained by more general time-dependent convex domains. The analyzed problems become more general in Section 2.2. Its main objective is to study a stochastic variational inequality featuring a product of the for $H\left(X\right)\partial\varphi\left(X\right)$ which will be called the set of *oblique subgradients*. The problem becomes challenging due to the presence of this new term, which imposes the use of some specific approaches because this new term preserves neither the monotony of the subdifferential operator nor the Lipschitz property of the matrix involved $H(X)$. In the forward case we first focus on the deterministic case, by considering a generalized Skorokhod problem with oblique reflection of the form

$$dx\left(t\right)+H\left(x\left(t\right)\right)dk\left(t\right)=f\left(t,x\left(t\right)\right)dt+dm\left(t\right),\quad t\geq 0;\quad dk\left(s\right)\in\partial\varphi\left(x\left(s\right)\right)\left(ds\right),$$

where the singular input $m:\mathbb{R}_{+}\rightarrow\mathbb{R}^{d}$ is a continuous function. The existence results are obtained via Yosida penalization techniques.

In Section 2.3 we renounce at the convex constraints for the equations found in the previous section, by considering, for the multivalued operator from the equation the product $H\partial^{-}\varphi$, where $\partial^{-}\varphi$ stands for the Fréchet subdifferential operator of a (ρ,γ)-semiconvex function φ. Even the main milestones are similar with the one found in Section 2.2, their approach is completely different since we have to construct from the beginning the tools which we shall use. The challenging part proves to be the study of a basic Cauchy problem. After we obtain the existence and uniqueness of its solution, we focus on a nonconvex Skorokhod problem with oblique Fréchet subgradients and, as applications, stochastic variational inequalities with nonconvex constraints are envisaged.

Section 2.4 studies obstacle problems for some stochastic differential equations of parabolic type. The key point consists in a weaker Hölder continuity condition for the diffusion process. Under the Gelfand-Lions triple setup, the authors obtain, under some "sufficiently rigid" barriers, a unique strong solution for the multivalued stochastic equation. Without that restricted condition, the notion of the solution can be extended and one obtain a weak-variational solution. Apart from the importance of its result, this section establishes also a rigorous construction of the framework for the formalism used in pioneering work Bensoussan, Răşcanu [15], where the measurability issues are omitted. Finally, in Section 2.5, the study aims to some infinite dimensional stochastic variational inequalities with oblique reflection. We prove the existence of a solution for a smooth multivalued problem with generalized reflection at the frontier and some applications to systems of PDEs are also provided.

Chapter 3 is structured also under the form of five sections and deals with the analysis of some backward stochastic variational inclusions, with - possibly - generalized reflection.

Section 3.1 treats the existence and uniqueness of the solution for the more general backward problem, featuring convex constraints and driven by a Brownian movement B,

$$(1) \qquad -dY_t + H(t, Y_t)\, \partial\varphi(Y_t)(dt) \ni F(t, Y_t, Z_t)\, dt - Z_t dB_t,\ t \in [0, T], \quad Y_T = \eta,$$

When we have only a time dependence for the matrix H we obtain the existence of a strong solution, together with the existence of an absolutely continuous feedback process. For the general case of a state dependence for H we use tightness criteria in order to get a solution for the equation. In Section 3.2 we renounce at the perturbing matrix H and derive some numerical Euler-Yosida schemes for the backward stochastic variational inequality considered in a Markovian framework. The convergence rate of the scheme is also provided. In Section 3.3 we prove the existence and uniqueness of the solution for an anticipated backward variational inclusion of the type (1), with the driver $F\left(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\eta(t)}\right)$, $t \in [0, T]$ and the terminal conditions $Y_t = \xi_t$, $Z_t = \zeta_t$, $t \in [T, T+\ell]$, $\mathbb{P}$–a.s. The instruments which permit to obtain the desired results come from Chapter 3, Section 3.1 and Peng,Yang [139].

Section 3.4 makes the step from the convex to the nonconvex setup for generalized backward stochastic variational inequalities. We investigate a mathematical model associated to the infection time in multistable gene networks. The mathematical processes are of hybrid switch type. The switch is governed by pure jump modes and linked to DNA bindings. The differential component follows backward stochastic dynamics reflected in some mode-dependent, nonconvex domains. First, we study the existence of solutions to the resulting stochastic variational inclusions, by reducing the model to a family of ordinary variational inclusions (see Confortola, Fuhrman, Jacod [49]) with generalized reflection in semiconvex domains. Second, by considering control-dependent drivers, we hint to some model-selection approach by embedding the (controlled) backward stochastic variational inclusion in a family of regular measures. Regularity and structural properties of these sets are given.

Using the reduction from Confortola, Fuhrman, Jacod [49], in Section 3.5, we propose an explicit, easily-computable algebraic criterion for approximate null-controllability of a class of general piecewise linear switch systems with a multiplicative noise. Second, we prove by examples that the notion of approximate controllability is strictly stronger than approximate null-controllability. A sufficient criterion for this stronger notion is also provided. The results are illustrated on a model derived from repressed bacterium operon (introduced in Krishna, Banerjee, Ramakrishnan, Shivashankar [104] and reduced in Crudu, Debussche, Rădulescu [56]).

Chapter 4 is devoted to some open problems and future research topics strongly related to the subjects analyzed during the present work.

The thesis ends with a rich list of 177 reference papers, all cited in the text.

The present thesis is based on 11 scientific papers, which were done within the projects:

- CNCSIS 1156 / 2005 (-2008), *Deterministic and stochastic differential models with states constraints. Control, invariance and numerical approximation*
- IDEI ID_395 / 2007 (-2010), *Differential systems with random perturbations; control and viability problems*
- PN-II-ID-PCE-2011-3-0843, no. 241/05.10.2011 (-2016), *Deterministic and stochastic systems with state constraints*
- PN-II-ID-PCE-2011-3-1038, no. 208/05.10.2011 (-2015), *Diagonal stability and flow invariance in control engineering. Techniques specialized for classes of dynamics, encompassed by a unified framework*
- FP7-PEOPLE-2007-1-1-ITN, no. 213841-2 / 2008 (-2012), *Deterministic and Stochastic Controlled Systems and Applications*

2 Forward Stochastic Variational Inequalities

2.1 Invariance and feedback approach

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis and $\{B_t : t \geq 0\}$ a $\mathbb{R}^k-$valued Brownian motion. Given a non-empty closed set $K \in \mathbb{R}^d$, a starting moment $t_0 \geq 0$ and a starting point $x_0 \in K$, it is known that, by adding a supplementary source (for example the convexity indicator function of a given convex set K) on the stochastic equation

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^{t\vee t_0} f(r, X_r^{t_0,x_0})dr + \int_{t_0}^{t\vee t_0} g(r, X_r^{t_0,x_0})dB_r, \quad t \geq 0, \tag{2}$$

we can maintain the solution $X_t^{t_0,x_0} \in K$, for all $t \geq t_0$. It is natural to see what are the conditions on the drift and diffusion coefficients such that the evolution of the state satisfies the constraint $X_t^{t_0,x_0} \in K$, for all $t \geq t_0$. We present, as a particular case, the situation when the coefficients of (2) are characterized by polyhedral representations.

Let us consider $\mathcal{K} = \{K(t) : t \geq 0\}$ a family of non-empty closed subsets $K(t) \subset \mathbb{R}^d$.

Definition 1 *We state that:*

- *The family $\mathcal{K}$ is strongly invariant for SDE (2) if, for all $t_0 \geq 0$, $x_0 \in K(t_0)$ and, for all the solutions $\{X_t^{t_0,x_0} : t \geq t_0\}$ it follows that $X_t^{t_0,x_0} \in K(t), \mathbb{P}-\ a.s.,\ \forall t \geq t_0$.*
- *The family $\mathcal{K}$ is weakly invariant (viable) for SDE (2) if, for every $t_0 \geq 0$ and $x_0 \in K(t_0)$ there exists a solution $\{X_t^{t_0,x_0} : t \geq t_0\}$ such that $X_t^{t_0,x_0} \in K(t), \mathbb{P}-a.s.,\ \forall t \geq t_0$.*

We give a characterization of the invariance in the moving sets $K(t)$, $t \geq 0$. Consider the continuous functions $f : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ and assume that there exist $L, M > 0$ and $\mu \in \mathbb{R}$ such that, for $\forall t \in [0, T]$, $\forall x, y \in \mathbb{R}^d$ we have

$$\begin{cases} i) & \langle x - y, f(t, x) - f(t, y)\rangle \leq \mu|x - y|^2, \quad \sup_{t\in[0,T]} |f(t, x)| \leq M(1 + |x|). \\ ii) & |g(t, x) - g(t, y)| \leq L|x - y|. \end{cases} \tag{3}$$

Recall the notations

- the distance from x to the set $K(t)$: $d(t, x) = d_{K(t)}(x) = \inf\{|x - y| : y \in K(t)\}$,
- $\mathbb{S}^d \subset \mathbb{R}^{d\times d}$, the space of symmetric non-negative matrices,
- $C_{pol}^{k,n}([0, T] \times \mathbb{R}^d)$, the set of functions $h : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ of class $C^{k,n}$ such that the function h and its derivatives $D_t^i h(t, x)$, $j \in \overline{0, k}$ and $D_x^\alpha h(t, x)$, $\alpha = (\alpha_1, ..., \alpha_d)$, $0 \leq \alpha_1 + ... + \alpha_d \leq n$, $\alpha_i \in \mathbb{N}$ for every i, have polynomial increasing to infinity in the space variable, that is there exist $C = C_T \geq 0$ and $p = p_T \in \mathbb{N}^*$ such that, for all $(t, x) \in [0, T] \times \mathbb{R}^d$,
$$\sum\nolimits_{i,\alpha} [|D_t^i h(t, x)| + |D_x^\alpha h(t, x)|] \leq C(1 + |x|^p).$$
- the infinitesimal generator associated with $\{X_t^{t_0,x_0} : t \geq t_0\}$:
$$\mathcal{A}(t)\varphi(x) = \frac{1}{2}\mathbf{Tr}\left[D_x^2\varphi(x)g(t, x)\, g^T(t, x)\right] + \langle f(t, x), \nabla_x\varphi(x)\rangle$$

Consider now the following parabolic PDE

$$(4) \qquad \frac{\partial u(t,x)}{\partial t} + \mathcal{A}(t)u(t,x) + G(t,x) = 0, \quad u(T,x) = \Theta(x), \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

where $G \in C^{0,0}_{pol}([0,T] \times \mathbb{R}^d)$ and $\Theta \in C^{0,0}_{pol}(\mathbb{R}^d)$. For the above PDE we use the definition of the viscosity solutions introduced by Crandal, Ishii, Lions [54].

2.1.1 Invariance conditions for a control security tube

We first analyze the classical toy model of a *control security tube.* We derive necessary and sufficient conditions that allow us to maintain the trajectory as a certain distance from a time-dependent point. If the drift coefficient has a linear form, given by a polyhedral representation, than the invariance conditions can be expressed using matrix measures.

Theorem 2 *Consider $\rho \in C^1([0,T];\mathbb{R}_+)$, $\rho > 0$, $a \in C^1([0,T];\mathbb{R}^d)$ and the time-dependent domain*

$$(5) \qquad K(t) = \overline{B(a(t),\rho(t))} = \{x \in \mathbb{R}^d : |x - a(t)| \le \rho(t)\}.$$

Equation (2) is $\overline{B(a(t),\rho(t))}-$ invariant if and only if, $\forall (t,x) \in [0,T] \times \mathbb{R}^d$ with $|x - a(t)| = \rho(t)$, we have:

$$(6) \qquad \begin{cases} \text{whenever } g^*(t,x)(x - a(t)) = 0 \text{ then} \\ 2\langle x - a(t), f(t,x)\rangle + |g(t,x)|^2 \le 2\langle x - a(t), a'(t)\rangle + 2\rho(t)\rho'(t). \end{cases}$$

Comments on the linear case. We consider a linear form for the drift coefficient in SDE (2) and let the set $K(t)$ defined by (5), with $a(t) \equiv 0$. Define $f(t,x) = A(t)x$, where $A : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is a continuous and bounded in norm matrix. The second condition from (6) can be rewritten as $2x^T A(t) x + |g(t,x)|^2 \le 2\rho(t)\rho'(t)$, for $|x| = \rho(t)$, where $2x^T A(t) x = x^T(A(t) + A^T(t))x =: x^T Q(t) x$, $Q(t)$ being a symmetric matrix. The Courant-Fischer theorem applied to $Q(t)$ gives us the dominant eigenvalue

$$\sigma_{\max}(Q(t)) = \max_{x\neq 0} \frac{x^T Q(t) x}{x^T x},$$

On the other hand, $\sigma_{\max}(Q(t)) = 2\sigma_{\max}\left(\frac{1}{2}\left(A(t) + A^T(t)\right)\right) = 2\mu_2(A(t))$, where

$$\mu_2(M) = \lim_{h\searrow 0} \frac{1}{h}\left[|I + hM| - 1\right] = \frac{1}{2}\sigma_{\max}(M + M^T)$$

represents the measure of matrix M with respect to the matrix norm (e.g. Bernstein [17, Fact 11.15.7]). Therefore, we have

$$\max_{\substack{|x|=\rho(t) \\ g^*(t,x)x=0}} 2x^T A(t) x \le \max_{|x|=\rho(t)} 2x^T A(t) x = 2\rho^2(t)\mu_2(A(t)).$$

Thus, from (6) we get the following sufficient condition for the invariance of the set $K(t)$:

$$(7) \qquad \mu_2(A(t)) + \frac{1}{2}\rho^{-2}(t) \max_{\substack{|x|=\rho(t) \\ g^*(t,x)x=0}} |g(t,x)|^2 \le \rho^{-1}(t)\rho'(t), \quad \forall t \in [0,T].$$

It is worth noticing that, for $g(t,x) \equiv 0$, $|x| = \rho(t)$, $\forall t \in [0,T]$, the necessary and sufficient condition (6) is equivalent to the inequality

$$(8) \qquad \mu_2(A(t)) \le \rho^{-1}(t)\rho'(t), \quad \forall t \in [0,T],$$

Inequality (8) is similar to the necessary and sufficient condition presented by paper Păstrăvanu, Matcovschi [137] for the invariance of $K(t)$ with respect to deterministic linear dynamics.

2.1.2 Feedback-based approach

Alternatively, keeping the above notations, we can reinterpret conditions (6) and the invariance problem from the perspective of finding a Lipschitz feedback law $U(t,x)$ which yields the time-dependent set $K(t)$ invariant for the SDE

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^{t\vee t_0} U(r, X_r^{t_0,x_0})dr + \int_{t_0}^{t\vee t_0} f(r, X_r^{t_0,x_0})dr + \int_{t_0}^{t\vee t_0} g(r, X_r^{t_0,x_0})dB_r, \quad t \geq 0. \tag{9}$$

The general case will be analyzed in the sequel, by considering the problem in the framework of multivalued (normal and oblique reflected) stochastic variational inequalities. For the moment we consider only the simple case of $K(t) \equiv \overline{B(0,\rho)}$, for every t. There exists a continuous feedback law $K \in L^0(\Omega; BV_{loc}([0,+\infty);\mathbb{R}^d))$ such that, for all $x \in \overline{B(0,\rho)}$, we have

$$\begin{cases} X_t^{t_0,x_0} = x_0, \quad \forall 0 \leq t \leq t_0, \\ X_t^{t_0,x_0} = x_0 - (K_t - K_{t_0}) + \displaystyle\int_{t_0}^{t} f(r, X_r^{t_0,x_0})dr + \int_{t_0}^{t} g(r, X_r^{t_0,x_0})dB_r, \quad t_0 \leq t, \\ X_t^{t_0,x_0} \in \overline{B(0,\rho)} \quad \text{and} \quad dK_t \in \partial I_{\overline{B(0,\rho)}}(X_t^{t_0,x_0})(dt), \quad \forall t \geq 0. \end{cases}$$

By $L^0(\Omega; BV_{loc}([0,+\infty);\mathbb{R}^d))$ we denote the space of random processes with values in the space of local bounded variation functions $BV_{loc}([0,+\infty);\mathbb{R}^d)$; the $\partial I_{\overline{B(0,\rho)}}$ represents the subdifferential operator of the convexity indicator function $I_{\overline{B(0,\rho)}}(x) = 0$ if $x \in \overline{B(0,\rho)}$ and $+\infty$ otherwise. In this context, the problem consists in finding an absolutely continuous control

$$K_t = \int_0^t U(r, X_r^{t_0,x_0})dr. \tag{10}$$

Assuming that such a control exists and maintaining the particular linear form of the drift coefficient introduced in the previous sub-section, then by (6), $\overline{B(0,\rho)}$ is invariant if and only if, for all $t \geq 0$ and $|x| = \rho$, each time when $g^*(t,x)\,x = 0$ we obtain $2\langle x, U(t,x) + A(t)x\rangle + |g(t,x)|^2 \leq 0$. Hence, in general, a $\overline{B(0,\rho)}$-invariance control of the form (10) does not exist, but if $g^*(t,x)\,x = 0$ for all $t \geq 0$ and $|x| = \rho$ then the feedback law

$$U(t,x) := -A(t)x - \frac{1}{2\rho^2}|g(t,x)|^2 x$$

yields the set $\overline{B(0,\rho)}$ invariant for the SDE (9). If the structure of $A(t)$ is of multidimensional type, then the linear feedback $U(t,x) = \theta(t)x$, with

$$\theta(t) := -\frac{1}{2\rho^2}\sup_{|x|=\rho}|g(t,x)|^2 - \max_{i=\overline{1,n}}|A_i|$$

assures the $\overline{B(0,\rho)}-$invariance for (9). Finally, it is important to mention that a time-independent feedback of the form $U(x) = \theta x$ can be used, where the constant θ is defined by

$$\theta := -\sup_{t\in[0,T]}\left\{\frac{1}{2\rho^2}\sup_{|x|=\rho}|g(t,x)|^2 + |A(t)|\right\}$$

and, respectively, by $\theta := -\dfrac{1}{2\rho^2}\sup_{\substack{|x|=\rho \\ t\in[0,T]}}|g(t,x)|^2 - \max_{i=\overline{1,n}}|A_i|$.

2.2 Generalized SVIs driven by convex constraints

As the main achievement of this section we prove the existence and uniqueness of the solution for the following stochastic variational inequality

$$
\text{(11)} \qquad \begin{cases} dX_t + H(X_t)\,\partial\varphi(X_t)(dt) \ni f(t, X_t)\,dt + g(t, X_t)\,dB_t, & t > 0, \\ X_0 = x_0, \end{cases}
$$

where B is a standard Brownian motion defined on a complete probability space and the new quantity $H(X)$ that appears acts on the set of subgradients; the product $H(X)\,\partial\varphi(X)$ will be called, from now on, the set of *oblique subgradients*. First, we focus on the deterministic case, considering a generalized Skorokhod problem with oblique reflection of the form

$$
\text{(12)} \qquad \begin{cases} x(t) + \displaystyle\int_0^t H(x(s))\,dk(s) = x_0 + \int_0^t f(s, x(s))\,ds + m(t), & t \geq 0, \\ dk(s) \in \partial\varphi(x(s))(ds), \end{cases}
$$

where the singular input $m : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function. The existence results are obtained via classical Yosida penalization techniques. We then continue with the study of some stochastic variational inequalities with oblique reflection.

2.2.1 A convex Skorokhod problem with oblique subgradients

Notations. Hypotheses.

Consider the deterministic generalized convex Skorokhod problem with oblique subgradients:

$$
\text{(13)} \qquad dx(t) + H(x(t))\,\partial\varphi(x(t))(dt) \ni dm(t), \quad t > 0, \quad x(0) = x_0,
$$

where

$$
\text{(14)} \qquad \varphi : \mathbb{R}^d \to (-\infty, +\infty] \text{ is a proper convex lower semicontinuous (l.s.c.) function}
$$

and

$$
\text{(15)} \qquad \begin{cases} (i) & x_0 \in Dom(\varphi) := \{x \in \mathbb{R}^d : \varphi(x) < +\infty\}, \\ (ii) & m \in C\left([0, +\infty); \mathbb{R}^d\right), \quad m(0) = 0. \end{cases}
$$

$H = (h_{i,j})_{d\times d} \in C_b^2\left(\mathbb{R}^d; \mathbb{R}^{d\times d}\right)$ is a matrix, such that for all $x \in \mathbb{R}^d$,

$$
\text{(16)} \qquad \begin{cases} (i) & h_{i,j}(x) = h_{j,i}(x), \quad \text{for every } i, j \in \overline{1, d}, \\ (ii) & \dfrac{1}{c}|u|^2 \leq \langle H(x)u, u\rangle \leq c\,|u|^2, \quad \forall\, u \in \mathbb{R}^d \text{ (for some } c \geq 1). \end{cases}
$$

Let $[H(x)]^{-1}$ be the inverse matrix of $H(x)$. Then $[H(x)]^{-1}$ has the same properties (16) as $H(x)$. Let $|H(x)| := \left(\sum_{i,j=1}^d |h_{i,j}(x)|^2\right)^{1/2}$ and denote

$$
b = \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|H(x) - H(y)|}{|x-y|} + \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|\,[H(x)]^{-1} - [H(y)]^{-1}\,|}{|x-y|},
$$

Define now by $\partial\varphi$ the subdifferential operator of φ, $\partial\varphi(x) := \{\hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x\rangle + \varphi(x) \leq \varphi(y)$, for all $y \in \mathbb{R}^d\}$ and $Dom(\partial\varphi) := \{x \in \mathbb{R}^d : \partial\varphi(x) \neq \emptyset\}$. We will use the notation $(x, \hat{x}) \in \partial\varphi$ in order to express that $x \in Dom(\partial\varphi)$ and $\hat{x} \in \partial\varphi(x)$.

If $E = \overline{E} \subset \mathbb{R}^d$ and $E^c = \mathbb{R}^d \setminus E$, then we denote, for $\varepsilon > 0$, $E_\varepsilon = \{x \in E : dist_{E^c}(x) \geq \varepsilon\} = \{x \in E : B(x, \varepsilon) \subset E\}$ the $\varepsilon-$interior of E. We impose the following supplementary assumptions:

(17) $$\begin{cases} (i) & D = Dom(\varphi) \text{ is a closed subset of } \mathbb{R}^d, \\ (ii) & \exists\, r_0 > 0,\ D_{r_0} \neq \emptyset \quad \text{and} \quad h_0 = \sup_{z \in D} dist_{D_{r_0}}(z) < +\infty, \\ (iii) & \exists\, L \geq 0 \text{ such that } |\varphi(x) - \varphi(y)| \leq L + L|x - y|\,, \text{ for all } x, y \in D. \end{cases}$$

A generalized Skorokhod problem.

Let $k : [t, T] \to \mathbb{R}^d$, where $0 \leq t \leq T$. We denote, $\|k\|_{[t,T]} := \sup\{|k(s)| : t \leq s \leq T\}$, and, for $t = 0$, $\|k\|_T := \|k\|_{[0,T]}$. Considering $\mathcal{D}[t, T]$ the set of the partitions of the time interval $[t, T]$, of the form $\Delta = (t = t_0 < t_1 < ... < t_n = T)$, let $S_\Delta(k) = \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)|$ and $\updownarrow k \updownarrow_{[t,T]} := \sup\limits_{\Delta \in \mathcal{D}} S_\Delta(k)$; if $t = 0$, denote $\updownarrow k \updownarrow_T := \updownarrow k \updownarrow_{[0,T]}$. Consider the space of bounded variation functions $BV([0, T]; \mathbb{R}^d) = \{k \mid k : [0, T] \to \mathbb{R}^d,\ \updownarrow k \updownarrow_T < +\infty\}$. Taking on the space of continuous functions $C\left([0, T]; \mathbb{R}^d\right)$ the usual supremum norm, remark the duality connection $(C([0, T]; \mathbb{R}^d))^* = \{k \in BV([0, T]; \mathbb{R}^d) \mid k(0) = 0\}$, with the duality given by the Riemann-Stieltjes integral. We will say that a function $k \in BV_{loc}([0, +\infty); \mathbb{R}^d)$ if, for every $T > 0$, $k \in BV([0, T]; \mathbb{R}^d)$.

Definition 3 *Given two functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$, we say that $dk(t) \in \partial\varphi(x(t))(dt)$ if*

$$\begin{cases} (a) & x, k : \mathbb{R}_+ \to \mathbb{R}^d \text{ are continuous}, \\ (b) & x(t) \in \overline{Dom(\varphi)}, \\ (c) & k \in BV_{loc}\left([0, +\infty); \mathbb{R}^d\right), k(0) = 0, \quad \text{and for } \forall 0 \leq s \leq t \leq T,\ \forall y \in C([0, T]; \mathbb{R}^d), \\ (d) & \displaystyle\int_s^t \langle y(r) - x(r), dk(r)\rangle + \int_s^t \varphi(x(r))\, dr \leq \int_s^t \varphi(y(r))\, dr. \end{cases}$$

We state that

Definition 4 *A pair of functions (x, k) is a solution of the Skorokhod problem with $H-$oblique subgradients (13) (and we write $(x, k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$) if $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ are continuous functions and*

(18) $$\begin{cases} (i) & x(t) + \displaystyle\int_0^t H(x(r))\, dk(r) = x_0 + m(t)\,, \quad \forall\, t \geq 0, \\ (ii) & dk(r) \in \partial\varphi(x(r))(dr)\,. \end{cases}$$

For the clarity of the presentation, some useful technical a priori estimates of the solution $(x, k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$ will be grouped together in a subsection entitled Technical results. Recall first the notation for modulus of continuity of a function $g : [0, T] \to \mathbb{R}^d$:

$$\mathbf{m}_g(\varepsilon) = \sup\{|g(u) - g(v)| : u, v \in [0, T]\,,\ |u - v| \leq \varepsilon\}\,.$$

Technical results.

We present some results with a priori estimates of the solution $(x,k) \in \mathcal{SP}\left(H\partial\varphi; x_0, m\right)$.

Lemma 5 *Let the assumptions (15), (16), (14) and (17) be satisfied. If $(x,k) \in \mathcal{SP}\left(H\partial\varphi; x_0, m\right)$, then for all $0 \le s \le t \le T$, there exists $C = C\left(b, c, L\right) > 0$ such that*

$$\begin{aligned}(19)\qquad \mathbf{m}_x\left(t-s\right) \le &\left[\left(t-s\right) + \mathbf{m}_m\left(t-s\right) + \sqrt{\mathbf{m}_m\left(t-s\right)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right)}\right]\\ &\times \exp\left\{C\left[1 + \left(t-s\right) + \left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s + 1\right)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right)\right]\right\}.\end{aligned}$$

Lemma 6 *Let the assumptions (15), (16), (14) and (17) be satisfied. If $(x,k) \in \mathcal{SP}\left(H\partial\varphi; x_0, m\right)$, $0 \le s \le t \le T$ and*

$$\sup_{r\in[s,t]} \left|x\left(r\right) - x\left(s\right)\right| \le 2\delta_0 = \frac{\rho_0}{2bc} \wedge \rho_0\ , \quad \textit{with } \rho_0 = \frac{r_0}{2\left(1 + r_0 + h_0\right)},$$

then

$$(20)\qquad \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le \frac{1}{\rho_0}\left|k\left(t\right) - k\left(s\right)\right| + \frac{3L}{\rho_0}\left(t-s\right)$$

and

$$(21)\qquad \left|x\left(t\right) - x\left(s\right)\right| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le \sqrt{t - s + \mathbf{m}_m\left(t-s\right)} \times e^{C_T\left(1 + \|m\|_T^2\right)},$$

where $C_T = C\left(b, c, r_0, h_0, L, T\right) > 0$.

Lemma 7 *Let the assumptions (15), (16), (14) and (17) be satisfied. Let $(x,k) \in \mathcal{SP}\left(H\partial\varphi; x_0, m\right)$, $0 \le s \le t \le T$ and $x\left(r\right) \in D_{\delta_0}$, for all $r \in [s,t]$. Then*

$$\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le L\left(1 + \frac{2}{\delta_0}\right)\left(t-s\right)$$

and, denoting by $C_T = C_T\left(b, c, r_0, h_0, L, T\right) > 0$, $\mathbf{m}_x\left(t-s\right) \le C_T \times \left[\left(t-s\right) + \mathbf{m}_m\left(t-s\right)\right]$.

Denote now $\mu_m\left(\varepsilon\right) = \varepsilon + \mathbf{m}_m\left(\varepsilon\right)$, for every $\varepsilon \ge 0$.

Lemma 8 *Let the assumptions (15), (16), (14) and (17) be satisfied and $(x,k) \in \mathcal{SP}\left(H\partial\varphi; x_0, m\right)$. Then, there exists a positive constant $C_T\left(\|m\|_T\right) = C\left(x_0, b, c, r_0, h_0, L, T, \|m\|_T\right)$, increasing function with respect to $\|m\|_T$, such that, for all $0 \le s \le t \le T$:*

$$(22)\qquad \begin{array}{ll}(a) & \|x\|_T + \updownarrow k\updownarrow_T \le C_T\left(\|m\|_T\right),\\ (b) & \left|x\left(t\right) - x\left(s\right)\right| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le C_T\left(\|m\|_T\right) \times \sqrt{\mu_m\left(t-s\right)}.\end{array}$$

We renounce now at the restriction that the function f is identically 0 and we consider the equation written under differential form

$$(23)\qquad dx\left(t\right) + H\left(x\left(t\right)\right)\partial\varphi\left(x\left(t\right)\right)\left(dt\right) \ni f\left(t, x\left(t\right)\right)dt + dm\left(t\right), \quad t > 0, \quad x\left(0\right) = x_0,$$

where

$$(24)\qquad \left\{\begin{array}{ll}(i) & \left(t,x\right) \longmapsto f\left(t,x\right) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ is a Carathéodory function}\\ (ii) & \displaystyle\int_0^T \left(f^{\#}\left(t\right)\right)^2 dt < +\infty, \quad \text{where} \quad f^{\#}\left(t\right) = \sup\nolimits_{x \in Dom(\varphi)} \left|f\left(t,x\right)\right|.\end{array}\right.$$

Existence and uniqueness of the solution for the Skorokhod problem.

Theorem 9 *Let the assumptions (15), (16), (14), (17) and (24) be satisfied. Then the differential equation (23) has at least one solution in the sense of Definition 4, i.e.* $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ *are continuous functions and*

$$(25)\qquad \begin{cases} (j) & x(t) + \int_0^t H(x(r))\, dk(r) = x_0 + \int_0^t f(r, x(r))\, dr + m(t), \quad \forall\, t \geq 0, \\ (jj) & dk(r) \in \partial\varphi(x(r))(dr). \end{cases}$$

Proposition 10 *Let the assumptions (16), (15), (14), (17) and (24) be satisfied. Assume also that there exists* $\mu \in L^1_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *such that, for all* $x, y \in \mathbb{R}^d$,

$$(26)\qquad |f(t,x) - f(t,y)| \leq \mu(t)\, |x - y|, \quad a.e.\ t \geq 0.$$

If $m \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$, *then the generalized convex Skorokhod problem with oblique subgradients (13) admits a unique solution* (x,k) *in the space* $C(\mathbb{R}_+;\mathbb{R}^d) \times [C(\mathbb{R}_+;\mathbb{R}^d) \cap BV_{loc}(\mathbb{R}_+;\mathbb{R}^d)]$. *Moreover, if* (x,k) *and* $(\hat{x}, \hat{k})$ *are two solutions, corresponding to* m, *respectively* $\hat{m}$, *then*

$$(27)\qquad |x(t) - \hat{x}(t)| \leq C e^{CV(t)} \left[|x_0 - \hat{x}_0| + \updownarrow m - \hat{m} \updownarrow_t \right],$$

where $V(t) = \updownarrow x \updownarrow_t + \updownarrow \hat{x} \updownarrow_t + \updownarrow k \updownarrow_t + \updownarrow \hat{k} \updownarrow_t + \int_0^t \mu(r)\, dr$ *and* C *is a positive constant depending only on* b *and* c.

Using the hypothesis from the uniqueness result we provide supplementary fine estimates on the approximating sequence $(x_\varepsilon)_{\varepsilon>0}$. We are now able to make the transition to the stochastic framework.

Corollary 11 *If* $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ *is a stochastic basis and* M *a* $\mathcal{F}_t-$*progressively measurable stochastic process such that* $M_\cdot(\omega) \in C^1(\mathbb{R}_+; \mathbb{R}^d)$, $\mathbb{P} - a.s.\ \omega \in \Omega$, *then, under the assumptions of Proposition 10,* $\mathbb{P} - a.s.\ \omega \in \Omega$, *the random generalized Skorokhod problem with oblique subgradients:*

$$\begin{cases} X_t(\omega) + \int_0^t H(X_t(\omega))\, dK_t(\omega) = x_0 + \int_0^t f(s, X_s(\omega))\, ds + M_t(\omega), \quad t \geq 0, \\ dK_t(\omega) \in \partial\varphi(X_t(\omega))(dt) \end{cases}$$

admits a unique solution $(X_\cdot(\omega), K_\cdot(\omega))$. *Moreover* X *and* K *are* $\mathcal{F}_t-$*progressively measurable stochastic processes.*

2.2.2 SVIs with oblique subgradients

Notations. Hypotheses

Our objective is to solve the following SVI with oblique reflection, driven by a convex subdifferential operator:

$$(28)\qquad \begin{cases} X_t + \int_0^t H(X_t)\, dK_t = x_0 + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s, \quad t \geq 0, \\ dK_t \in \partial\varphi(X_t)(dt), \end{cases}$$

where $x_0 \in \mathbb{R}^d$ and, by denoting $f^{\#}(t) := \sup_{x\in Dom(\varphi)} |f(t,x)|$ and $g^{\#}(t) := \sup_{x\in Dom(\varphi)} |g(t,x)|$,

$$(29)\qquad \begin{cases} (i) & (t,x) \longmapsto f(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ and } (t,x) \longmapsto g(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k} \text{ are} \\ & \quad \text{Carathéodory functions (i.e. measurable w.r.t. } t \text{ and continuous w.r.t. } x\text{)}, \\ (ii) & \displaystyle\int_0^T (f^{\#}(t))^2 dt + \int_0^T (g^{\#}(t))^4 dt < +\infty, \end{cases}$$

We also add Lipschitz continuity conditions:

$$(30)\qquad \begin{cases} \exists\, \mu \in L^1_{loc}(\mathbb{R}_+), \ \ \exists\, \ell \in L^2_{loc}(\mathbb{R}_+) \text{ s.t. } \forall\, x,y \in \mathbb{R}^d, \quad a.e.\ t \geq 0, \\ |f(t,x) - f(t,y)| \leq \mu(t)\,|x-y| \quad \text{and} \quad |g(t,x) - g(t,y)| \leq \ell(t)\,|x-y|\,. \end{cases}$$

Definition 12 *(I) Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ and a $\mathbb{R}^k-$valued $\mathcal{F}_t-$Brownian motion $\{B_t : t \geq 0\}$, a pair $(X,K) : \Omega \times [0,+\infty) \to \mathbb{R}^d \times \mathbb{R}^d$ of continuous $\mathcal{F}_t-$progressively measurable stochastic processes is a strong solution of the SDE (28) if, $\mathbb{P}-a.s.$*

$$(31)\qquad \begin{cases} i) & X_t \in \overline{Dom(\varphi)}, \ \ \forall\, t \geq 0, \ \ \varphi(X_\cdot) \in L^1_{loc}(\mathbb{R}_+), \\ ii) & K_\cdot \in BV_{loc}\left([0,+\infty); \mathbb{R}^d\right), \quad K_0 = 0, \\ iii) & X_t + \displaystyle\int_0^t H(X_s)\,dK_s = x_0 + \int_0^t f(s,X_s)\,ds + \int_0^t g(s,X_s)\,dB_s, \ \ \forall\, t \geq 0, \\ iv) & \forall\, 0 \leq s \leq t, \ \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous}: \\ & \quad \displaystyle\int_s^t \langle y(r) - X_r, dK_r\rangle + \int_s^t \varphi(X_r)\,dr \leq \int_s^t \varphi(y(r))\,dr. \end{cases}$$

That is $(X_\cdot(\omega), K_\cdot(\omega)) \in \mathcal{SP}(H\partial\varphi; x_0, M_\cdot(\omega))$, $\mathbb{P}-a.s.\ \omega \in \Omega$, with

$$M_t = \int_0^t f(s,X_s)\,ds + \int_0^t g(s,X_s)\,dB_s\,.$$

(II) If there exists a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)_{t\geq 0}$, a $\mathbb{R}^k-$valued $\mathcal{F}_t-$Brownian motion $\{B_t : t \geq 0\}$ and a pair $(X_\cdot, K_\cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$ of $\mathcal{F}_t-$progressively measurable continuous stochastic processes such that

$$(X_\cdot(\omega), K_\cdot(\omega)) \in \mathcal{SP}(H\partial\varphi; x_0, M_\cdot(\omega)), \quad \mathbb{P}-a.s.\ \omega \in \Omega,$$

then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$ is called a weak solution of the SVI (28).

Denote by $S^p_d[0,T]$, $p \geq 0$, the space of progressively measurable continuous stochastic processes $X : \Omega \times [0,T] \to \mathbb{R}^d$, such that $\|X\|_{S^p_d} = (\mathbb{E}\,\|X\|^p_T)^{\frac{1}{p}\wedge 1} < +\infty$, if $p > 0$ and $\|X\|_{S^0_d} = \mathbb{E}\,[1 \wedge \|X\|_T]$, where $\|X\|_T = \sup_{t\in[0,T]} |X_t|$. The space $(S^p_d[0,T], \|\cdot\|_{S^p_d})$, $p \geq 1$, is a Banach space and $S^p_d[0,T]$, $0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1,Z_2) = \|Z_1 - Z_2\|_{S^p_d}$ (when $p = 0$ the metric convergence coincides with the probability convergence). Denote by $\Lambda^p_{d\times k}(0,T)$, $p \in [0,+\infty)$, the space of progressively measurable stochastic processes $Z : \Omega \times (0,T) \to \mathbb{R}^{d\times k}$ such that

$$\|Z\|_{\Lambda^p} = \begin{cases} \left[\mathbb{E}\left(\displaystyle\int_0^T \|Z_s\|^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}\wedge 1}, & \text{if } p > 0, \\ \mathbb{E}\left[1 \wedge \left(\displaystyle\int_0^T \|Z_s\|^2 ds\right)^{\frac{1}{2}}\right], & \text{if } p = 0. \end{cases}$$

The space $(\Lambda^p_{d\times k}(0,T), \|\cdot\|_{\Lambda^p})$, $p \geq 1$, is a Banach space and $\Lambda^p_{d\times k}(0,T)$, $0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1,Z_2) = \|Z_1 - Z_2\|_{\Lambda^p}$.

Existence and uniqueness

Theorem 13 deals with the existence of a weak solution in the sense of Definition 12, while Theorem 14 proves the uniqueness of a strong solution.

Theorem 13 *Let the assumptions (16), (14), (17) and (29) be satisfied. Then the SVI (28) has at least one weak solution* $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$ *.*

Proof. The proof is divided in three main steps. We construct a sequence of approximating equations, whose unique sequence of solutions is tight in $C([0,T]\,;\mathbb{R}^{2d+1})$, which permits us to make use of the Prohorov and Skorokod theorems. Finally, we pass to the limit in order to obtain a weak solution for the SVI (28). ■

We now prove the uniqueness of the solution in this metric space S_d^0 of progressively measurable continuous stochastic processes $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$.

Theorem 14 *If the assumptions (16), (14), (17), (29) and (30) are satisfied, then the SVI (28) has a unique strong solution* $(X, K) \in S_d^0 \times S_d^0$*.*

Proof. It is sufficient to prove the *pathwise uniqueness*, since by Ikeda, Watanabe [94, Theorem 1.1, page 149] *the existence of a weak solution* and *the pathwise uniqueness* implies the existence of a strong solution. ■

2.3 Multivalued differential equations driven by oblique Fréchet subgradients

2.3.1 $(\rho,\gamma)-$semiconvex functions

Definition 15 *Let $\gamma \geq 0$. A set E is $\gamma-$semiconvex if for all $x \in Bd\,(E)$ there exists $\hat{x} \in \mathbb{R}^d \backslash \{0\}$ such that*

$$\langle \hat{x}, y - x \rangle \leq \gamma\, |\hat{x}|\, |y - x|^2\,, \quad \forall y \in E.$$

Definition 16 *Let E be a non empty closed subset of $\mathbb{R}^d$. We say that E satisfies the "uniform exterior ball condition" (for short, r_0-UEBC) if*

- *$N_E\,(x) \neq \{0\}$, for all $x \in Bd\,(E)$, where $N_E\,(x) = \left\{u \in \mathbb{R}^d : \lim_{\varepsilon \searrow 0} \frac{dist_E(x+\varepsilon u)}{\varepsilon} = |u|\right\}$ is the normal exterior cone to E in $x \in Bd(E)$.*
- *there exists $r_0 > 0$ such that, for every $x \in Bd\,(E)$ and each $u \in N_E\,(x)$, $|u| = r_0$ we have*

$$dist_E\,(x + u) = r_0 \quad \textit{or, equivalently,} \quad B\,(x + u, r_0) \cap E = \emptyset,$$

We present a result which establishes a one-to-one correspondence between the $\gamma-$semiconvexity property of a set and its $\frac{1}{2\gamma}-$UEBC. Let us now define the $\varepsilon-$neighborhood of a given set $E \subset \mathbb{R}^d$ by $\mathcal{E}_\varepsilon(E) := \left\{z \in \mathbb{R}^d : dist_E\,(z) < \varepsilon\right\}$ and its closed $\varepsilon-$neighborhood is $\overline{\mathcal{E}}_\varepsilon(E)$. The $\varepsilon-$interior of the closed set E is $\mathcal{I}_\varepsilon(E) := \{x \in E \mid dist_{Bd(E)}(x) \geq \varepsilon\}$. Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ and denote $\mathrm{Dom}\,(\varphi) = \left\{v \in \mathbb{R}^d : \varphi\,(v) < +\infty\right\}$.

Definition 17 *Define the Fréchet subdifferential of φ at $u \in \mathbb{R}^d$ by*

$$\partial^- \varphi\,(u) = \left\{u^* \in \mathbb{R}^d : \liminf_{v \to u; v \neq u} \frac{\varphi\,(v) - \varphi\,(u) - \langle u^*, v - u\rangle}{|v - u|} \geq 0\right\},$$

if $u \in \mathrm{Dom}\,(\varphi)$, and $\partial^-\varphi\,(u) = \emptyset$ if $u \notin \mathrm{Dom}\,(\varphi)$.

Let us introduce now the following notations:

$a)$ $\mathrm{Dom}\,(\partial^-\varphi) = \left\{u \in \mathbb{R}^d : \partial^-\varphi\,(u) \neq \emptyset\right\}$,
$b)$ $\partial^-\varphi = \{(u, u^*) : u \in \mathrm{Dom}\,(\partial^-\varphi)\,,\ u^* \in \partial^-\varphi\,(u)\}$,
$c)$ $\nabla^-\varphi\,(u)$ is the minimal norm element of the set $\partial^-\varphi\,(u)$, when $\partial^-\varphi\,(u) \neq \emptyset$.

Definition 18 *Let $\rho, \gamma \geq 0$. The function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is a $(\rho,\gamma)-$semiconvex function if $\overline{\mathrm{int}\,(\mathrm{Dom}\,(\varphi))} = \overline{\mathrm{Dom}\,(\varphi)}$ is $\gamma-$semiconvex, $\mathrm{Dom}\,(\partial^-\varphi) \neq \emptyset$ and, for every $(u, u^*) \in \partial^-\varphi$ and $v \in \mathbb{R}^d$:*

$$\langle u^*, v - u\rangle + \varphi\,(u) \leq \varphi\,(v) + (\rho + \gamma\,|u^*|)\,|v - u|^2\,.$$

The following result shows that the Fréchet subdifferential operator $\partial^-\varphi$ is locally bounded in the interior of the domain of φ. This quite strong property permits us to prove also that the interior of $\mathrm{Dom}\,(\varphi)$ coincides with the interior of $\mathrm{Dom}\,(\partial^-\varphi)$. However, in order to be able to approach, under not very restricted/strong assumptions on the initial data the initial multivalued evolution equation, a more refined version of Proposition 19 is mandatory. This is the reason why Proposition 20 plays a very important role for our study.

Proposition 19 *Let* $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ *be a* $(\rho, \gamma)-$*semiconvex function. If we consider* $u_0 \in \mathrm{Dom}\,(\varphi)\,,\ r_0, M_0 > 0$*, satisfying* $\varphi\,(u_0 + r_0 v) \leq M_0,\ \forall\,|v| \leq 1$*, then there exist* $\rho_0 > 0$ *and* $\hat{b} \geq 0$ *such that*

$$\rho_0|\hat{u}| \leq \langle \hat{u}, u - u_0\rangle + \hat{b} + \hat{b}(1 + |\hat{u}|)|u - u_0|^2, \quad \forall(u, \hat{u}) \in \partial^-\varphi. \tag{32}$$

Moreover there exist $M \geq 0$ *and* $\delta_0 \in (0, r_0]$ *such that*

$$|\hat{u}| \leq M, \quad \forall u \in \overline{B\,(u_0, \delta_0)} \subset \mathrm{Dom}\,(\varphi) \text{ and } \hat{u} \in \partial^-\varphi\,(u)\,. \tag{33}$$

We present now a result that provides useful information regarding the functions φ_ε, J_ε and A_ε, defined below, in a similar manner with the corresponding counterpart, found in the convex framework, as we can see, for example, in Barbu [6, Chapter 1] or Brézis [38, Chapter II]. The information will prove to be essential for the theorem which treats the existence of a solution for our Cauchy problem.

Proposition 20 *Let consider* $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ *a l.s.c. function and* $a, b, c \geq 0$ *be such that*

$$\varphi\,(v) + a\,|v|^2 + b\,|v| + c \geq 0, \quad \forall v \in \mathbb{R}^d.$$

If $0 < \varepsilon < \dfrac{1}{2a}$*, then, for every* $u \in \mathbb{R}^d$*, there exists* $u_\varepsilon \in \mathrm{Dom}\,(\varphi)$ *such that*

$$\frac{1}{2\varepsilon}\,|u - u_\varepsilon|^2 + \varphi\,(u_\varepsilon) = \inf_{v \in \mathbb{R}^d} \left\{ \frac{1}{2\varepsilon}\,|u - v|^2 + \varphi\,(v) \right\} =: \varphi_\varepsilon(u). \tag{34}$$

Moreover, we obtain:

$a)$ $\quad J_\varepsilon\,(u) := u_\varepsilon \in \mathrm{Dom}\,(\partial^-\varphi)$ *and* $A_\varepsilon\,(u) := \dfrac{1}{\varepsilon}\,(u - u_\varepsilon) \in \partial^-\varphi\,(u_\varepsilon)$.

$b)$ *The following inequality holds, for all* $u \in \mathbb{R}^d$*,* $u_0 \in \mathrm{Dom}\,(\varphi)$ *and* $0 < \varepsilon < \frac{1}{4a+1}$*,*

$$|J_\varepsilon\,(u) - u|^2 \leq \frac{1}{1 - \varepsilon\,(4a+1)}\,|u - u_0|^2 + \frac{4\varepsilon}{1 - \varepsilon\,(4a+1)}\left[\beta\,(|u|) + b^2 + \varphi(u_0)\right], \tag{35}$$

where $\beta\,(r) = ar^2 + br + c$*. In particular,* J_ε *and* A_ε *are globally sublinear functions for* $0 < \varepsilon \leq \frac{1}{4a+2}$*, i.e.*

$$|J_\varepsilon(u)| \leq C(1 + |u|), \quad |A_\varepsilon(u)| \leq \frac{C}{\varepsilon}(1 + |u|), \quad \forall u \in \mathbb{R}^d, \text{ where } C = C(a, b, c, u_0).$$

Moreover, if $u \in \overline{B\,(u_0, r_0)}$*,* $r_0 > 0$*, then*

$$|J_\varepsilon\,(u) - u| \leq \frac{r_0 + \sqrt{\varepsilon}C_0}{\sqrt{1 - \varepsilon\,(4a+1)}},$$

where $C_0 = 2\sqrt{\beta\,(r_0 + |u_0|) + b^2 + \varphi(u_0)}$*. Also, taking* $u = u_0$ *in (35) it follows*

$$\begin{array}{ll} (b_1) & \lim_{\varepsilon \to 0} J_\varepsilon\,(u_0) = u_0, \quad \forall u_0 \in \mathrm{Dom}\,(\varphi)\,, \\ (b_2) & \mathrm{Dom}\,(\partial^-\varphi) \text{ is dense in } \mathrm{Dom}\,(\varphi)\,, \\ (b_3) & \overline{\mathrm{Dom}\,(\partial^-\varphi)} = \overline{\mathrm{Dom}\,(\varphi)}. \end{array} \tag{36}$$

c) In addition to its lower semicontinuity property, we suppose now that φ is a (ρ,γ) −semiconvex function. We fix $u_0 \in \mathrm{Dom}(\varphi)$ and $\lambda_0 > 0$. If

$$(37)\qquad \begin{array}{ll} (i) & 0 < r_0 \leq \bar{r}_0 := \dfrac{1}{36}\left(\dfrac{\lambda_0}{1+\lambda_0}\right)^2 \dfrac{1}{\left(1+(\rho+\gamma)\lambda_0\right)^2}, \\ (ii) & 0 < \varepsilon \leq \bar{\varepsilon}_0 := \dfrac{1}{4a+2} \wedge \dfrac{1-r_0}{4a+1} \wedge \sqrt{r_0} \wedge \dfrac{r_0^2}{1+C_0^2}, \end{array}$$

where $C_0 = 2\sqrt{\beta\left(r_0+|u_0|\right)+b^2+\varphi(u_0)}$, then, for all $u, v \in \overline{B\left(u_0, r_0\right)}$, it follows

$$(38)\qquad \begin{array}{ll} (c_1) & \left|J_\varepsilon\left(u\right) - J_\varepsilon\left(v\right)\right| \leq \left(1+(\rho+\gamma)\lambda_0\right)\left|u-v\right|; \\ (c_2) & \left|A_\varepsilon\left(u\right) - A_\varepsilon\left(v\right)\right| \leq \dfrac{2+(\rho+\gamma)\lambda_0}{\varepsilon}\left|u-v\right|. \end{array}$$

In particular, the minimizing point $J_\varepsilon\left(u\right) (= u_\varepsilon)$ of $\inf_{v\in\mathbb{R}^d}\left\{\frac{1}{2\varepsilon}\left|u-v\right|^2 + \varphi\left(v\right)\right\}$ is unique for $0 < \varepsilon \leq \bar{\varepsilon}_0$ and $u \in \mathcal{E}_{\bar{r}_0}(D)$ (we denoted $D := \mathrm{Dom}\left(\varphi\right)$); also, $\varphi_\varepsilon \in C^1(\mathcal{E}_{\bar{r}_0}(D))$ and $\nabla\varphi_\varepsilon\left(u\right) = A_\varepsilon\left(u\right) = \dfrac{1}{\varepsilon}\left(u - J_\varepsilon u\right) \in \partial^-\varphi\left(J_\varepsilon u\right)$. Moreover, $\nabla\varphi_\varepsilon$ and J_ε are Lipschitz on every bounded subset of $\mathcal{E}_{\bar{r}_0}(D)$ and

$$(c_3)\qquad \mathrm{int}\left(\mathrm{Dom}\left(\varphi\right)\right) = \mathrm{int}\left(\mathrm{Dom}\left(\partial^-\varphi\right)\right).$$

2.3.2 Multivalued differential equations with oblique subgradients. Main results

We study for the beginning the following Cauchy problem

$$(39)\qquad x'\left(t\right) + H\left(t, x\left(t\right)\right)\partial^-\varphi\left(x\left(t\right)\right) \ni g\left(t, x(t)\right), \text{ a.e. } t \in [0,T], \quad x\left(0\right) = x_0 \in \mathrm{Dom}\left(\varphi\right),$$

with an arbitrary fixed $T > 0$; $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function, which satisfies

$$(H_g) : \begin{cases} i) & \forall N > 0,\ \exists G_N \geq 0 : |g(t,x) - g(t,y)| \leq G_N\left|x-y\right|,\ \forall\left|x\right|,\left|y\right| \leq N, \\ ii) & \exists L_1, L_2 \in L^1(0,T;\mathbb{R}_+), \text{ s.t. } \left|g(t,x)\right| \leq L_1(t) + L_2(t)\left|x\right|,\ \forall x \in \mathbb{R}^d, \text{ a.e. } t \in [0,T] \end{cases}$$

and

$$(H_\varphi) : \varphi : \mathbb{R}^d \to (-\infty, +\infty] \text{ is a proper, lower semicontinuous, } (\rho,\gamma) - \text{semiconvex function,}$$

The matrix $H(\cdot,\cdot) = \left(h_{i,j}(\cdot,\cdot)\right)_{d\times d}$ and its inverse $\left[H\left(\cdot,\cdot\right)\right]^{-1}$ are from $C_b^{1,2}\left([0,T]\times\mathbb{R}^d;\mathbb{R}^{d\times d}\right)$ [1] and satisfy similar assumptions (H_H) to the ones found in the convex case.

Definition 21 *A pair (x,h) of functions $x, h : [0,T] \longrightarrow \mathbb{R}^d$ is a solution of the oblique reflected problem (39) (and we will write $(x,h) \in \mathcal{OR}\left(H\partial^-\varphi; x_0, g\right)$) if*

$$(40)\qquad \begin{cases} i) & x\left(t\right) \in \overline{\mathrm{Dom}\left(\varphi\right)}, \quad \forall t \geq 0,\ x \in C(\left[0,T\right];\mathbb{R}^d) \text{ and } h, \varphi\left(x\right) \in L^1\left(0,T;\mathbb{R}^d\right), \\ ii) & x\left(t\right) + \displaystyle\int_0^t H\left(r, x\left(r\right)\right)h\left(r\right)dr = x_0 + \int_0^t g\left(r, x(r)\right)dr, \quad \forall t \in [0,T], \\ iii) & h\left(t\right) \in \partial^-\varphi\left(x\left(t\right)\right), \quad a.e.\ t \in [0,T]. \end{cases}$$

[1] A matrix $G = (g_{ij})_{d\times d} \in C_b^{1,2}\left([0,T]\times\mathbb{R}^d;\mathbb{R}^{d\times d}\right)$ if $g_{ij} : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ are functions of class $C^{1,2}$ and there exists a positive boundedness constant C_G such that $|g_{ij}(t,x)| + |\frac{\partial}{\partial t}g_{ij}(t,x)| + |\nabla_x g_{ij}(t,x)| + |D^2_{xx}g_{ij}(t,x)| \leq C_G$, for all $(t,x) \in [0,T]\times\mathbb{R}^d$, $\forall i,j \in \overline{1,d}$.

We focus our attention on the main results on multivalued nonconvex ODEs. We first prove the uniqueness of the solution. The existence result is more technical; it is divided into four parts. First, one constructs a suitable sequence of approximating equations and, using a localization technique, we obtain the convergence of its sequence of solutions. Passing to the limit we obtain a local solution for the initial generalized evolution equation. Finally, by imposing an additional assumption regarding the $\mathrm{Dom}\,(\varphi)$, we study the extension to a global solution.

Uniqueness; properties of the solution

Theorem 22 *Let the assumptions* $(H\varphi)$, (Hg) *and* (H_H) *be satisfied. Then the generalized nonconvex differential system with oblique subgradients (39) admits a unique solution* (x,h) *in the sense of Definition 21. Moreover, if* $(x,h)\in\mathcal{OR}\left(H\partial^-\varphi;x_0,g\right)$ *and* $(\hat{x},\hat{h})\in\mathcal{OR}\left(H\partial^-\varphi;x_0,\hat{g}\right)$ *then*

$$|x\left(t\right)-\hat{x}\left(t\right)|\leq Ce^{CU(t)}\left[|x_0-\hat{x}_0|+\int_0^t|g\left(r,x(r)\right)-\hat{g}(r,x(r))|dr\right],\quad\forall 0\leq t\leq T, \tag{41}$$

where $U\left(t\right)=\int_0^t|x\left(r\right)|dr+\int_0^t|\hat{x}\left(r\right)|dr+\left(1+\gamma\right)\int_0^t\left|h\left(r\right)\right|dr+\left(1+\gamma\right)\int_0^t|\hat{h}\left(r\right)|dr+2\rho t+G_N t$, *with* $N\geq||x||_T\vee||\hat{x}||_T$ *and* C *is a constant depending only on* L_H *and* c_H.

Existence of the solution

Theorem 23 (Local existence) *Consider* $x_0\in\mathrm{Dom}\,(\varphi)$ *and let the assumptions* $(H\varphi)$, (Hg) *and* (H_H) *be satisfied. Then, there exists* $T^*\in(0,T]$ *and a pair of functions* $(x,h)\in C([0,T^*];\mathbb{R}^d)\times L^2(0,T^*;\mathbb{R}^d)$ *which is a solution of problem (39) on* $[0,T^*]$.

Theorem 24 (Global existence) *Under the assumptions of Theorem 23, if we suppose, in addition, that* $\mathrm{Dom}(\varphi)$ *is bounded and closed, then there exists a pair of functions* $(x,h)\in C([0,T];\mathbb{R}^d)\times L^2(0,T;\mathbb{R}^d)$ *which is a solution of problem (39) on the entire time interval* $[0,T]$.

We can generalize the results we have just obtained by extending our analysis for local semiconvex sets and local semiconvex functions. We first update Definition 15 and Definition 18, as follows.

Definition 25 *We say that a non-empty set* $E\subset\mathbb{R}^d$ *is locally semiconvex if there exists a non-decreasing function* $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ *such that, for all* $x\in Bd\,(E)$, $|x|\leq R$, *there exists* $\hat{x}\in\mathbb{R}^d\backslash\{0\}$ *such that*

$$\langle\hat{x},y-x\rangle\leq\gamma(R)\,|\hat{x}|\,|y-x|^2,\quad\forall y\in E,\ \ |y|\leq R.$$

Definition 26 *We say that the function* $\varphi:\mathbb{R}^d\to(-\infty,+\infty]$ *is a locally semiconvex function if there exist non-decreasing functions* $\rho,\gamma:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\overline{\mathrm{int}\,(\mathrm{Dom}\,(\varphi))}=\overline{\mathrm{Dom}\,(\varphi)}$ *is locally semiconvex (*γ *being the function from Definition 25),* $\mathrm{Dom}\,(\partial^-\varphi)\neq\emptyset$ *and, for every* $R\geq 0$, $(u,u^*)\in\partial^-\varphi$ *and* $v\in\mathbb{R}^d$, *satisfying* $|u|\leq R$, $|v|\leq R$, *we have*

$$\langle u^*,v-u\rangle+\varphi\left(u\right)\leq\varphi\left(v\right)+\left(\rho(R)+\gamma(R)\left|u^*\right|\right)|v-u|^2.$$

Concerning the existence and uniqueness of a solution for Eq.(39) under these new assumptions the results follow the same lines that we presented when we worked with the global semiconvexity hypothesis. The uniqueness results is identical with Theorem 22 since the localization will be made for $R = ||x||_T \vee ||\hat{x}||_T$, where $x, \hat{x}$ are two supposed solutions of the equation. For the existence of a solution we will state below an adapted result, Theorem 23 and Theorem 24 becoming one.

Theorem 27 *Consider $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ a proper, l.s.c., locally semiconvex function and let the assumptions (Hg) and (H_H) be satisfied. For every initial datum $x_0 \in \mathrm{Dom}\,(\varphi)$ and each $R > |x_0|$ there exists $T_1 > 0$ and a pair of functions $(x, h) \in C([0, T_1]; \mathbb{R}^d) \times L^2(0, T_1; \mathbb{R}^d)$ which is a solution of problem (39) on $[0, T_1]$. Moreover, $x(t) \in \mathrm{Dom}\,(\varphi) \cap \overline{B(0, R)}$, $\forall t \in [0, T_1]$. If, in addition, $\mathrm{Dom}\,(\varphi)$ is closed and the maximal local solution of (39) is bounded, then it is a global solution on the entire interval $[0, T]$.*

2.3.3 A nonconvex Skorokhod problem with generalized reflection

Setting the problem

In this section we consider a generalized Skorokhod problem, driven by Fréchet oblique reflected subgradients. This Cauchy problem can be, formally, written as:

$$\left\{\begin{array}{l} dx\,(t) + H(t, x(t))\partial^- \varphi\,(x\,(t))\,(dt) \ni g(t, x(t))dt + dm\,(t)\,, \quad \text{a.e. } t \geq 0, \\ x\,(0) = x_0 \in \mathrm{Dom}(\varphi), \end{array}\right. \tag{42}$$

where $m \in C\left(\mathbb{R}_+; \mathbb{R}^d\right)$, $m\,(0) = 0$ and with the assumptions (H_H), (H_g) and (H_φ) imposed in Section 2.3.2 still holding.

Definition 28 *A pair (x, k) of continuous functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ is a solution of equation (42) (and we will write $(x, k) \in \mathcal{GSP}\,(H\partial^- \varphi; x_0, g, m))$ if*

$$\left\{\begin{array}{rl} i) & x\,(t) \in \overline{\mathrm{Dom}\,(\varphi)}, \quad \forall\, t \geq 0 \text{ and } \varphi\,(x\,(\cdot)) \in L^1_{loc}\,(\mathbb{R}_+)\,, \\ ii) & k \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right),\ k\,(0) = 0, \\ iii) & x\,(t) + \displaystyle\int_0^t H(r, x(r))dk\,(r) = x_0 + \int_0^t g(r, x(r))dr + m\,(t)\,, \quad \forall t \geq 0, \\ iv) & \displaystyle\int_s^t \langle y\,(r) - x\,(r)\,, dk\,(r)\rangle + \int_s^t \varphi\,(x\,(r))\,dr \leq \int_s^t \varphi\,(y\,(r))\,dr \\ & \quad + \displaystyle\int_s^t |y\,(r) - x\,(r)|^2\,(\rho dr + \gamma d\updownarrow k \updownarrow_r)\,, \quad \forall 0 \leq s \leq t,\ \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous.} \end{array}\right. \tag{43}$$

In order to have the uniform boundedness and the continuity of the solution for the generalized Skorokhod problem we should introduce some additional assumptions regarding the geometrical properties of $\mathrm{Dom}\,(\varphi)$. They are useful for obtaining the results found also in the convex framework. Let us introduce the notion of drop of vertex x and running direction v. Let $x, v \in \mathbb{R}^d$ and $r > 0$. The closed convex hull

$$D_x\,(v, r) := \mathrm{conv}\left\{\{x\} \cup \overline{B\,(x + v, r)}\right\} = \left\{x + t\,(u - x)\ \mid u \in \overline{B\,(x + v, r)},\ t \in [0, 1]\right\}$$

is called $(|v|\,, r)$*-drop of vertex x and running direction v*. Remark that, if $|v| \leq r$, then $D_x\,(v, r) = \overline{B\,(x + v, r)}$.

Definition 29 *The set $E \subset \mathbb{R}^d$ satisfies the uniform interior drop condition if there exist $r_0, v_0 > 0$ such that for all $x \in E$ there exists $v_x \in \mathbb{R}^d$ with $|v_x| \leq v_0$ and*

$$D_x(v_x, r_0) \subset E$$

(we also say that E satisfies the uniform interior (v_0, r_0)-drop condition).

Proposition 30 *If the set E, with its interior non-empty, satisfies the uniform interior (v_0, r_0)-drop condition then E satisfies the shifted uniform interior ball condition ($(\gamma,\sigma,\delta)-$SUIBC or $\gamma-$SUIBC, for short). More precisely, there exist $\gamma \geq 0$ and $\delta, \sigma > 0$, such that, for every $y \in E$, there exist $\lambda_y \in (0,1]$ and $v_y \in \mathbb{R}^d$, $|v_y| \leq 1$ satisfying*

$$\begin{array}{ll} (i) & \lambda_y - \left(|v_y| + \lambda_y\right)^2 \gamma \geq \sigma, \\ (ii) & \overline{B(x + v_y, \lambda_y)} \subset E, \quad \forall x \in E \cap \overline{B(y,\delta)}. \end{array} \tag{44}$$

We enhance now assumption (H_φ) concerning the l.s.c., semiconvex function φ by considering the following new hypothesis, which will be used furthermore:

(45)

$$(H_\varphi): \begin{cases} i) & \varphi : \mathbb{R}^d \to (-\infty, +\infty] \text{ is a proper, lower semicontinuous, semiconvex function,} \\ ii) & \exists K > 0 \quad \text{such that} \quad |\varphi(x) - \varphi(y)| \leq K + K|x - y|, \quad \forall x, y \in \mathrm{Dom}(\varphi), \\ iii) & \mathrm{Dom}(\varphi) \text{ is bounded and it satisfies the } \gamma - \textit{SUIBC}. \end{cases}$$

Existence and uniqueness of the solution of the Skorokhod problem

We first prove the uniqueness of the solution.

Theorem 31 *Suppose that hypotheses (H_H), (H_g) and (H_φ) hold. If $m \in BV_{loc}(\mathbb{R}_+;\mathbb{R}^d)$ then the nonconvex Skorokhod problem with oblique subgradients (42) admits at most one solution (x,k) in the space $C(\mathbb{R}_+;\mathbb{R}^d) \times \left[C(\mathbb{R}_+;\mathbb{R}^d) \cap BV_{loc}(\mathbb{R}_+;\mathbb{R}^d)\right]$. Moreover, if $(x,k) \in \mathcal{GSP}(H\partial^-\varphi; x_0, g, m)$ and $(\hat{x}, \hat{k}) \in \mathcal{GSP}(H\partial^-\varphi; \hat{x}_0, g, \hat{m})$, then*

$$|x(t) - \hat{x}(t)| \leq Ce^{CV(t)}\left[|x_0 - \hat{x}_0| + \updownarrow m - \hat{m} \updownarrow_t\right], \quad \text{a.e. } t \geq 0, \tag{46}$$

where $V(t) = \updownarrow x \updownarrow_t + \updownarrow \hat{x} \updownarrow_t + (1+\gamma)\updownarrow k \updownarrow_t + (1+\gamma)\updownarrow \hat{k} \updownarrow_t + 2\rho t + G_N t$, with $N \geq ||x||_T \vee ||\hat{x}||_T$ and C is a constant depending on L_H and c_H.

We can advance now to the part treating the existence of a solution for Eq.(42).

Proposition 32 *Let the assumptions (H_H) and (H_φ) be satisfied and $(x,k) \in \mathcal{GSP}(H\partial^-\varphi; x_0, 0, m)$. Then, there exists a positive constant $C_T(\|m\|_T) = C(x_0, L_H, c_H, K, T, \|m\|_T)$, increasing function with respect to $\|m\|_T$, such that, for all $0 \leq s \leq t \leq T$,*

$$\begin{array}{ll} (a) & \|x\|_T + \updownarrow k \updownarrow_T \leq C_T(\|m\|_T), \\ (b) & |x(t) - x(s)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \leq C_T(\|m\|_T)\sqrt{t - s + \mathbf{m}_m(t-s)}. \end{array} \tag{47}$$

We are now able to provide the result which assures the existence of a solution for the Skorokhod equation (42).

Theorem 33 *Suppose that hypotheses $(H_H - i \;\&\; ii)$, $(H_g - i \;\&\; ii)$, $(H_\varphi - i,\; ii \;\&\; iii)$ hold and let $m \in C(\mathbb{R}_+;\mathbb{R}^d)$, $m(0) = 0$. Assume also that, for $T > 0$,*

$$\int_0^T (g^\#(t))^2 dt < +\infty, \quad \text{where} \quad g^\#(t) := \sup_{x \in \mathrm{Dom}(\varphi)} |g(t,x)|.$$

Then the nonconvex Skorokhod problem (42) admits a unique solution $(x,k) \in \mathcal{GSP}(H\partial^-\varphi; x_0, g, m)$, in the sense of Definition 28.

2.3.4 Applications. SVIs with oblique Fréchet subgradients

In order to consider general stochastic variational inequalities with oblique Fréchet subgradients, we first present a problem which makes, in fact, the transition from the (deterministic) Skorokhod problem analyzed in the previous section to its stochastic counterpart. The approach follows the same stapes like the ones developed in the convex framework.

2.4 Obstacle problems for parabolic SDEs with Hölder continuous diffusion

2.4.1 Problem formulation. Main assumptions

Our research is devoted to a qualitative analysis (existence and uniqueness results, asymptotic behavior and a maximum principle for the strong solution) of the following stochastic variational inequality, considered in a infinite-dimensional setting and featuring weak assumptions on the coefficients:[2]

(48)
$$\begin{cases} (i) & u\in L^p_{ad}(\Omega\times(0,T);V)\cap L^2_{ad}(\Omega;C([0,T];H)),\\ (ii) & u(\omega,t)\in K(t),\ \forall t\in[0,T],\ \mathbb{P}-a.s.\ \omega\in\Omega,\\ (iii) & \text{there exists } k\in L^2_{ad}(\Omega;C([0,T];H)),\\ & \text{a } H\text{-valued bounded variation stochastic process, such that,}\\ & u(t)+k(t)=u_0+\displaystyle\int_0^t A(r,u(r))dr+\int_0^t g(r,u(r))dW_r,\ \text{for all } t\in[0,T],\mathbb{P}-a.s.\ \omega\in\Omega,\\ (iv) & \displaystyle\int_s^t\langle v(r)-u(r),dk(r)\rangle\le 0,\ \mathbb{P}-a.s., \forall v\in C([0,T];H), v(r)\in K(r)\ \text{for all } r\in[0,T],\\ & \qquad\qquad\qquad\qquad \text{for all } s,t\in[0,T],\ s\le t,\\ (v) & u(0,x)=u_0(x)\quad \text{in } D,\end{cases}$$

where $D\subset\mathbb{R}^N$ is an open bounded set, with its smooth frontier Γ (for example, of class C^2). Also, notice that conditions $(ii-iv)$ can be written, formally, in the equivalent variational form, as:[3]

$$(ii')\quad du(t)+A(t,u(t))dt+\partial I_{K(t)}(u(t))dt\ni g(t,u(t))dW(t).$$

Consider $H=L^2(D)$ and let V be a separable reflexive Banach space, with continuous dense embedding in H, such that, for all $u\in V$, $u^+\in V$, where $u^+(x):=\max\{u(x),0\}$. In addition, we assume the following assumption on V: for every Lipschitz function $\eta\in C^\infty(\mathbb{R})$, satisfying $\eta(0)=0$, we have, for all $v\in V$,

(49)
$$\eta(v)(\cdot)(:=\eta(v(\cdot)))\in V\quad\text{and}\quad \|\eta(v)\|_V\le C(1+\|v\|_V).$$

Denote by $|\cdot|_2, \|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively, the norms from H,V and V^*, respectively; $\langle v^*,v\rangle$ is the duality between V^* and V, while $(\cdot,\cdot)$ represents the inner product from H. Moreover, for $v\in L^r(D)$ and $u\in L^q(D)$, $\frac{1}{r}+\frac{1}{q}=1$, $r,q\ge 1$, let $(v,u)=\int_D v(x)u(x)dx$. The usual norm in $L^r(D)$ is given by $|\cdot|_r$. Finally, for $k,r,s\in\mathbb{N}$, denote by $H^k(D)$, $H^k_0(D)$ and $H^{s,r}((0,T)\times D)$, respectively, the usual Sobolev spaces on D and $(0,T)\times D$, respectively.

Introduce now the assumptions on Eq.(48) in a manner which will cover both the Lipschitz and the Hölder continuity of the diffusion coefficient. Whenever it is necessary, we precise the exact situation we deal with. In this manner, we present an unitary analysis, which recuperates the known results for the Lipschitz continuity of the diffusion. In order to obtain the existence of

[2] The spaces $L^2_{ad}(\Omega;C([0,T];H))\subset L^2(\Omega;C([0,T];H))$ and $L^p_{ad}(\Omega\times(0,T);X)\subset L^p(\Omega\times(0,T);X)$, $1\le p<\infty$, with $X=H$ or V, represents the closed linear subspaces of the non-anticipatives stochastic processes, adapted to the stochastic basis introduced in hypothesis $(\mathbf{H}_3)$.

[3] The function $I_{K(t)}(u)=0$ if $u\in K(t)$ and $+\infty$ if $u\in H\setminus K(t)$ is the convexity indicator function of the set $K(t)$ and the maximal monotone graph $\partial I_{K(t)}$ is its subdifferential operator, which is given by $\partial I_{K(t)}(u)=\{u^*\in H:(u^*,v-u)\le 0,\ \forall v\in K(t)\}=\{u^*\in H:u^*(x)\in\partial I_{K(t,x)}(u(x))\text{ a.e.}\}$.

an weak variational solution, some specific hypothesis will be given at the beginning of Section 2.4.4.

H$_1$ **:** Let $p>1$ and $A(t,\cdot):V\to V^*$, $t\in[0,T]$ verifying, for every $u,v,w\in V$,

(i) there exist $\mu,\gamma>0$ and $\lambda,\lambda_1,\nu\in\mathbb{R}$ such that

$i_1)$ $\langle A(t,u),u\rangle+\lambda_1|u|_2^2+\nu\geq\gamma\|u\|_V^p$ a.e. $t\in(0,T)$;

$i_2)$ $\langle A(t,u),b(u)\rangle+\lambda_1(1+|b(u)|_2)(1+|A(t,\psi)|_2+\|\psi\|_V+\|u\|_V)\geq 0$, a.e. $t\in(0,T)$, for every increasing function $b\in C^\infty(\mathbb{R})$ satisfying $b(V)\subset V$ and every $\psi\in V$ such that $b(\psi)=0$ and $A(t,\psi)\in V$;

$i_3)$ $\langle A(t,u)-A(t,v),b(u-v)\rangle+\lambda(u-v,b(u-v))\geq 0$, a.e. $t\in(0,T)$, for every increasing function $b\in C^\infty(\mathbb{R})$ satisfying $b(V)\subset V$ and $b(0)=0$; remark that (i_2) appears as a consequence of $(i_1)\&(i_3)$ if $p=2$.

$i_4)$ $\|A(t,u)\|_{V^*}\leq\mu(1+\|u\|_V^{p-1})$, a.e. $t\in(0,T)$.

(ii) $\theta\longmapsto\langle A(t,u+\theta v),w\rangle:\mathbb{R}\to\mathbb{R}$ is continuous, a.e. $t\in(0,T)$.

(iii) $t\longmapsto A(t,u):[0,T]\to V^*$ is Lebesgue measurable on $[0,T]$.

H$_2$ **:** $g:[0,T]\times D\times\mathbb{R}\to\mathbb{R}$ is measurable and there exist $L,M>0$ and $\alpha\in[1/2,1]$ such that, $\forall r,q\in\mathbb{R}$, a.e. $(t,x)\in[0,T]\times D$,

$$|g(t,x,r)-g(t,x,q)|\leq L|r-q|^\alpha \quad\text{and}\quad |g(t,x,r)|\leq M(1+|r|).$$

H$_3$ **:** $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\geq 0})$ is a complete filtered probability space (a stochastic base) and W is a H-valued Wiener process, with the covariance operator $Q\in\mathcal{L}^1(H)$ (Q is a nuclear operator)[4] and q the kernel of Q. Assume **Tr** $(q)\in L_1^\alpha(D)$, where

$$\alpha_1=\begin{cases}\frac{1}{1-\alpha}, & \text{if } \alpha\in[1/2,1)\\ +\infty, & \text{if } \alpha=1.\end{cases}$$

Recall that the operator $Q\in\mathcal{L}(H)$ is defined by

$$\mathbb{E}\left[(W(t)-W(s),h)(W(t)-W(s),k)\right]=(t-s)(Qh,k),\quad\forall\, 0\leq s\leq t,\ h,k\in H;$$

H$_4$ **:** Let $\psi_1,\psi_2:[0,T]\times D\to\mathbb{R}$, satisfying $\psi_1,\psi_2\in H^{1,2}((0,T)\times D)\cap L^p(0,T;V)$, $\psi_1(t,x)\leq\psi_2(t,x)$, $\forall(t,x)\in[0,T]\times D$ and consider the time-dependent constraints set characterized by:

$$\begin{aligned}
K(t)&=\{h\in H: h(x)\geq\psi_1(t,x),\ \text{a.e. } x\in D\}=:[[\psi_1(t),+\infty))\quad\text{or}\\
K(t)&=\{h\in H: h(x)\leq\psi_2(t,x),\ \text{a.e. } x\in D\}=:((-\infty,\psi_2(t)]]\quad\text{or}\\
K(t)&=\{h\in H: \psi_1(t,x)\leq h(x)\leq\psi_2(t,x),\ \text{a.e. } x\in D\}=:[[\psi_1(t),\psi_2(t)]].
\end{aligned}$$

[4] A linear operator Q, defined over a separable Hilbert space H (with the inner product $(\cdot,\cdot)$ and the endowed norm $|\cdot|_H$), is compact if Q maps any bounded set of H to a relatively compact set of H (Q maps any weakly convergent sequence in H to a strongly convergent sequence in H). Q is a nulcear operator if $(Q^*Q)^{1/2}$ is a non-negative operator with finite trace i.e. for some (and hence all) orthonormal bases $\{e_i\}_i$ of H, the sum $\|Q\|_1:=\mathrm{Tr}\left[(Q^*Q)^{1/2}\right]=\sum_{i=1}^\infty((Q^*Q)^{1/2}e_i,e_i)$ is convergent. The space of the nuclear operators, equipped with the norm $\|\cdot\|_1$, is $\mathcal{L}^1(H)$. Let now $\mathcal{L}^2(H)$ be the space of bounded operators with finite Hilbert-Schmidt norm $\|Q\|_2:=\mathrm{Tr}(QQ^*)=\sum_{i=1}^\infty|Qe_i|_H^2$. The Hilbert-Schmidt operators are nuclear operators. Denote $\mathcal{L}_Q^2(H)=\{v\in\mathcal{L}(H): vQ^{1/2}\in\mathcal{L}^2(H)\}$ and $\|v\|_Q^2:=\sum_{i=1}^\infty|vQ^{1/2}e_i|_H^2=\mathrm{Tr}(vQv^*)$.

Moreover, suppose there exist $\xi_1, \xi_2, \xi_3 \in H^{1,0}((0,T) \times D)$, $\xi_i \geq 0$ a.e., $\forall i = 1,2,3$, $\xi_3 \leq 1$ a.e., such that $\psi \in L^p(0,T;V) \cap H^{2,0}((0,T) \times D)$ and $A(\psi) \in L^2((0,T) \times D)$, where

$$(50) \qquad \psi := \begin{cases} \psi_1 + \xi_1, & \text{if } K(t) = [[\psi_1(t), +\infty)) \\ \psi_2 - \xi_2, & \text{if } K(t) = ((-\infty, \psi_2(t)]] \\ (1-\xi_3)\psi_1 + \xi_3\psi_2, & \text{if } K(t) = [[\psi_1(t), \psi_2(t)]] \end{cases}$$

H$_5$: $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $\mathbb{P}$-a.s., $u_0 \in K(0) \in \{[[\psi_1(x), +\infty)), ((-\infty, \psi_2(x)]], [[\psi_1(x), \psi_2(x)]]\}$, depending on the case ($\psi_i(x) := \psi_i(0,x)$, $i \in \{1,2\}$).

Finally, we add some additional working hypothesis. We first analyze the problem under the hypotheses ($\mathbf{H}_6 - (i)$) and, in Section 2.4.3, we obtain, along with the uniqueness of the solution, the existence of an absolutely continuous feedback process k. In Section 2.4.4 we renounce at the mentioned assumption ($\mathbf{H}_6 - (i)$) and we provide the existence of an weak-variational solution for the problem.

H$_6$:

(i) $g(t,x,\psi_1(t,x)) = g(t,x,\psi_2(t,x)) = 0$, $\forall (t,x) \in [0,T] \times D$.

(ii) If g is Hölder continuous, with the Hölder exponent $\alpha \in [1/2, 1)$, we suppose that V is a Hilbert space, continuously densely embedded into H and:

$a)$ $A(t,u) = A(u) := A_0(u) + \hat{A}_0(u)$, with $A_0 \in \mathcal{L}(V, V^*)$, $\hat{A}_0 : H \to H$ is a continuous operator. The operator $A(u)$ verifies ($\mathbf{H}_1$), with $p = 2$.

$b)$ $g(t,x,r) \equiv g(x,r)$, $\psi_i(t,x) \equiv \psi_i(x)$, $i \in \{1,2\}$, $K(t) \equiv K \subset H$.

$c)$ $u_0(\omega, x) = u_0(x)$, $\mathbb{P} - a.s.$, $u_0 \in K$.

$d)$ $\psi_i, \psi \in V$, $A(\psi) \in H$.

Let $G(t,u)$, $u \in H$, $t \in [0,T]$ be the linear operator $G(t,u) : \mathrm{Dom}(G(t,u)) \subset H \to H$ given by:

$$(51) \qquad \begin{cases} (i) & (G(t,u)h)(x) = g(t,x,u(x))h(x) \\ (ii) & \mathrm{Dom}(G(t,u)) = \{h \in H : G(t,u)h \in H\}, \ \forall t \in [0,T]. \end{cases}$$

2.4.2 Framework for the existence and uniqueness results

We first present a result which offers useful regularizations of the term characterizing the obstacle. Following the suggestions from Barbu [6, Chapter 2], given a maximal monotone graph $\beta \subset \mathbb{R} \times \mathbb{R}$, we introduce the *mollifier approximation* $\beta^\varepsilon \in C^\infty(\mathbb{R})$ of its Yosida approximation $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$, which is given as the unique solution of the inclusion $\beta_\varepsilon(r) \in \beta(r - \varepsilon\beta_\varepsilon(r))$. More precisely, for $\rho \in C_0^\infty(\mathbb{R})$, with $\int_{\mathbb{R}} \rho(\theta) d\theta = 1$, $\rho(\theta) = \rho(-\theta) \geq 0$, $\forall \theta \in \mathbb{R}$ and $\rho(\theta) = 0$, for all $|\theta| \geq 1$, the $1/\varepsilon$-Lipschitz function $\beta^\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined by

$$\beta^\varepsilon(r) := \int_{-\infty}^{\infty} \beta_\varepsilon(r - \varepsilon^2\theta)\rho(\theta)d\theta - \int_{-\infty}^{\infty} \beta_\varepsilon(-\varepsilon^2\theta)\rho(\theta)d\theta.$$

Before advancing to the core of this section, we present the Itô's formula, adapted to the current infinite dimensional setting. For more details, the interested reader is invited to consult Pardoux [130, Theorem 4.2.] (the study is focused on the Lipschitz diffusion setup) or Viot [168, Theorem 1.3] (the results are obtained under a Hölder assumption for the diffusion).

2.4.3 The case of an absolutely continuous feedback process

Theorem 34 *Suppose that the hypothesis* ($\mathbf{H}_1$-$\mathbf{H}_6$) *hold. Then there exists a unique pair of stochastic processes* $(u,h) \in [L^p_{ad}(\Omega\times(0,T);V)\cap L^2_{ad}(\Omega;C([0,T];H))]\times L^2_{ad}(\Omega\times(0,T);H)$ *such that, for all* $t\in[0,T]$, $\mathbb{P}$*-a.s.,*

$$\text{(52)}\qquad \begin{cases} (i) \quad u(t)\in K(t), \qquad (ii) \quad h(t)\in\partial I_{K(t)}(u(t)) \ a.e. \ in \ \Omega\times(0,T), \\ (iii) \quad u(t)+\int_0^t A(s,u(s))ds+\int_0^t h(s)ds = u_0+\int_0^t G(s,u(s))dW(s), \end{cases}$$

the equality (52-iii) taking place in V^*. *In addition, if* (u_1,h_1) *and* (u_2,h_2), *respectively, are the solutions corresponding to the initial data* u_{01} *and* u_{02}, *respectively, then, for every* $0\le s\le t\le T$,

- *for* $1/2\le\alpha\le 1$,

$$\text{(53)}\qquad \mathbb{E}(|u_1(t)-u_2(t)|_1|\mathcal{F}_s)\le e^{\lambda(t-s)}|u_1(s)-u_2(s)|_1,\ \mathbb{P}\text{-}a.s.,$$

- *for* $\alpha=1$ *we have*

$$\text{(54)}\qquad \begin{cases} (i) \quad \mathbb{E}(|u_1(t)-u_2(t)|_2^2|\mathcal{F}_s)\le|u_1(s)-u_2(s)|_2^2 e^{(t-s)(\lambda+\frac{L^2}{2}|\mathbf{Tr}q|_\infty)},\ \mathbb{P}\text{-}a.s., \\ (ii) \quad \mathbb{E}\left(\sup_{t\in[0,T]}|u_1(t)-u_2(t)|_2^2\right)\le 4\mathbb{E}(|u_{01}-u_{02}|_2^2)C(T), \end{cases}$$

where $C(T)=\exp(T(4|\lambda|+14L^2|\mathbf{Tr}q|_\infty))$.

2.4.4 Weak variational solutions

We weaken now the assumptions on the barriers ψ_1 and ψ_2 by giving up to the hypothesis ($\mathbf{H}_6-(i)$). Moreover, in the same spirit of ($\mathbf{H}_6-(ii_a)$), assume $A(t,u):=A_0(u)-F(t,u)$, with $A_0\in\mathcal{L}(V,V^*)$, $A_0=A_0^*$, $\langle A_0(v),v\rangle\ge\alpha_0\,||v||_V^2$, $\forall v\in V$ ($\alpha_0>0$) and $F(t,\cdot):V\to H$, $\forall t$ is Lipschitz and has sublinear growth with respect to the second variable. Moreover, assume that the embedding $V\subset H$ is a compact one. Denote by $\mathcal{W}^{\mathbb{F}}(\Omega\times(0,T);K)$ the space of adapted (with respect to $\mathbb{F}$) stochastic processes $v\in L^2_{ad}(\Omega;C([0,T];H))\cap L^2_{ad}(\Omega\times(0,T);V)$ of the form $v(t)=v_0+\eta(t)+\int_0^t\tilde{v}(s)dW(s)$, with $v_0\in H$, $\eta\in L^2_{ad}(\Omega;C([0,T];V^*))$, $\eta(0)=0$, $\|\eta\|_{BV([0,T];H)}<+\infty\ \ \mathbb{P}-a.s.$ and $\tilde{v}\in L^2_{ad}(\Omega\times(0,T);\mathcal{L}^2_Q(H))$, such that $v(\omega,t)\in K$, $\mathbb{P}$-a.s., for all $t\in[0,T]$.

Let us construct the operator

$$\begin{aligned} H(v,u) &= \frac{1}{2}\mathbb{E}|v_0-u_0|_2^2+\mathbb{E}\int_0^T\langle v(r)-u(r),A_0(u(r))-F(r,u(r))\rangle\,dr \\ &+\mathbb{E}\int_0^T\langle v(r)-u(r),d\eta(r)\rangle+\frac{1}{2}\mathbb{E}\int_0^T\mathbf{Tr}(\tilde{v}(r)-\widetilde{G}(u(r)))Q(\tilde{v}(r)-\widetilde{G}(u(r)))dr. \end{aligned}$$

and propose the following definition for the notion of weak variational solution. For more details on the notion of stochastic weak variational solutions, the reader can consult the pioneering work on this topic from Răşcanu [143], Bensoussan, Răşcanu [15] and, also, some more recent studies given by Maticiuc, Răşcanu [122] and Pardoux, Răşcanu [136] where this type of solutions for backward SDEs is addressed.

Definition 35 *A collection $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}, \bar{W}, \bar{u})$ is an weak variational solution for Eq.(48) if the following hold:*

$$
(55) \qquad \begin{cases} (i) & \bar{W} \text{ is a } H\text{-valued Wiener process on } (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}}), \text{ with its covariance } Q, \\ (ii) & \bar{u} \in L^2_{ad}(\bar{\Omega} \times (0,T); V), \\ (iii) & \bar{u}(t) \in K, \bar{\mathbb{P}}\text{-a.s., a.e. } t \in (0,T) \\ (iv) & H(v, \bar{u}) \geq 0, \text{ for every } v \in \mathcal{W}^{\bar{\mathbb{F}}}(\bar{\Omega} \times (0,T); K). \end{cases}
$$

We formulate the main statement of this section. The proof is based on tightness results and on the suitable choice of the working space.

Theorem 36 *Suppose that the embedding $V \subset H$ is a compact one. Under the hypothesis ($\boldsymbol{H}_1$-$\boldsymbol{H}_5$ and ($\boldsymbol{H}_6 - (ii)$)), the problem (48) admits at least one weak variational solution, in the sense of Definition 35.*

2.4.5 Behavior of the strong solution

We present below a qualitative analysis of the solution for our obstacle problem. We provide a maximum principle and we investigate the exponential stability of the strong solution.

A maximum principle

We first give some conditions for the existence of a unique strong solution $u \in L^2_{ad}(\Omega; C([0,T]; H)) \cap L^p_{ad}(\Omega \times (0,T); V)$ for the following problem

$$
(56) \qquad \begin{cases} du(t) + A(t,u)dt = g(t,u)dW(t), \quad \mathbb{P}\text{-}a.s., \text{ in } (0,T) \times D, \\ u(0) = u_0, \quad \mathbb{P}\text{-}a.s., \text{ in } D, \quad u(t) \in K(t), \ \forall t \in [0,T], \ \mathbb{P}\text{-}a.s. \end{cases}
$$

Problem (56) is equivalent with finding a solution for (52), with its second component $h = 0$. We situate our research in the framework of Section 2.4.1.

Case $K(t) = [[\psi_1(t), +\infty))$. We have the following existence and uniqueness result.

Theorem 37 *Suppose that all hypothesis ($\boldsymbol{H}_1$) to ($\boldsymbol{H}_6$) hold and, moreover, $A(\psi_1) \leq 0, \mathbb{P}\text{-}a.s.$, a.e. in $(0,T) \times D$. Then, the problem (56) admits a unique solution*

$$u \in L^p_{ad}(\Omega \times (0,T); V) \cap L^2_{ad}(\Omega; C([0,T]; H)).$$

We provide similar results for the cases $K(t) = ((-\infty, \psi_2(t)]]$ and $K(t) = [[\psi_1(t), \psi_2(t)]]$.

Mean exponential stability

We first remark that, for every $T > 0$, there exist a unique adapted stochastic process $u : \Omega \times [0, +\infty) \to H$, such that $u|_{[0,T]}$ is a solution of the problem (52).

Theorem 38 *Under the hypothesis ($\boldsymbol{H}_1$) to ($\boldsymbol{H}_6$), the following estimates hold:*

I. If $1/2 \leq \alpha \leq 1$, we have:

$$
(57) \qquad \begin{cases} (i) & \mathbb{E}(|u_1(t) - u_2(t)|_1 | \mathcal{F}_s) \leq e^{\lambda(t-s)} |u_1(s) - u_2(s)|_1, 0 \leq s \leq t \\ (ii) & \mathbb{E}\left(\displaystyle\int_0^t |u_1(s) - u_2(s)|_1 ds\right) \leq \dfrac{e^{\lambda t} - 1}{\lambda} \mathbb{E}|u_{01} - u_{02}|_1 \\ (iii) & \mathbb{E}\left(\displaystyle\int_0^\infty |u_1(s) - u_2(s)|_1 ds\right) \leq \dfrac{-1}{\lambda} \mathbb{E}|u_{01} - u_{02}|_1, \text{ for } \lambda < 0 \end{cases}
$$

II. If $\alpha = 1$, denote $\tilde{C} := \lambda + \frac{L^2}{2}|\mathbf{Tr}q|_\infty$ and we have:

(58)
$$\begin{cases} (i) & \mathbb{E}(|u_1(t) - u_2(t)|_2^2|\mathcal{F}_s) \le |u_1(s) - u_2(s)|_2^2 e^{\tilde{C}(t-s)} \\ (ii) & \mathbb{E}\left(\int_0^t |u_1(s) - u_2(s)|_2^2 ds\right) \le \dfrac{e^{\tilde{C}t} - 1}{\tilde{C}} \mathbb{E}|u_{01} - u_{02}|_2^2 \\ (iii) & \mathbb{E}\left(\int_0^\infty |u_1(s) - u_2(s)|_2^2 ds\right) \le \dfrac{1}{|\tilde{C}|} \mathbb{E}|u_{01} - u_{02}|_2^2, \text{ if } \tilde{C} < 0 \\ (iv) & \mathbb{E}\sup_{t\in[0,T]} |u_1(t) - u_2(t)|_2^2 \le C(T)\mathbb{E}|u_{01} - u_{02}|_2^2,\ C(T) := 4\exp T(4|\lambda| + 14L^2|\mathbf{Tr}q|_\infty). \end{cases}$$

Conclusion 39

1. *If $\lambda < 0$, then the solutions are exponentially stable in expectation and, for $\alpha = 1$, $\lambda + \frac{L^2}{2}|\mathbf{Tr}q|_\infty < 0$, they are exponentially stable in quadratic mean.*

2. *If $\lambda < 0$, then $|u_1(t) - u_2(t)|_1$ is a supermartingale, due to the inequality (53-i):*

$$\mathbb{E}(|u_1(t) - u_2(t)|_1|\mathcal{F}_s) \le |u_1(s) - u_2(s)|_1,\ \forall 0 \le s \le t,\ \mathbb{P}\text{-}a.s.$$

If $\alpha = 1$ and $\lambda + \frac{L^2}{2}|\mathbf{Tr}q|_\infty < 0$, then $|u_1(t) - u_2(t)|_2^2$ is a supermartingale, due to the inequality (54-i).

3. *From Doob's convergence theorem and (57-i), with $\lambda < 0$, it follows that*

$$\lim_{t\to\infty} |u_1(t) - u_2(t)|_1 = 0,\ \mathbb{P}\text{-}a.s.,$$

i.e., the solutions are asymptotic stable almost surely. From (58-i), with $\lambda + \frac{L^2}{2}|tr\ q|_\infty < 0$, we have that

$$\lim_{t\to\infty} |u_1(t) - u_2(t)|_2 = 0,\ \mathbb{P}\text{-}a.s.$$

2.5 Parabolic variational inequalities with generalized reflecting directions

2.5.1 Preliminaries, notations and basic assumptions

If $[a,b]$ is a real, closed interval and $\mathbb{Y}$ is a Banach space, then we denote by $L^p\left(a,b;\mathbb{Y}\right), C\left([a,b];\mathbb{Y}\right)$, $BV\left([a,b];\mathbb{Y}\right)$ and $AC\left([a,b];\mathbb{Y}\right)$ the usual spaces of p-integrable, continuos,with bounded variation, and, respectively, absolutely continuous $\mathbb{Y}$-valued function on $[a,b]$. By $W^{1,p}([a,b];\mathbb{Y})$ we shall denote the space of $y \in L^p(a,b;\mathbb{Y})$ such that $y' \in L^p(a,b;\mathbb{Y})$, where y' is the derivative in the sense of distributions. We denote by $\mathcal{L}\left(\mathbb{Y}\right)$ the Banach space of linear operators $A:\mathbb{Y}\rightarrow\mathbb{Y}$, with the norm $\left\Vert A\right\Vert_{\mathcal{L}(\mathbb{Y})} := \sup\left\{\left\Vert Ay\right\Vert_{\mathbb{Y}} : \left\Vert y\right\Vert_{\mathbb{Y}} = 1\right\}$.

Throughout this section we shall our work in the Gelfand triple framework. More precisely, we consider two real separable Hilbert spaces $\mathbb{V}$ and $\mathbb{H}$ such that $\mathbb{V}\subset\mathbb{H}\cong\mathbb{H}^*\subset\mathbb{V}^*$, with continuous and dense embeddings, where $\mathbb{V}^*$ denoted the dual of $\mathbb{V}$. Moreover, assume that the inclusion $\mathbb{V}\subset\mathbb{H}$ is a compact one. The norm from $\mathbb{V}$ is denoted by $||\cdot||$, the one from $\mathbb{H}$ is $|\cdot|$ and $\mathbb{V}^*$ is endowed with the norm $||\cdot||_*$. The scalar product of $\mathbb{H}$ is $(\cdot,\cdot)$ and the duality pairing between $\mathbb{V}$ and $\mathbb{V}^*$ is given by $\langle\cdot,\cdot\rangle$. Let $\gamma_1,\gamma_2>0$ be some positive constants of boundedness, corresponding to the above inclusions $||y||_* \le \gamma_1\left|y\right| \le \gamma_2\left||y|\right|$, $\forall y\in\mathbb{V}$.

We study the following type of evolution equation, driven by oblique reflected subgradients, as we can see below:

$$\left\{\begin{array}{l} y'(t)+Ay(t)+\Theta(t,y(t))\partial\varphi(y(t))\ni f(t,y(t)),\\ y(0)=y_0\in\mathbb{H},\ t\in[0,T]\,, \end{array}\right. \tag{59}$$

where:

(A_φ) : $\varphi:\mathbb{H}\rightarrow(-\infty,+\infty]$ is a proper, convex, lower semicontinuous function.

(A_A) : $A\in\mathcal{L}(\mathbb{V},\mathbb{V}^*)$, $A=A^*$ s.t., for some constants $\alpha_0,\alpha_1>0$, $\forall y\in\mathbb{V}$, $\langle Ay,y\rangle\ge\alpha_0\left||y|\right|^2$ and

$$(Ay,\nabla\varphi_\varepsilon(y))\ge-\alpha_1(1+|\nabla\varphi_\varepsilon(y)|)(1+|y|),\ \forall y\in D(A_H). \tag{60}$$

Here $D(A_H):=\{v\in\mathbb{V}:Av\in\mathbb{H}\}$ and $A_Hv=Av$, $\forall v\in D(A_H)$.

(A_f) : $f:[0,T]\times\mathbb{H}\rightarrow\mathbb{H}$ is a Carathéodory function.

(A_Θ) : $\Theta:[0,T]\times\mathbb{H}\rightarrow\mathcal{L}(\mathbb{H})$, such that $(t,y)\mapsto\Theta(t,y)h:[0,T]\times\mathbb{H}\rightarrow\mathbb{H}$ is a continuous function, for all $h\in\mathbb{H}$.

Assume also that there exist $u_0\in\mathbb{V}$ and $\hat{u}_0\in\mathbb{H}$ such that $(u_0,\hat{u}_0)\in\partial\varphi$. The above hypothesis assure that problem (59) is well posed.

2.5.2 Existence and uniqueness of a solution

Concerning the Cauchy problem (59), we first prove the existence of at least one solution. For its uniqueness we will renounce at the dependence on the state for the perturbing term Θ and we consider some particular systems of PDEs. We assume (A_φ), (A_f) and (A_Θ) still holding and we enhance them by adding the additional hypothesis:

$$(H_\Theta^1)\qquad\left\{\begin{array}{ll}(i) & \forall\,(t,y)\in[0,T]\times\mathbb{H},\ \Theta(t,y):\mathbb{H}\rightarrow\mathbb{H}\text{ is a self-adjoint linear operator;}\\ (ii) & \exists\beta_0,\beta_1>0\text{ s.t., }\forall(t,y,h)\in[0,T]\times\mathbb{H}\times\mathbb{H},\quad \beta_0\left|h\right|^2\le(\Theta(t,y)h,h)\le\beta_1\left|h\right|^2.\end{array}\right.$$

$$(H_f^1)\qquad |f(t,y)|\le\mu_1(t)+\mu_2(t)\left|y\right|,\ \forall y\in\overline{\mathrm{Dom}(\varphi)},\text{ a.e. } t\quad(\text{for }\mu_1,\mu_2\in L^2\left(0,T;\mathbb{R}_+\right)).$$

$$(H_\varphi^1)\qquad \mathrm{Dom}(\varphi)\cap\mathbb{V}\neq\emptyset.$$

Without losing the generality, for convenience only, we can suppose that $\varphi(y)\ge\varphi(0)=0$, which easily implies that $0=\varphi_\varepsilon(0)\le\varphi_\varepsilon(y)$, for every $\varepsilon>0$ and $y\in\mathbb{H}$.

Existence of a solution

Define first the notion of solution for Eq.(59).

Definition 40 *A pair of functions $y, k : [0,T] \to \mathbb{H}$ is a (strong) solution of the oblique reflected evolution equation (59) if*

$$\begin{aligned} &(i) \quad y \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}),\\ &(ii) \quad k \in C([0,T];\mathbb{H}) \cap BV([0,T];\mathbb{H}), k(0)=0,\\ &(iii) \quad y(t)+\int_0^t Ay(s)ds+\int_0^t \Theta(s,y(s))dk(s)=y_0+\int_0^t f(s,y(s))ds,\\ &(iv) \quad \int_s^t \langle z(r)-y(r),dk(r)\rangle+\int_s^t \varphi(y(r))dr \le \int_s^t \varphi(z(r))dr, \forall s\le t,\ \forall z\in C([0,T];\mathbb{H}) \end{aligned} \tag{61}$$

We provide now the main result of this section.

Theorem 41 *Consider the hypothesis (A_φ), (A_f), (A_Θ), (H_f^1), (H_Θ^1) and (H_φ^1) be satisfied. Then, the evolution equation with oblique reflecting subgradients (59) admits at least one strong solution (y,k), in the sense of Definition 40. Moreover, the feedback reflecting process k is an absolutely continuous one, that is there exists $h \in L^2(0,T;\mathbb{H})$ such that $k(t)=\int_0^t h(s)ds$.*

Parabolic variational inequalities. Uniqueness of the solution

In order to prove the uniqueness of a solution we restrict the study of the general problem (59) and analyze a scenario given by the consideration of a particular system of multivalued PDEs. For doing this, let D be a domain with a smooth frontier (for example, of class C^2) from $\mathbb{R}^d$, $\mathbb{H} = L^2(D;\mathbb{R}^k) \equiv (L^2(D))^k$, $\mathbb{V} = H_0^1(D;\mathbb{R}^k) \equiv (H_0^1(D))^k$ and consider $\Theta : [0,T] \to \mathbb{R}^{k\times k}$, $\Theta(t) := diag(\Theta_1(t),...,\Theta_k(t))$, with $\Theta_i \in C^1([0,T];\mathbb{R})$. Assume also that $0 < c \le \Theta_i(t) \le C$, $\left|\frac{d}{dt}\Theta_i(t)\right| \le C$, for some positive constants c, C and for all $t\in[0,T]$, $i=\overline{1,k}$. For a convex, proper, l.s.c. function $\varphi : \mathbb{R}^k \to (-\infty,+\infty]$, consider the semilinear system of multivalued parabolic PDEs:

$$\begin{cases} \dfrac{du_i(x,t)}{dt} - \Delta u_i(x,t) + \Theta(t)\partial\varphi(u(x,t)) \ni f(t,u(x,t)), \text{ on } Q := D\times(0,T),\ i=\overline{1,k},\\ u(x,0)=u_0(x), \text{ on } D,\\ u(x,t)=0, \text{ on } \partial D\times(0,T), \end{cases} \tag{62}$$

where $u=(u_1,...,u_k) : D\times[0,T] \to \mathbb{R}^k$ and $u_0 \in H_0^1(D;\mathbb{R}^k)$, $f\in L^2(D\times(0,T);\mathbb{R}^k)$.

Theorem 42 *Under the setup introduced above, there exists a unique solution for problem (62).*

3 Backward Stochastic Variational Inequalities

3.1 Multivalued BSDEs with oblique subgradients

3.1.1 Setting the problem

Let $T>0$ be fixed and consider the backward stochastic variational inequality with oblique reflection (for brevity, $BSVI\left(H\left(t,y\right)\partial\varphi(y)\right)$, $BSVI\left(H\left(t\right)\partial\varphi(y)\right)$ or $BSVI\left(H\left(y\right)\partial\varphi(y)\right)$, respectively, if the matrix H depends only on time or on the state of the system, respectively), $\mathbb{P}-a.s.$,

$$\text{(63)}\qquad \begin{cases} Y_t+\displaystyle\int_t^T H\left(s,Y_s\right)dK_s=\eta+\int_t^T F\left(s,Y_s,Z_s\right)ds-\int_t^T Z_s dB_s, \quad t\in\left[0,T\right], \\ dK_s\in\partial\varphi\left(Y_s\right)\left(ds\right), \end{cases}$$

where $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq0})$ is a stochastic basis and $\{B_t:t\geq0\}$ is an $\mathbb{R}^k-$valued Brownian motion. Moreover, denoting $\mathcal{F}_t:=\mathcal{F}_t^B=\sigma(\{B_s:0\leq s\leq t\})\vee\mathcal{N}$, we assume the following.

(H_1) The terminal datum $\eta\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^d)$.

(H_2) $H(\cdot,\cdot,y):\Omega\times[0,T]\to\mathbb{R}^{d\times d}$ is $\mathcal{F}_t-$progressively measurable for every $y\in\mathbb{R}^d$; there exist $L_H,a_H,b_H>0$ such that, $\mathbb{P}-a.s.\ \omega\in\Omega$, $H=\left(h_{i,j}\right)_{d\times d}\in C\left(\left[0,T\right]\times\mathbb{R}^d;\mathbb{R}^{d\times d}\right)$ and, for all $t\in[0,T]$ and $y,\tilde{y}\in\mathbb{R}^d$, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$\text{(64)}\qquad \begin{cases} (i) & h_{i,j}\left(t,y\right)=h_{j,i}\left(t,y\right), \quad \forall i,j\in\overline{1,d}, \\ (ii) & a_H\left|u\right|^2\leq\left\langle H\left(t,y\right)u,u\right\rangle\leq b_H\left|u\right|^2, \quad \forall u\in\mathbb{R}^d, \\ (iii) & \left|H(t,\tilde{y})-H\left(t,y\right)\right|\leq L_H|\tilde{y}-y|. \end{cases}$$

(H_3) The function $\varphi:\mathbb{R}^d\to(-\infty,+\infty]$ is a proper lower semicontinuous convex function.

(H_4) The generator function $F\left(\cdot,\cdot,y,z\right):\Omega\times\left[0,T\right]\to\mathbb{R}^d$ is $\mathcal{F}_t-$progressively measurable for every $(y,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times k}$ and there exist $L,\ell\in L^2\left(0,T;\mathbb{R}_+\right)$ such that

$$\text{(65)}\qquad \begin{cases} (i) & \textit{Lipschitz conditions: for all } y,y'\in\mathbb{R}^d,\ z,z'\in\mathbb{R}^{d\times k},\ d\mathbb{P}\otimes dt-a.e.: \\ & |F(t,y',z)-F(t,y,z)|\leq L\left(t\right)|y'-y|, \quad |F(t,y,z')-F(t,y,z)|\leq\ell\left(t\right)|z'-z|; \\ (ii) & \textit{Boundedness condition:} \quad \mathbb{E}\displaystyle\int_0^T\left|F\left(t,0,0\right)\right|^2dt<+\infty. \end{cases}$$

Definition 43 *Given two functions $x,g:[0,+\infty)\to\mathbb{R}^d$ we say that $dg\left(t\right)\in\partial\varphi\left(x\left(t\right)\right)\left(dt\right)$ on $\mathbb{R}_+$ if, for all $T>0$,*

$$\begin{array}{ll} (a) & x\in C([0,T];\mathbb{R}^d),\ g\in BV([0,T];\mathbb{R}^d),\ g(0)=0, \\ (b) & \displaystyle\int_0^T\varphi\left(x\left(t\right)\right)dt<+\infty,\ \textit{for all } T\geq0, \quad \textit{and, for } \forall s\leq t,\ \forall y\in C([0,T];\mathbb{R}^d), \\ (c) & \displaystyle\int_s^t\left\langle y\left(r\right)-x(r),dg\left(r\right)\right\rangle+\int_s^t\varphi\left(x\left(r\right)\right)dr\leq\int_s^t\varphi\left(y\left(r\right)\right)dr. \end{array}$$

Let us now introduce the definition of solution for Eq.(63). For the case $H\left(t,y\right)\equiv H\left(t\right)$ we obtain the existence of a strong solution while, for $H\left(t,y\right)$ we obtain a weak solution for Eq.(63). In second situation, the equation must be considered in a suitable way.

Definition 44 *Given $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ a fixed stochastic basis and $\{B_t : t \geq 0\}$ an $\mathbb{R}^k$−valued Brownian motion, we say that a triplet (Y, Z, K) is a strong solution for the $BSVI\,(H\,(t)\,\partial\varphi(y))$ if*

$$(Y, Z, K) \in S_d^0\,[0,T] \times \Lambda_{d\times k}^0\,(0,T) \times S_d^0\,[0,T]\,,$$

such that, $\mathbb{P} - a.s.\ \omega \in \Omega$, $K_\cdot(\omega) \in BV([0,T]\,;\mathbb{R}^d)$ and

$$\begin{cases} Y_t + \displaystyle\int_t^T H\,(s)\,dK_s = \eta + \int_t^T F\,(s, Y_s, Z_s)\,ds - \int_t^T Z_s dB_s, \quad \forall t \in [0,T]\,, \\ dK_s \in \partial\varphi\,(Y_s)\,(ds)\,. \end{cases}$$

Consider now the case when the matrix H depends also on the state of the system. Renouncing at the dependence of F on the process Z, one can reconsider the BSVI with oblique reflection in the following manner, $\mathbb{P} - a.s.\ \omega \in \Omega$,

$$\text{(66)}\qquad \begin{cases} Y_t + \displaystyle\int_t^T H\,(s, X_s, Y_s)\,dK_s = g(X_T) + \int_t^T F\,(s, X_s, Y_s)\,ds - (M_T - M_t)\,,\ \forall t \in [0,T]\,, \\ dK_s \in \partial\varphi\,(Y_s)\,(ds)\,, \end{cases}$$

where $\{X_t : t \in [0,T]\}$ is an $\mathbb{R}^k$-valued progressively measurable continuous stochastic process, $g : \mathbb{R}^k \to \mathbb{R}^d$ is a continuous function and M is a continuous martingale (possible with respect to its natural filtration if not any other filtration available). The role played by X is an appropriate restriction of the ω-dependence of H and F, which are now deterministic functions. Assume that:

(H_2') $H : [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a continuous function which satisfies (64), uniformly with respect to $x \in \mathbb{R}^k$;

(H_4') $F : [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous function, for which there exists a constant $L > 0$ such that, for all $t \in [0,T]$,

$$|F(t,x,y') - F(t,x,y)| \leq L|y' - y|, \quad \text{for all } y, y' \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^k.$$

Definition 45 *If there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a quadruple $(X, Y, M, K) : \Omega \times [0,T] \to \mathbb{R}^k \times (\mathbb{R}^d)^3$ such that*

- (a) *M is a continuous martingale with respect to the filtration given by $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, $\mathcal{F}_t := \mathcal{F}_t^{X,Y,M} = \sigma(\{X_s, Y_s, M_s : 0 \leq s \leq t\}) \vee \mathcal{N}$, $t \in [0,T]$,*
- (b) *X, Y, K are càdlàg stochastic processes, adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$,*
- (c) *relation (66) is verified for every $t \in [0,T]$, $\mathbb{P} - a.s.\ \omega \in \Omega$,*

then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X, Y, M, K)$ is called a weak solution of the $BSVI(H(t,y)\partial\varphi(y))$.

We are able now formulate the main results of this section.

Theorem 46 *Considering hypothesis $(H_2'), (H_3)$ and (H_4') satisfied, the $BSVI\,(H\,(t,y)\,\partial\varphi(y))$ (66) admits at least one weak solution $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, X_t, Y_t, M_t, K_t)_{t\in[0,T]}$, in the sense of Definition 45. In addition, we can choose the process X as being a diffusion process from the class which places the problem in a Markovian framework, the drift and the diffusion coefficients b and σ being some given continuous Lipschitz functions.*

In the case $H(s,y) = H(s)$ (that is H is independent of the state), denoting

$$\nu_t = \int_0^t L(s) \left[\mathbb{E}^{\mathcal{F}_s} |\eta|^p\right]^{1/p} ds \quad \text{and} \quad \theta = \sup_{t\in[0,T]} \left(\mathbb{E}^{\mathcal{F}_t} |\eta|^p\right)^{1/p}, \tag{67}$$

we have the following result, which assures the existence of a strong solution.

Theorem 47 *Let $p \geq 2$ and the assumptions $(H_1), (H_2), (H_3)$ and (H_4) be satisfied. If $l(t) \equiv l < \sqrt{a_H}$, $H \in C^1\left([0,T]; \mathbb{R}^{d\times d}\right)$ and*

$$\mathbb{E}e^{\delta\theta} + \mathbb{E}\left|\varphi(\eta)\right| < +\infty, \tag{68}$$

for all $\delta > 0$, then the $BSVI\,(H(t)\,\partial\varphi(y))$ admits a unique strong solution $(Y,Z,K) \in S_d^2[0,T] \times \Lambda^2_{d\times k}(0,T) \times S_d^0[0,T]$. Moreover, for all $\delta > 0$,

$$\mathbb{E}\sup_{s\in[0,T]} e^{\delta p\nu_s} |Y_s|^p + \mathbb{E}\left(\int_0^T e^{2\delta\nu_s} |Z_s|^2 ds\right)^{p/2} < +\infty \tag{69}$$

and there exists a positive constant $C = C(a_H, b_H, L_H)$, independent of the terminal time T, such that, $\mathbb{P}-a.s.\ \omega \in \Omega$,

$$|Y_t| \leq C(1 + \left[\mathbb{E}^{\mathcal{F}_t} |\eta|^p\right]^{1/p}), \quad \text{for all } t \in [0,T].$$

In addition, the process $K \in S_d^2[0,T]$ and it can be represented as

$$K_t = \int_0^t U_s ds, \quad \text{with } U \in \Lambda_d^2(0,T) \text{ and } U_s \in \partial\varphi(Y_s), \text{ a.e. on } [0,T], \mathbb{P}-a.s.\ \omega \in \Omega.$$

Finally, the following estimate holds:

$$\mathbb{E}\sup_{t\in[0,T]} |Y_t|^2 + \mathbb{E}\int_0^T |U_t|^2 dt + \mathbb{E}\int_0^T |Z_t|^2 dt \leq C\left(\mathbb{E}|\eta|^2 + \mathbb{E}\left|\varphi(\eta)\right| + \mathbb{E}\int_0^T |F(t,0,0)|^2 dt\right). \tag{70}$$

The proofs of the above results are detailed along the next sections. First, we focuse on the construction of a sequence of approximating equations and a priori estimates of their solutions. The estimates will be valid for both cases covered by Theorem 46 and Theorem 47. After this, the proof is split between two sections, each one being devoted to the particularities brought by Theorem 46 and Theorem 47.

3.1.2 Approximating problems and a priori estimates

In order to prove the existence of the solution (strong or weak) we can assume, without loosing the generality, that $\varphi(y) \geq \varphi(0) = 0$.

Technical results

We first introduce, grouped under the form of a small section, some useful results which will be used during the study.

We can now start, simultaneously, the proofs of Theorem 47 and Theorem 46 by obtaining some a priori estimates for the solutions of the approximating equations.

Proof. Let $p > 1$.

Step 1. *Boundedness under the assumption*

$$0 \le \ell(t) \equiv \ell < \sqrt{a}.$$

Consider an approximating BSDE, with its solution $(Y^\varepsilon, Z^\varepsilon)$ and denote $U^\varepsilon = \nabla\varphi_\varepsilon(Y^\varepsilon)$. There exists a positive constant $C = C(a, b, \Lambda, l, L(\cdot))$, independent of ε, such that

$$\mathbb{E}\sup_{s\in[0,T]}|Y_s^\varepsilon|^2 + \mathbb{E}\int_0^T (|U_r^\varepsilon|^2 + |Z_r^\varepsilon|^2)dr \le C\left[\mathbb{E}|\eta|^2 + \mathbb{E}\varphi(\eta) + \mathbb{E}\int_0^T |F(r,0,0)|^2 dr\right]. \tag{71}$$

Step 2. *Convergences under the assumption*

$$0 \le \ell(t) \equiv \ell < \sqrt{a}.$$

The estimates of Step 1 imply that there exist a sequence $\{\varepsilon_n : n \in \mathbb{N}^*\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and six progressively measurable stochastic processes Y, Z, U, F, χ, h such that

$$Y_0^{\varepsilon_n} \to Y_0, \quad \text{in } \mathbb{R}^d, \quad Z^{\varepsilon_n} \rightharpoonup Z, \quad \text{weakly in } L^2(\Omega\times(0,T);\mathbb{R}^{d\times k}),$$

and, weakly in $L^2(\Omega\times(0,T);\mathbb{R}^d)$,

$$\begin{cases} Y^{\varepsilon_n} \rightharpoonup Y, & \nabla\varphi_{\varepsilon_n}(Y^{\varepsilon_n}) \rightharpoonup U, & H(\cdot, Y^{\varepsilon_n}) \rightharpoonup h, \\ H(\cdot, Y^{\varepsilon_n})\nabla\varphi_{\varepsilon_n}(Y^{\varepsilon_n}) \rightharpoonup \chi & \text{and} & F(\cdot, Y^{\varepsilon_n}, Z^{\varepsilon_n}) \rightharpoonup F. \end{cases}$$

The convergence $Y^{\varepsilon_n} \rightharpoonup Y$ implies that, on the sequence $\{\varepsilon_n : n \in \mathbb{N}^*\}$, $J_{\varepsilon_n}(Y^{\varepsilon_n}) \rightharpoonup Y$, weakly in $L^2(\Omega\times(0,T);\mathbb{R}^d)$. From the approximating BSDE we have that, at the limit,

$$Y_t + \int_t^T \chi_s ds = \eta + \int_t^T F_s ds - \int_t^T Z_s dB_s.$$

The continuity of the three integrals from the above equation imply also the continuity of the process Y, but the previous convergences are not yet sufficient to conclude that (Y, Z) is a solution of the considered equation. The remaining problems consist in proving that, for every $s \in [0,T]$, $\mathbb{P} - a.s.\ \omega \in \Omega$,

$$\chi_s = h_s U_s, \quad h_s = H(s, Y_s), \quad Us \in \partial\varphi(Y_s) \quad \text{and} \quad F_s = F(s, Y_s, Z_s).$$

■

Starting with this point, the proofs of Theorem 47 and Theorem 46 will take two separate paths.

3.1.3 Strong existence and uniqueness for $H(t,y) \equiv H_t$

We will continue in this section the proof of Theorem 47.

Proof. We continue the proof of the existence of a solution. Under the assumptions of Step 3 (Section 3.1.2) we prove that $\{Y^\varepsilon : 0 < \varepsilon \le 1\}$ is a Cauchy sequence. We obtain that the triplet (Y, Z, K) is the unique strong solution of the $BSVI(H(t), \varphi, F)$. ■

3.1.4 Weak existence for $H(t,y)$

We will continue in this section the proof of Theorem 46. All the a priori estimates obtained in Section 3.1.2. remain valid. In Section 3.1.3. we proved that the approximating sequence is a Cauchy sequence when the matrix H does not depend on the state of the system and, as a consequence, we derived the existence and uniqueness of a strong solution for $BSVI\left(H\left(t\right),\varphi,F\right)$. In the current setup, allowing the dependence on Y we will situate ourselves in a Markovian framework and we will use tightness criteria in order to prove the existence of a weak solution for $BSVI\left(H\left(t,y\right),\varphi,F\right)$.

First let $b:[0,T]\times\mathbb{R}^k\to\mathbb{R}^k$, $\sigma:[0,T]\times\mathbb{R}^k\to\mathbb{R}^{k\times k}$ be two continuous functions satisfying the classical Lipschitz conditions, which imply the existence of a non-exploding solution for the following SDE

$$X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dB_r,\quad t\le s\le T. \tag{72}$$

Let now consider the continuous generator function $F:[0,T]\times\mathbb{R}^k\times\mathbb{R}^d\to\mathbb{R}^d$ and assume there exist $L\in L^2\left(0,T;\mathbb{R}_+\right)$ such that, for all $t\in[0,T]$ and $x\in\mathbb{R}^k$,

$$|F(t,x,y')-F(t,x,y)|\le L\left(t\right)|y'-y|,\quad \textit{for all } y,y'\in\mathbb{R}^d, \tag{H_4'}$$

Given a continuous function $g:\mathbb{R}^k\to\mathbb{R}^d$, satisfying a sublinear growth condition, consider now the $BSVI\left(H\left(t,y\right),\varphi,F\right)$

$$\begin{cases} Y_s^{t,x}+\displaystyle\int_s^T H(r,Y_r^{t,x})dK_r^{t,x}=g(X_T^{t,x})+\int_s^T F(r,X_r^{t,x},Y_r^{t,x})dr-\int_s^T Z_r^{t,x}dB_r, \quad t\le s\le T,\\ dK_r^{t,x}\in\partial\varphi(Y_r^{t,x})\left(dr\right), \quad \text{for every } r. \end{cases} \tag{73}$$

Consider now the Skorokhod space $\mathcal{D}([0,T];\mathbb{R}^m)$ of càdlàg functions $y:[0,T]\to\mathbb{R}^m$ (i.e. right continuous and with left-hand side limit). It can be shown (see Billingsley [20]) that, although $\mathcal{D}([0,T];\mathbb{R}^m)$ is not a complete space with respect to the Skorokhod metric, there exists a topologically equivalent metric with respect to which it is complete and that the Skorokhod space is a Polish space. The space of continuous functions $C([0,T];\mathbb{R}^m)$, equipped with the supremum norm topology is a subspace of $\mathcal{D}([0,T];\mathbb{R}^m)$; the Skorokhod topology restricted to $C([0,T];\mathbb{R}^m)$ coincides with the uniform topology. We will use on $\mathcal{D}([0,T];\mathbb{R}^m)$ the Meyer-Zheng topology, which is the topology of convergence in measure on $[0,T]$, weaker than the Skorokhod topology. The Borel $\sigma-$field for the Meyer-Zheng topology is the canonical $\sigma-$field as for Skorokhod topology. Note that for the Meyer-Zheng topology, $\mathcal{D}([0,T];\mathbb{R}^m)$ is a metric space but not a Polish space.

We continue now the proof of Theorem 46.

Proof. For any fixed $n\ge1$ consider the following approximating equation, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$Y_t^n+\int_t^T H\left(s,Y_s^n\right)\nabla\varphi_{1/n}\left(Y_s^n\right)ds=g(X_T^{t,x})+\int_t^T F\left(s,X_s,Y_s^n\right)ds-\int_t^T Z_s^n dB_s,\quad \forall t\in\left[0,T\right]. \tag{74}$$

We prove a weakly convergence in the sense of the Meyer-Zheng topology, that is the laws converge weakly if we equip the space of paths with the topology of convergence in $dt-$measure. We obtain a collection $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathbb{P}},\mathcal{F}_t^{\bar Y,\bar M},\bar Y_t,\bar M_t,\bar K_t)_{t\in[0,T]}$ which is a weak solution of Eq.(73), in the sense of Definition (45), and the proof is now complete. ■

3.2 A numerical approximation scheme for multivalued BSDEs

3.2.1 Notations. Hypothesis. Preliminaries

We shall consider a finite horizon $T > 0$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{t \leq T}$ whose natural filtration is denoted $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$. The analyzed problem is considered in a Markovian framework, as follows. Consider the following data for the forward continuous stochastic process:

- some continuous coefficient functions $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times d}$, $g : \mathbb{R}^m \to \mathbb{R}^n$ and $F : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$, which satisfies the following standard assumptions: for some constants $\alpha \in \mathbb{R}$, $L, \beta, \gamma \geq 0$ and for all $t \in [0,T]$, $x, \tilde{x} \in \mathbb{R}^m$, $y, \tilde{y} \in \mathbb{R}^n$ and $z, \tilde{z} \in \mathbb{R}^{n\times d}$:

$$(75) \qquad \begin{cases} (i) & |b(x) - b(\tilde{x})| + \|\sigma(x) - \sigma(\tilde{x})\| \leq L\,|x - \tilde{x}|\,, \\ (ii) & \langle y - \tilde{y}, F(t,x,y,z) - F(t,x,\tilde{y},z)\rangle \leq \alpha\,|y - \tilde{y}|^2\,, \\ (iii) & |F(t,x,y,z) - F(t,x,y,\tilde{z})| \leq \beta\,\|z - \tilde{z}\|\,, \end{cases}$$

  and there exist some constants $M > 0$ and $p,\, q \in \mathbb{N}$ such that, for all $t \in [0,T]$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$:

$$(76) \qquad |g(x)| \leq M(1 + |x|^q) \quad \text{and} \quad |F(t,x,y,0)| \leq M(1 + |x|^p + |y|).$$

- a proper convex lower semicontinuous function $\varphi : \mathbb{R}^n \to (-\infty, +\infty]$ satisfying, for $M > 0$ and $r \in \mathbb{N}$ and all $x \in \mathbb{R}^m$:

$$(77) \qquad |\varphi(g(x))| \leq M(1 + |x|^r).$$

3.2.2 Approximations schemes for BSVIs

We consider a partition of $[0,T]$, $\pi = \{t_i = ih : 0 \leq i \leq n\}$, with $h := T/n,\ n \in \mathbb{N}^*$, on which we approximate the solution of the backward stochastic variational inequality. For the numerical simulations of the forward part of the solution, the most standard approach consists in approximating the SDE in a proper way on each interval $[t_i, t_{i+1}]$ by the classical Euler scheme (see, e.g. Kloeden, Platen [103]):

$$\begin{cases} X^h_{t_{i+1}} = X^h_{t_i} + b\left(X^h_{t_i}\right)h + \sigma\left(X^h_{t_i}\right)\left(B_{t_{i+1}} - B_{t_i}\right),\ i = \overline{0, n-1} \\ X^h_0 = X_0. \end{cases}$$

The above numerical scheme is easy to implement since it requires only the simulation of d-independent Gaussian variables for the Brownian increments, providing a weak error of h order. For $t \in [t_i, t_{i+1}]$ let

$$X^h_t = X^h_{t_i} + b(X^h_{t_i})\,(t - t_i) + \sigma(X^h_{t_i})\,(B_t - B_{t_i})\,.$$

We shall analyze the one-dimensional BSDE, without losing the generality of the study. Using the Yosida approximation $\nabla\varphi_\varepsilon$ of the multivalued operator $\partial\varphi$, with $\varepsilon = h^a$ and $a \in (0, 1/2)$ (the way of choosing this constant will be detailed later), we deduce that the following approximate equation

$$(78) \quad Y^h_t + \int_t^T \nabla\varphi_{h^a}(Y^h_r)dr = g(X_T) + \int_t^T F(r, X_r, Y^h_r, Z^h_r)dr - \int_t^T Z^h_r dB_r,\ \forall t \in [0,T]\,,\ \mathbb{P} - a.s.,$$

admits a unique solution $\left(Y_t^h, Z_t^h\right) \in S_1^2[0,T] \times \Lambda_{1\times d}^2(0,T)$. We propose the following implicit discretization procedure, which define the pair $(\tilde{Y}^h, \tilde{Z}^h)$ inductively, for $i = \overline{n-1,0}$:

$$\begin{cases} \tilde{Y}_T^h := g(X_T^h), \ \tilde{Z}_T^h = 0, \\ \tilde{Y}_{t_i}^h := \mathbb{E}^{i,h}(\tilde{Y}_{t_{i+1}}^h) + h\left[F(t_i, X_{t_i}^h, \tilde{Y}_{t_i}^h, \tilde{Z}_{t_i}^h) - \nabla\varphi_{h^a}(\tilde{Y}_{t_i}^h)\right], \\ \tilde{Z}_{t_i}^h := \dfrac{1}{h}\mathbb{E}^{i,h}(\tilde{Y}_{t_{i+1}}^h(B_{t_{i+1}} - B_{t_i})), \\ \tilde{U}_{t_i}^h := \nabla\varphi_{h^a}(\mathbb{E}^{i,h}(\tilde{Y}_{t_{i+1}}^h)), \end{cases} \tag{79}$$

where $\mathbb{E}^{i,h}(\cdot) := \mathbb{E}\left(\cdot\,|\mathcal{F}_{t_i}^h\right)$ and $\mathcal{F}_{t_i}^h := \sigma(X_{t_j}^h : 0 \le j \le i)$.

Remark 48 *Observe that $\tilde{Y}_{t_i}^h$ is defined implicitly as the solution of a fixed point problem. Since the involved functions are Lipschitz, it is well defined. Moreover, for small values of $h > 0$ it can be estimated numerically in an accurate way.*

Remark 49 *We can also use an explicit scheme to define*

$$\tilde{Y}_{t_i}^h := \mathbb{E}^{i,h}(\tilde{Y}_{t_{i+1}}^h) + h\mathbb{E}^{i,h}\left[F(t_i, X_{t_i}^h, \tilde{Y}_{t_{i+1}}^h, \tilde{Z}_{t_i}^h) - \nabla\varphi_{h^a}(\tilde{Y}_{t_{i+1}}^h)\right].$$

The advantage of this scheme is that it does not require a fixed point procedure but, from a numerical point of view, adding a term in the conditional expectation makes it more difficult to estimate. Therefore the implicit scheme can be more tractable in practice.

Consider now a continuous version of the approximating scheme (79). From the martingale representation theorem there exists a square integrable process $\tilde{Z}^h$ such that

$$\tilde{Y}_{t_{i+1}}^h = \mathbb{E}^i(\tilde{Y}_{t_{i+1}}^h) + \int_{t_i}^{t_{i+1}} \tilde{Z}_s^h dB_s, \tag{80}$$

and, therefore, we define, for $t \in (t_i, t_{i+1}]$,

$$\bar{Y}_t^h := \tilde{Y}_{t_i}^h - (t - t_i)\left[F(t_i, X_{t_i}^h, \tilde{Y}_{t_i}^h, \tilde{Z}_{t_i}^h) - \nabla\varphi_{h^a}(\tilde{Y}_{t_i}^h)\right] + \int_{t_i}^t \tilde{Z}_s^h dB_s. \tag{81}$$

To approximate Z_t^h we use

$$\bar{Z}_t^h := \frac{1}{h}\mathbb{E}^i\left[\int_{t_i}^{t_{i+1}} Z_s^h ds\right], \ t \in [t_i, t_{i+1})$$

rather than $Z_{t_i}^h$, which is the best approximation of Z^h by adapted processes which are constant on each interval $[t_i, t_{i+1})$. In order to prove an error estimate of the scheme first we use the solution $\left(Y_t^h, Z_t^h\right)_{t\in[0,T]}$ of the approximating equation (78).

Proposition 50 *Under the assumptions (75)-(77), there exists $C > 0$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}|Y_t - Y_t^h|^2 + \mathbb{E}\int_0^T |Z_t - Z_t^h|^2 dt \le C\Gamma(T)\, h^a, \tag{82}$$

where $\Gamma(T) := \mathbb{E}\left[1 + |g(X_T)|^2 + |X_T|^r + \int_0^T F\left(0, X_s^h, 0, 0\right) ds\right]$.

We have the following result.

Theorem 51 *Under suitable assumptions, there exists the constant $C > 0$ which depends only on the Lipschitz constants of the coefficients, such that:*

$$\sup_{t\in[0,T]} \mathbb{E}|Y_t - \tilde{Y}_t^h|^2 + \mathbb{E}\int_0^T \left[|Y_t - \tilde{Y}_t^h|^2 + |Z_t - \tilde{Z}_t^h|^2\right] dt \le Ch^{a\wedge(1-2a)}. \tag{83}$$

3.3 Anticipated BSVIs with generalized reflection

3.3.1 Setting the problem

Let $T>0$ be arbitrary but fixed, $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$ be a complete stochastic basis and $\{B_t : t\geq 0\}$ an $\mathbb{R}^k$–valued Brownian motion. Set $\mathcal{F}_t := \sigma(\{B_s : 0\leq s\leq t\})\vee\mathcal{N}_{\mathbb{P}}$ Our aim is to study the existence and uniqueness of the backward stochastic variational inequality with oblique reflection and with generators depending not only the present values of the solutions but also the future:

$$
\left\{
\begin{array}{l}
-dY_t + H(t)\,\partial\varphi(Y_t)\,dt \ni f\left(t,Y_t,Z_t,Y_{t+\delta(t)},Z_{t+\eta(t)}\right)dt - Z_t dB_t \quad t\in[0,T]\,,, \\
Y_t=\xi_t\,,\quad Z_t=\zeta_t\,,\quad t\in[T,T+\ell]\,,\quad \mathbb{P}\text{–a.s.}
\end{array}
\right.
\tag{84}
$$

Definition 52 *We say that a triplet (Y,Z,U) is a strong solution of the anticipated oblique BSVI (84) if*

$$
(Y,Z,U)\in S_d^2[0,T+\ell]\times\Lambda_{d\times k}^2(0,T+\ell)\times\Lambda_d^2[0,T+\ell]\,,
$$

such that, $\mathbb{P}$–a.s.,

$$
\left\{
\begin{array}{l}
Y_t+\displaystyle\int_t^T H(s)\,U_s ds=\xi_T+\int_t^T f\left(s,Y_s,Z_s,Y_{s+\delta(s)},Z_{s+\eta(s)}\right)ds-\int_t^T Z_s dB_s,\quad t\in[0,T]\,,\\
U_t\in\partial\varphi(Y_t)\,,\quad t\in[0,T]\,,\\
Y_t=\xi_t\,,\quad Z_t=\zeta_t\,,\quad t\in[T,T+\ell]\,.
\end{array}
\right.
$$

Assume that:

(H_1) The functions $\delta,\eta:[0,T]\to\mathbb{R}_+$ are continuous and there exists a constant $\ell\geq 0$ such that, for all $t\in[0,T]$, we have

$$
t+\delta(t)\leq T+\ell\,,\quad t+\eta(t)\leq T+\ell
$$

and there exists $L_1\geq 0$ such that for all $t\in[0,T]$ and for all nonnegative and integrable function g, we have

$$
\int_t^T g(s+\delta(s))\,ds\leq L_1\int_t^{T+\ell}g(s)\,ds\,,\quad \int_t^T g(s+\eta(s))\,ds\leq L_1\int_t^{T+\ell}g(s)\,ds\,.
$$

(H_2) $(\xi,\zeta)\in S_d^2[T,T+\ell]\times\Lambda_{d\times k}^2(T,T+\ell)\,.$

(H_3) $H(\cdot,\cdot):\Omega\times[0,T]\to\mathbb{R}^{d\times d}$ is $\mathcal{F}_t$–progressively measurable. There exists $L_H,a_H,b_H>0$ such that $\mathbb{P}$–a.s.

$$
H=(h_{i,j})_{d\times d}\in C^1\big([0,T];\mathbb{R}^{d\times d}\big)
$$

and for all $t\in[0,T]$, $\mathbb{P}$–a.s.,

$$
\left\{
\begin{array}{ll}
(i) & h_{i,j}(t)=h_{j,i}(t)\,,\quad \text{for any } i,j\in\overline{1,d},\\
(ii) & a_H|u|^2\leq\langle H(t)\,u,u\rangle\leq b_H|u|^2\,,\quad \text{for any } u\in\mathbb{R}^d,
\end{array}
\right.
\tag{85}
$$

(H_4) The function $\varphi:\mathbb{R}^d\to(-\infty,+\infty]$ is proper, lower semicontinuous and convex such that we have $\mathbb{E}\sup_{t\in[T,T+\ell]}|\varphi(\xi_t)|<+\infty$.

(H_5) For all $s \in [0,T]$, the function $f(\cdot, s, \cdot, \cdot, \cdot, \cdot)$ is defined on $\Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \times L^2(\Omega, \mathcal{F}_r, \mathbb{P}; \mathbb{R}^d) \times L^2(\Omega, \mathcal{F}_{r'}, \mathbb{P}; \mathbb{R}^{d\times k})$ with values in $L^2(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^d)$, where $r, r' \in [s, T+\ell]$.

Moreover, assume there exist $\rho \in L^2(0,T;\mathbb{R}_+)$, $L_2, L_3, L_4, L_5 \geq 0$ such that, for all $s \in [0,T]$, $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times k}$, $u, u' \in S_d^2[s, T+\ell]$, $v, v' \in \Lambda^2_{d\times k}(s, T+\ell)$, $r, \bar{r} \in [s, T+\ell]$, we have $d\mathbb{P} \otimes ds$–a.e.

$$
(86)\qquad \begin{cases}
(i) & |f(s,y,z,u_r,v_{\bar r}) - f(s,y',z,u_r,v_{\bar r})| \leq L_2 |y-y'|, \\
(ii) & |f(s,y,z,u_r,v_{\bar r}) - f(s,y,z',u_r,v_{\bar r})| \leq L_3 |z-z'|, \\
(iii) & |f(s,y,z,u_r,v_{\bar r}) - f(s,y,z,u'_r,v_{\bar r})| \leq L_4\, \mathbb{E}^{\mathcal{F}_s}(|u_r - u'_r|), \\
(iv) & |f(s,y,z,u_r,v_{\bar r}) - f(s,y,z,u_r,v'_{\bar r})| \leq L_5\, \mathbb{E}^{\mathcal{F}_s}(|v_{\bar r} - v'_{\bar r}|), \\
(v) & |f(s,0,0,0,0)| \leq \rho(s).
\end{cases}
$$

The main result of this section is given within the next theorem.

Theorem 53 *Let the assumptions* $(\mathrm{H}_1 - \mathrm{H}_5)$ *be satisfied and* $L_3^2 + L_1 L_5^2 < a_H$ *. Then there exists a unique solution* (Y, Z, U) *of (84), in the sense of Definition 52, such that,* $\mathbb{P}$*–a.s., for all* $t \in [0,T]$*,*

$$
(87)\qquad \mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} e^{\beta s}|Y_s|^2 + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta r}|U_r|^2 dr + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta r}|Z_r|^2 dr \leq C\cdot\Theta_1,
$$

where

$$
(88)\qquad \Theta_1 := \mathbb{E}^{\mathcal{F}_t}\Big(e^{\beta T}|\xi_T|^2 + e^{\beta T}\varphi(\xi_T) + \int_t^T e^{\beta r}|\rho(r)|^2 dr + \int_T^{T+\ell} e^{\beta r}|\zeta_r|^2 dr + \int_T^{T+\ell} e^{\beta r}|\xi_r|^2 dr\Big)
$$

The proof will be the subject of the next sections. The existence will obtained using the classical Moreau–Yosida approximation of the function φ by convex C^1 functions φ_ϵ, $\epsilon > 0$.

3.3.2 A priori estimates on the penalized equation

We consider the approximating anticipated BSDE

$$
(89)\qquad \begin{cases}
Y_t^\epsilon + \displaystyle\int_t^T H(s)\nabla\varphi_\epsilon(Y_s^\epsilon)\,ds = \xi_T + \int_t^T f\big(s, Y_s^\epsilon, Z_s^\epsilon, Y^\epsilon_{s+\delta(s)}, Z^\epsilon_{s+\eta(s)}\big)ds - \int_t^T Z_s^\epsilon dB_s, \\
\qquad\qquad t\in[0,T]. \\
Y_t^\epsilon = \xi_t, \quad Z_t^\epsilon = \zeta_t, \quad t \in [T, T+\ell].
\end{cases}
$$

If we denote $\tilde f(t,y,z,u,v) := f(t,y,z,u,v) - H(t)\nabla\varphi_\epsilon(y)$, then it is immediately that $\tilde f$ satisfies the assumption (H_5) with the same L_3, L_4, ρ and L_2 replaced by the constant $(L_2 + b_H/\epsilon)$. According to Peng, Yang [139, Theorem 4.2] the anticipated BSDE (89) has a unique solution $(Y^\epsilon, Z^\epsilon) \in S_d^2[0, T+\ell] \times \Lambda^2_{d\times k}(0, T+\ell)$.

Proposition 54 *Let the assumptions* $(\mathrm{H}_1 - \mathrm{H}_5)$ *be satisfied and* $L_3^2 + L_1 L_5^2 < a_H$ *. Then there exists a positive constant* C *independent of* ϵ *such that,* $\mathbb{P}$*–a.s., for all* $t \in [0,T]$ *and* $\epsilon > 0$*,*

$$
(90)\qquad \begin{cases}
(i) & \mathbb{E}^{\mathcal{F}_t}\big(\sup_{s\in[t,T]} e^{\beta s}|Y_s^\epsilon|^2\big) + \mathbb{E}^{\mathcal{F}_t}\displaystyle\int_t^T e^{\beta r}|Z_r^\epsilon|^2 dr \leq C\cdot\Theta_1, \\
(ii) & e^{\beta t}\varphi(J_\epsilon(Y_t^\epsilon)) + \mathbb{E}^{\mathcal{F}_t}\displaystyle\int_t^T e^{\beta r}\varphi(J_\epsilon(Y_r^\epsilon))\,dr \leq C\cdot\Theta_1, \\
(iii) & \mathbb{E}^{\mathcal{F}_t}\displaystyle\int_t^T e^{\beta r}|\nabla\varphi_\epsilon(Y_r^\epsilon)|^2 dr \leq C\cdot\Theta_1
\end{cases}
$$

and

$$
\text{(91)} \qquad \begin{cases} (iv) & e^{\beta t}|Y_t^\epsilon - J_\epsilon\left(Y_t^\epsilon\right)|^2 \leq C\epsilon \cdot \Theta_1\,, \\ (v) & \mathbb{E}^{\mathcal{F}_t} \displaystyle\int_t^T e^{\beta r}|Y_r^\epsilon - J_\epsilon\left(Y_r^\epsilon\right)|^2 dr \leq C\epsilon^2 \cdot \Theta_1\,, \end{cases}
$$

where Θ_1 is given by (88).

In what follows we prove that the sequence (Y^ϵ, Z^ϵ) is a Cauchy one.

Proposition 55 *Let the assumptions $(\mathrm{H}_1 - \mathrm{H}_5)$ be satisfied and $L_3^2 + L_1L_5^2 < a_H$. Then there exists a positive constant C independent of ϵ such that, $\mathbb{P}$–a.s., for all $t \in [0,T]$ and $\epsilon, \epsilon' > 0$,*

$$
\text{(92)} \qquad \mathbb{E}^{\mathcal{F}_t}\big(\sup_{s\in[t,T]} e^{\beta s}|Y_s^\epsilon - Y_s^{\epsilon'}|^2\big) + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta r}|Z_r^\epsilon - Z_r^{\epsilon'}|^2 dr \leq C\left(\epsilon + \epsilon'\right)\cdot \Theta_1\,,
$$

where the quantity Θ_1 is given by (88).

3.3.3 Proof of the existence and uniqueness of the solution

We start with the proof of the main result.
Proof of Theorem 53. *Existence.* From Proposition 55 we obtain that there exist $(Y,Z) \in S_d^2\left[0,T\right] \times \Lambda_{d\times k}^2\left(0,T\right)$ such that

$$
\lim_{\epsilon\to 0} Y^\epsilon = Y, \quad \text{in } S_d^2\left[0,T\right] \quad \text{and} \quad \lim_{\epsilon\to 0} Z^\epsilon = Z, \quad \text{in } \Lambda_{d\times k}^2\left(0,T\right).
$$

We obtain that there exists $U \in \Lambda_d^2\left(0,T\right)$ such that $\nabla\varphi_{\epsilon_n}\left(Y^{\epsilon_n}\right) \rightharpoonup U$, weakly in $\Lambda_d^2\left(0,T\right)$, as $\epsilon_n \to 0$. The application $\Gamma : \Lambda_d^2\left(0,T+\ell\right) \to \mathbb{R}^d$ defined by $\Gamma(X) := \mathbb{E}\int_0^T H\left(r\right)X_r dr$ is weak continuous and, as consequence,

$$
\lim_{n\to\infty} \mathbb{E}\int_0^T H\left(r\right)\nabla\varphi_{\epsilon_n}\left(Y_r^{\epsilon_n}\right)dr = \mathbb{E}\int_0^T H\left(r\right)U_r dr\,.
$$

We can pass to the limit in the approximating equation (89) and we obtain that

$$
\begin{cases} Y_t + \displaystyle\int_t^T H\left(s\right)U_s ds = \xi_T + \int_t^T f\big(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\eta(s)}\big)ds - \int_t^T Z_s dB_s,\ t \in [0,T]\,, \\ Y_t = \xi_t\,, \quad Z_t = \zeta_t\,, \quad t \in [T, T+\ell]\,. \end{cases}
$$

■

The next result shows the effect of the anticipated time on the first component of the solution Y.

Proposition 56 *We suppose that the functions $\delta, \delta' : [0,T] \to \mathbb{R}_+$ satisfy (H_1). Let the assumptions $(\mathrm{H}_2 - \mathrm{H}_5)$ be satisfied and $L_3^2 < a_H$. In addition, we assume that assumptions (86–iii) is replaced by*

$$
(jjj) \quad |f(s,y,z,u_r,v_{\bar r}) - f(s,y,z,u_r',v_{\bar r})| \leq L_4\left|\mathbb{E}^{\mathcal{F}_s}\left(u_r - u_r'\right)\right|.
$$

Let (Y,Z,U) and (Y',Z',U') be the solutions of equation (84) corresponding to δ and respectively δ' and with f independent of $Z_{t+\eta(t)}$.

If $\delta(t) \le \delta'(t)$, for any $t \in [0,T]$, then there exists $C > 0$ such that

$$|Y_t - Y_t'|^2 \le C \int_t^T (\delta'(s) - \delta(s))\, ds \cdot \Theta_2\,,$$

where $\Theta_2 := \mathbb{E}^{\mathcal{F}_t}\Big(e^{\beta T}|\xi_T|^2 + e^{\beta T}\varphi(\xi_T) + \int_t^T e^{\beta r}|\rho(r)|^2 dr + \int_T^{T+\ell} e^{\beta r}|\xi_r|^2 dr\Big)$.

The classical comparison theorem has the following specific form for the anticipated oblique BSVI (84). The proof of the next result follows closely Peng, Yang [139, Theorem 5.1] and the arguments used along Proposition 54 and Theorem 53.

Proposition 57 *We suppose that $\xi^1, \xi^2 \in S_1^2[T, T+\ell]$ and the functions f^1, f^2 satisfy (H_5) and are independent of $Z_{t+\eta(t)}$. Let the assumptions (H_1) and $(\mathrm{H}_3 - \mathrm{H}_4)$ be satisfied such that $L_3^2 < a_H$. In addition, we assume that f^2 is increasing with respect to the last variable.*

Let (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) be the solutions of equation (84) corresponding to (ξ^1, f^1) and respectively (ξ^2, f^2).

If $\xi^1(t) \ge \xi^2(t)$ for any $t \in [T, T+\ell]$ and $f^1(s,y,z,u_r) \ge f^2(s,y,z,u_r)$, for all $s \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^k$, $u \in S_1^2[s, T+\ell]$, $r \in [s, T+\ell]$, then we have $Y_t^1 \ge Y_t^2$, a.e., $\mathbb{P}$–a.s.

3.4 BSVI with nonconvex, switch-dependent reflection model

3.4.1 Motivation. Infection time in multi-stable gene networks

It is well-known that, in prokaryotes, genes are switched between different states (e.g. on/off) by interactions between specific proteins which intervene at the level of regulation and specific DNA sequences. To better understand the mathematical model we are going to present hereafter, let us concentrate on a basic network presenting bistability of protein concentration and derived from bacteriophage lambda. Our simplifying mathematical approach considers a two-scale model (see, for instance Crudu, Debussche, Muller, Radulescu [55]). The state space of this component is obviously discrete consisting of standard vectors basis of $\mathbb{R}^4$ $(E = \{e_1, e_2, e_3, e_4\})$. Switching between these states is given at random times generated according to the propensity function computed starting from the current state (e.g. Gillespie [75]). While in lysogenic state, repressor and dimer concentrations are given by an ordinary differential equation (ODE). In lytic state, transcription of the repressor is turned off. In an attempt to distinguish between symbiotic (lysogenic) behavior and the excision of the virus, it is, therefore, natural to set the lysogenic domain $\mathcal{O}$ to be the exterior of some regions "around $(0,0)$" (or other stable points). Assume, for the time being that, at some time $T > 0$, the phage lambda has been functioning on a lysogenic pathway starting at some time t_0. Then, the trajectory has been reflected such that to remain in the lysogenic domains $\mathcal{O}_i$. While many type of reflection can be considered, we will assume here that the virus is driven by the best reachable stable state. Reverse-engeneering from time T in order to detect the time of infection leads to the attempt of solving a backward differential equation adapted to the underlying DNA (Markov) mechanism and reflected in the nonconvex domains $\mathcal{O}_i$. We emphasize that, in our framework, the domains are allowed to vary in time (at time t they depend on the mode Γ_{t-}).

3.4.2 Preliminaries. Setting the problem

We briefly recall the construction of a particular class of Markov pure jump, non explosive processes on a space Ω and taking their values in a metric space $(E, \mathcal{B}(E))$. For the explicit construction of Ω (using the Hilbert cube), the reader is referred to Davis [59, Section 23]. Here, $\mathcal{B}(E)$ denotes the Borel σ-field of E. The elements of the space E are referred to as modes. In all generality, $E \subset \mathbb{R}^{m'}$, for some $m' \geq 1$. The process is completely described by a couple (λ, Q) constituting on:

(i) a Lipschitz continuous jump rate $\lambda : E \longrightarrow \mathbb{R}_+$ such that $\sup_{\theta \in E} |\lambda(\theta)| \leq c_0$ and

(ii) a transition measure $Q : E \longrightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ stands for the set of probability measures on $(E, \mathcal{B}(E))$ such that:

$$\begin{cases} (ii_1) & Q(\gamma, \{\gamma\}) = 0; \\ (ii_2) & \text{for each bounded, uniformly continuous } h, \text{ there exists a continuous } \eta_h : \mathbb{R} \to \mathbb{R}_+, \\ & \text{such that } \eta_h(0) = 0 \text{ and } \left| \int_E h(\theta) Q(\gamma, d\theta) - \int_E h(\theta) Q(\gamma', d\theta) \right| \leq \eta_h(|\gamma - \gamma'|). \\ & \text{(The distance } |\gamma - \gamma'| \text{ is the usual Euclidian one on } \mathbb{R}^m.\text{)} \end{cases}$$

Given an initial mode $\gamma_0 \in E$, the first jump time has a conditional law $\mathbb{P}^{\gamma_0}(T_1 \geq t) = \exp(-t\lambda(\gamma_0))$. The process $\Gamma_t := \gamma_0$, on $t < T_1$. The post-jump location γ_1 has $Q(\gamma_0, \cdot)$ as conditional distribution. Next, we select the inter-jump time increment $T_2 - T_1$ to be such that $\mathbb{P}^{\gamma_0}(T_2 - T_1 \geq t \,/ T_1, \gamma_1) = \exp(-t\lambda(\gamma_1))$ and let us set $\Gamma_t := \gamma_1$, if $t \in [T_1, T_2)$. The post-jump location γ_2 satisfies the relation $\mathbb{P}^{\gamma_0}(\gamma_2 \in A \,/\, T_2, T_1, \gamma_1) = Q(\gamma_1, A)$, for all Borel set $A \subset E$. And so on. Similar construction can be given for a non-zero initial starting time (i.e. a pair (t, γ_0)).

We now look at the stochastic process Γ under $\mathbb{P}^{\gamma_0}$ and denote by $\mathbb{F}^0$ its natural filtration $\left(\mathcal{F}_{[0,t]} := \sigma\left\{\Gamma_r : r \in [0,t]\right\}\right)_{t\geq 0}$. The predictable σ-algebra will be denoted by $\mathcal{P}^0$ and the progressive σ-algebra by $Prog^0$. As usual, we introduce the random measure q on $\Omega \times [0,+\infty] \times E$ by setting

$$q(\omega, A) = \sum_{k\geq 1} 1_{\left(T_k(\omega), \Gamma_{T_k(\omega)}(\omega)\right)\in A}, \text{ for all } \omega \in \Omega,\ A \in \mathcal{B}([0,+\infty]) \times \mathcal{B}(E).$$

The compensator of q is $\widehat{q}(ds,)\, d\theta := \lambda(\Gamma_{s-})\, Q(\Gamma_{s-}, d\theta)\, ds$ and the compensated martingale measure is given by

$$\widetilde{q}(dsd\theta) := q(dsd\theta) - \lambda(\Gamma_{s-})\, Q(\Gamma_{s-}, d\theta)\, ds.$$

Following the general theory of integration with respect to random measures (see, for example Ikeda, Watanabe [94]), we denote by $\mathcal{L}^r\left(q; \mathbb{R}^N\right)$ the space of all $\mathcal{P}^0 \otimes \mathcal{B}(E)$ - measurable, $\mathbb{R}^N-$valued functions $H_s(\omega,\theta)$ on $\Omega \times \mathbb{R}_+ \times E$ such that, for all $T < +\infty$,

$$\mathbb{E}^{\gamma_0}\left[\int_0^T \int_E |H_s(\theta)|^r\, q(dsd\theta)\right] = \mathbb{E}^{\gamma_0}\left[\int_0^T \int_E |H_s(\theta)|^r\, \lambda(\Gamma_{s-})\, Q(\Gamma_{s-}, d\theta)\, ds\right] < +\infty.$$

Here, $N \in \mathbb{N}^*$ and $r \geq 1$ is a real parameter. By abuse of notation, whenever no confusion is at risk, the family of processes satisfying the above condition for a fixed $T > 0$ will still be denoted by $\mathcal{L}^r\left(q; \mathbb{R}^M\right)$.To keep arguments simple, we will be dealing with a finite set of modes $E = \{1, 2, ..., p\}$, for some $p \geq 2$. Moreover, at some point we will assume that the observations on the DNA are only made up to time T_M for some $M \in \mathbb{N}^*$. Then, we will need to modify q to take into account this condition as well as a terminal time $T > 0$.

Let us introduce the rest of the elements which will construct the backward multivalued inclusion. Consider a function $\varphi_{\mathcal{O}} : \mathbb{R}^m \to (-\infty, +\infty]$ such that $\mathrm{Dom}(\varphi_{\mathcal{O}}) := \{y \in \mathbb{R}^m : \varphi(y) < +\infty\}$. For the function $\varphi_{\mathcal{O}}$ assume $\mathrm{Dom}(\varphi_{\mathcal{O}}) = \mathcal{O}$ and, as before, $\mathrm{Dom}(\partial^- \varphi_{\mathcal{O}}) := \{x \in \mathbb{R}^m : \partial^- \varphi_{\mathcal{O}}(x) \neq \emptyset\}$. The reader is invited to note that the domains appearing in our example (cf. Fig. 1) are not convex. Given two non-negative real constants $\rho, \beta \geq 0$, we consider a family of mode-indexed, (ρ,β)-semiconvex functions $\varphi_{\mathcal{O}_\gamma} : \mathbb{R}^m \to (-\infty, +\infty]$ and assume

$$(\mathrm{A}_O) \qquad \overline{\mathrm{Dom}\left(\varphi_{\mathcal{O}_\gamma}\right)} = \mathcal{O}_\gamma \text{ is bounded,}$$

for all $\gamma \in E$. The oblique direction will be given by a continuous symmetric matrix-valued function $H : \mathbb{R}_+ \times \mathbb{R}^m \longrightarrow \mathcal{S}_m^+$ satisfying

$$(\mathrm{A}_H) \qquad \begin{cases} i) & |H(t,y) - H(t,y')| + |\left(H(t,y)\right)^{-1} - \left(H(t,y')\right)^{-1}| \leq c_H |y - y'|, \\ ii) & \dfrac{1}{c_H}|u|^2 \leq \langle H(t,y)\, u, u\rangle \leq c_H |u|^2, \text{ for all } u \in \mathbb{R}^m, \end{cases}$$

for some $c_H > 0$ and all $t \in \mathbb{R}_+$, $(y, y') \in \mathbb{R}^{2m}$. Here, $\mathcal{S}_m^+$ stands for the family of symmetric, positive-definite real-valued matrix of $m \times m$ type.

We consider that the driver function $f : \mathbb{R}_+ \times E \times E \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is globally continuous, bounded and there exists some constant $c_f > 0$ such that

$$(\mathrm{A}_F) \qquad |f(t,\gamma,\gamma',y,z) - f(t,\gamma,\gamma',y',z')| \leq c_f\left(|y - y'| + |z - z'|\right),$$

for all $(t, \gamma, \gamma', y, , y', z, z') \in \mathbb{R}_+ \times E \times E \times \mathbb{R}^{4m}$.

In connection to our model, for some fixed terminal time $T > 0$, we consider the following backward stochastic variational inclusion with mode-dependent reflection :

$$\left\{\begin{array}{l} -dY_t^{T,\xi} + H(t, Y_t^{T,\xi})\partial^- \varphi_{\mathcal{O}_{\Gamma_{t-}}}(Y_t^{T,\xi})dt \ni \displaystyle\int_E f(t,\gamma,\Gamma_{t-}, Y_{t-}^{T,\xi}, Z_t^{T,\xi}(\gamma))\widehat{q}(dt,d\gamma) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \displaystyle\int_E Z_t^{T,\xi}(\gamma)\, q\,(dtd\gamma)\,, \\ Y_T^{T,\xi} = \xi \in \mathbb{L}^0\left(\Omega, \mathcal{F}_T, \mathbb{P}^{\gamma_0}; \mathbb{R}^m\right), \end{array}\right. \tag{93}$$

$\mathbb{P}^{\gamma_0}-$almost everywhere. We consider an additional cemetery state $\Delta \in \mathbb{R}^m$ acting as an indicator of the infection time. As we will see afterwards, this equation can be linked to a system of reflected ordinary differential equations. With this in mind, the coherence of our solution will have to be ensured at jumping times. In other words, one would need the solution $Y_t^{T,\xi}$ to belong to $\mathcal{O}_{\Gamma_{t-}}$ and will check this condition at switching times. Should this condition fail to hold, the trajectory will be sent to Δ (lysogenic pathway is not coherent with the model prior to this time) and remains at Δ for any time before.

The definition of a solution is given, as usual, by a triplet $\left(Y^{T,\xi}, Z^{T,\xi}, K^{T,\xi}\right)$ in which the latter components take into account the adaptedness, respectively a feedback correction for $Y^{T,\xi}$.

Definition 58 *A solution of (93) consists of a triplet* $(Y_t^{T,\xi}, Z_t^{T,\xi}, K_t^{T,\xi})$ *such that:*

$$\left\{\begin{array}{ll} (i) & \text{1. The process } Y_\cdot^{T,\xi} \text{ is càdlàg and continuous except, maybe, at switching times.} \\ & \text{2. For } \mathbb{P}^{\gamma_0}\times Leb\text{-almost all } (\omega,t) \text{ such that } T_n(\omega) \le t < T_{n+1}(\omega)\,,\ Y_t^{T,\xi} \in \mathcal{O}_{\Gamma_{T_n}}. \\ & \text{3. If } Y_{T_n}^{T,\xi}(\omega) \notin \mathcal{O}_{\Gamma_{T_{n-1}}}, \text{ then } Y_s^{T,\xi}(\omega) = \Delta, \text{ for almost all } s < T_n(\omega)\,. \\ (ii) & \text{1. The process } Z_\cdot^{T,\xi}(\cdot) \text{ is } \mathbb{R}^m\text{-valued, } \mathbb{F}-\text{predictable and} \\ & \text{2. } \mathbb{E}^{\gamma_0}\left[\displaystyle\int_0^T\int_E |Z_t^{T,\xi}(\gamma)|\widehat{q}(dt,d\gamma)\right] < +\infty. \\ (iii) & \text{1. The process } K_\cdot^{T,\xi} \text{ is } \mathbb{F}-\text{adapted and } \displaystyle\int_0^T |K_t^{T,\xi}|^2 dt < +\infty, \ \mathbb{P}^{\gamma_0}\text{-almost everywhere.} \\ & \text{2. For } \mathbb{P}^{\gamma_0}\times Leb\text{-almost all } (\omega,t) \text{ such that } T_n(\omega)\le t< T_{n+1}(\omega)\,, \text{ one has} \\ & \quad (K_t^{T,\xi}(\omega)\,, Y_t^{T,\xi}(\omega)) \in (\partial^-\varphi_{\mathcal{O}_{\Gamma_{T_n(\omega)}(\omega)}}(Y_t^{T,\xi}(\omega)) \times \mathbb{R}^m) \cup \{(0,\Delta)\}\,. \\ (iv) & \text{One has, } \mathbb{P}^{\gamma_0}\times Leb\text{-almost everywhere,} \\ & \quad Y_t^{T,\xi} + \displaystyle\int_t^T H(Y_s^{T,\xi})K_s^{T,\xi}ds + \sum\nolimits_{n\ge 0,\, t<T_n\le T} Z_{T_n}^{T,\xi}(\Gamma_{T_n}) \\ & \qquad = \xi + \displaystyle\int_t^T\int_E f(s,\gamma,Y_s^{T,\xi}, Z_s^{T,\xi})\lambda(\Gamma_s)\,Q(\Gamma_s, d\gamma)\,ds. \end{array}\right.$$

Moreover, unless stated otherwise, we will assume that the mode process jumps at most $M > 0$ times prior to $T > 0$, i.e.

$$\mathbb{P}^{\gamma_0}\left(T_{M+1} = +\infty\right) = 1. \tag{A_M}$$

3.4.3 Measurability issues, driver and compensator

Before giving the reduction of our equation to a system of ODE, we need to introduce some notations making clear the stochastic structure of several concepts : final data, predictable and càdlàg adapted processes as well as the driver and the compensator of the initial random measure. The notations in this subsection follow the ordinary differential approach from Confortola, Fuhrman, Jacod [49, Proof of Theorem 3]. Since we are only interested in what happens on $[0,T]$, we introduce a cemetery state $(\infty,\overline{\gamma})$ which will incorporate all the information after $T\wedge T_M$. It is clear that the conditional law of T_{n+1} given (T_n,Γ_{T_n}) is now composed by an exponential part on $[T_n\wedge T,T]$ and an atom at ∞. Similarly, the conditional law of $\Gamma_{T_{n+1}}$ given $(T_{n+1},T_n,\Gamma_{T_n})$ is the Dirac mass at $\overline{\gamma}$ if $T_{n+1}=\infty$ and given by Q otherwise. Finally, under the assumption A_M, after T_M, the marked point process is concentrated at the cemetery state.

We set $\overline{E}_T:=([0,T]\times E)\cup\{(\infty,\overline{\gamma})\}$. For every $n\geq 1$, we let $\overline{E}_{T,n}\subset\left(\overline{E}_T\right)^{n+1}$ be the set of all marks of type $e=((t_0,\gamma_0),...,(t_n,\gamma_n))$ where

$$\begin{cases} t_0=0,\ \ (t_i)_{0\leq i\leq n} \text{ is non-decreasing;}\\ \text{for every } 0\leq i\leq n-1,\ \text{if } t_i\leq T,\ \text{then } t_i<t_{i+1};\\ \text{for every } 0\leq i\leq n-1,\ \text{if } t_i>T,\ \text{then } (t_i,\gamma_i)=(\infty,\overline{\gamma}), \end{cases} \tag{94}$$

and endow it with the family of all Borel sets $\mathcal{B}_n$. For these sequences, the maximal time is denoted by $|e|:=t_n$. Moreover, by abuse of notation, we set $\gamma_{|e|}:=\gamma_n$. Whenever $T\geq t>|e|$, we set

$$e\oplus(t,\gamma):=((t_0,\gamma_0),...,(t_n,\gamma_n),(t,\gamma))\in\overline{E}_{T,n+1}. \tag{95}$$

By defining

$$e_n:=((0,\gamma_0),(T_1,\Gamma_{T_1}),...,(T_n,\Gamma_{T_n})), \tag{96}$$

we get an $\overline{E}_{T,n}-$valued random variable, corresponding to our mode trajectories.

Let us now express the different notions (final condition, adapted process, predictable process, etc.) with respect to this framework.

The final data ξ is a $\mathcal{F}_T-$measurable random variable and, thus, for every $n\geq 0$, there exists a $\mathcal{B}_n/\mathcal{B}(\mathbb{R}^m)-$measurable function $\overline{E}_{T,n}\ni e\mapsto\xi^n(e)\in\mathbb{R}^m$ such that:

$$\begin{cases} \text{If } |e|=\infty,\ \text{then } \xi^n(e)=0.\\ \text{Otherwise, on } T_n(\omega)\leq T<T_{n+1}(\omega),\ \xi(\omega)=\xi^n(e_n(\omega)). \end{cases} \tag{97}$$

A càdlàg process Y **continuous except, maybe, at switching times** T_n is given by the existence of a family of $\mathcal{B}_n\otimes\mathcal{B}([0,T])/\mathcal{B}(\mathbb{R}^m)$-measurable functions y^n such that:

$$\begin{cases} \text{For all } e\in\overline{E}_{T,n},\ y^n(e,\cdot)\ \text{is continuous on } [0,T]\ \text{and constant } [0,T\wedge|e|].\\ \text{If } |e|=\infty,\ \text{then } y^n(e,\cdot)=0.\\ \text{Otherwise, on } T_n(\omega)\leq t<T_{n+1}(\omega),\ y_t(\omega)=y^n(e_n(\omega),t),\ \text{for all } t\leq T. \end{cases} \tag{98}$$

Similar, an $\mathbb{R}^m-$valued $\mathbb{F}$-**predictable process** Z defined on $\Omega\times[0,T]\times E$ is given by the existence of a family of $\mathcal{B}_n\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(E)/\mathcal{B}(\mathbb{R}^m)-$measurable functions z^n satisfying

$$\begin{cases} \text{If } |e|=\infty,\ \text{then } z^n(e,\cdot,\cdot)=0.\\ \text{Otherwise, on } T_n(\omega)<t\leq T_{n+1}(\omega),\ z_t(\omega,\gamma)=z^n(e_n(\omega),t,\gamma),\ \text{for all } t\leq T \text{ and } \gamma\in E. \end{cases} \tag{99}$$

To deduce the form of **the compensator**, one takes into account (A_M) and simply writes:

$$\begin{cases} \text{If } n\leq M-1,\\ \widehat{q}^n_e(dt,d\gamma):=\lambda(\gamma_{|e|})Q(\gamma_{|e|},d\gamma)1_{|e|<\infty,t\in[|e|,T]}Leb(dt)+\delta_{\overline{\gamma}}(d\gamma)\,\delta_\infty(dt)\,1_{(|e|<\infty,t>T)\cup|e|=\infty},\\ \text{If } n\geq M,\ \text{then } \widehat{q}^n_e(dt,d\gamma)=\delta_{\overline{\gamma}}(d\gamma)\,\delta_\infty(dt)\\ \widehat{q}(\omega,dt,d\gamma):=\sum\limits_{n=0}\widehat{q}^n_{e_n(\omega)}(dt,d\gamma)\,1_{T_n(\omega)<t\leq T_{n+1}(\omega)\wedge T}. \end{cases} \tag{100}$$

Finally, given a predictable process $z := (z^n)$, **the driver** is given by a family of $\mathcal{B}_n \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m) / \mathcal{B}(\mathbb{R}^m)$ −measurable functions

$$f(z^n)^n : \overline{E}_{T,n} \times [0,T] \times E \times \mathbb{R}^{2m} \longrightarrow \mathbb{R}^m$$

such that:

$$\text{(101)} \quad \begin{cases} \text{If } |e| < \infty \text{ and } n \leq M-1, \text{ then, for all } (e,t,\gamma,y,y',w,w') \in \overline{E}_{T,n} \times [0,T] \times E \times \mathbb{R}^{4m}, \\ |f(z^n)^n(e,t,\gamma,y,w) - f(z'^{,n})^n(e,t,\gamma,y',w')| \\ \quad \leq c\left(|y-y'| + |w-w'| + \sum_{\gamma' \in E} |z^n(e,t,\gamma') - z'^n(e,t,\gamma')| \, Q(\gamma_{|e|}, d\gamma')\right). \\ \text{Otherwise, } f(z^n)^n(e,\cdot,\cdot,\cdot,\cdot) = 0. \end{cases}$$

In this case, we identify the driver as follows.

$$\text{(102)} \quad \begin{cases} \text{Whenever } T_n < t \leq T_{n+1}, \text{ we have} \\ f(t,\gamma,\Gamma_{t-},y,\zeta) = f(z^n)^n(e_n(\omega), t, \gamma, y, \zeta(\gamma) - z^n(e_n(\omega), t, \gamma)). \end{cases}$$

3.4.4 A scheme based on reflected solutions for ordinary differential equations. The iterating differential inclusion

We consider a càdlàg process Y continuous except, maybe, at switching times T_n. Then, as explained before, this can be identified with a family (y^n). At jumping times T_{n+1}, the process Y is something like

$$Y_{T_{n+1}} = y^{n+1}\left(e_n \oplus \left(T_{n+1}, \Gamma_{T_{n+1}}\right), T_{n+1}\right). \tag{103}$$

We construct, for every $n \geq 0$,

$$\widehat{y}^{n+1}(e,t,\gamma) := y^{n+1}(e \oplus (t,\gamma), t) \, 1_{|e|<t} \tag{104}$$

and $Y_{T_{n+1}}$ can be obtained by simple integration of the previous quantity with respect to the conditional law of $\left(T_{n+1}, \Gamma_{T_{n+1}}\right)$ knowing $\mathcal{F}_{T_n}$.

We introduce the following scheme. We let ξ be a final condition. We "correct" $\xi = (\xi^n)$ given by (97) as to be in the admissible domains as follows:

$$\xi^n_{adm}(e) := \xi^n(e) \, 1_{\xi^n(e) \in \mathcal{O}_{\gamma_{|e|}}} + \Delta 1_{\xi^n(e) \in \mathcal{O}^c_{\gamma_{|e|}}}.$$

It is obvious that, should the data not be in the target domain, there is no point in solving the reflected BSDE. In this case, we simply set the solution to be a constant point Δ designed to be a flag signaling that infection cannot precede the current time. We consider the family of (ordinary) differential inclusions

$$\text{(105)} \quad \begin{cases} y^M(e_M(\omega), t) = \xi^M_{adm}(e_M(\omega)), \\ \text{For } n \leq M-1, \ \xi^{n,+}_{adm}(e_n(\omega)) := \begin{cases} \xi^n_{adm}(e_n(\omega)), & \text{if } y^{n+1}(e_{n+1}(\omega), 0) \in \mathcal{O}_{\gamma_{|e_n(\omega)|}}, \\ \Delta, & \text{otherwise}, \end{cases} \\ -dy^n(e_n(\omega), t) + H(t, y^n(e_n(\omega), t)) \, \partial^- \varphi_{\mathcal{O}_{\gamma_{|e_n(\omega)|}}}(y^n(e_n(\omega), t)) \, dt \ni \\ \quad + \sum_{\gamma \in E} f(\widehat{y}^{n+1})^n(e_n(\omega), s, \gamma, y^n(e_n(\omega), s), -y^n(e_n(\omega), s)) \, \widehat{q}^n_{e_n(\omega)}(ds, \{\gamma\}), \\ y^n(e_n(\omega), T) = \xi^{n,+}_{adm}(e_n(\omega)). \end{cases}$$

Proposition 59 *Let us assume that (A_O, A_H, A_F and A_M) hold true. Then, the càdlàg process $Y = (y^n)$ continuous, except at switching times, is a solution for (93) if and only if it satisfies the system (105).*

The elements of proof are provided in Section 3.4.6. The basic idea is to employ the structure presented in the previous subsection. Indeed, since Z only acts at jumping times, there is a simple relation linking z^n to y^n and $\widehat{y}^{n+1}$. The conclusion follows by plugging this z into the equation.

In this subsection we turn our attention to the solvability of such reflected differential inclusions. To this purpose, let us freeze the regular domain $\mathcal{O}$ of a (ρ, β)-semiconvex function $\varphi_{\mathcal{O}}$ satisfying the assumption (A_O). The inclusion has the form:

$$\text{(106)} \qquad \begin{cases} -dy\left(t\right) + H(t, y\left(t\right))\partial^{-}\varphi_{\mathcal{O}}\left(y\left(t\right)\right)dt \ni \int_E \overline{f}\left(t, \gamma', y\left(t\right)\right)\nu\left(dt, d\gamma'\right), \\ y\left(T\right) = \eta, \end{cases}$$

where ν stands for the compensator $\widehat{p}$. Throughout the remaining of the section and unless stated otherwise, the function $\overline{f} : \mathbb{R}_+ \times E \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is assumed to be globally continuous, bounded and Lipschitz continuous in space, uniformly with respect to the time and $\gamma \in E$.

Definition 60 *A solution of (106) consists of a couple (y, k) satisfying the following:*

$$\begin{cases} (i) & \text{1. The function } y \in C\left([0,T];\mathbb{R}^m\right) \text{ is continuous and for Leb -almost all } t,\ y\left(t\right) \in \mathcal{O}. \\ & \text{2. The application } [0,T] \ni t \mapsto \varphi_{\mathcal{O}}\left(y\left(t\right)\right) \text{ is integrable w.r.t. Lebesgue measure.} \\ (ii) & \text{1. The function } k \in \mathbb{L}^2\left([0,T];\mathbb{R}^m\right) \text{ is square integrable w.r.t. Lebesgue measure.} \\ & \text{2. For Leb}-\text{almost all } t \in [0,T], \text{ one has } k\left(t\right) \in \partial^{-}\varphi_{\mathcal{O}}\left(y\left(t\right)\right). \\ (iii) & \text{The equality } y\left(t\right) + \int_t^T H\left(s, y\left(s\right)\right)k\left(s\right)ds \ = \eta + \int_t^T \int_E \overline{f}\left(s, \gamma', y\left(s\right)\right)\nu\left(ds, d\gamma'\right), \\ & \text{holds true, Leb}-\text{almost everywhere.} \end{cases}$$

Theorem 61 *We assume (A_O) and (A_H) to hold true (where $\mathcal{O}_\gamma$ is replaced with $\mathcal{O}$). Then, for every $\eta \in \mathcal{O}$, there exists a unique couple of deterministic functions $(y, k) \in C([0,T];\mathbb{R}^m) \times \mathbb{L}^2\left([0,T];\mathbb{R}^m\right)$ which satisfies (106), in the sense of Definition 60.*

3.4.5 Targeted design. Occupation measures

We consider our differential mechanism to be governed by an exogenous control parameter that can be associated to temperature and/or catalysts conditions. The construction of the differential component is given by a driver depending on a predictable control process u. We let U be a compact metric space and assume that the driver function $f : \mathbb{R}_+ \times E \times E \times \mathbb{R}^m \times \mathbb{R}^m \times U \longrightarrow \mathbb{R}^m$ is globally continuous, bounded and there exists $c > 0$ such that

$$(\mathrm{A}'_F) \qquad |f(t, \gamma, \gamma', y, z, u) - f(t, \gamma, \gamma', y', z', u)| \leq c(|y - y'| + |z - z'|),$$

for all $(t, \gamma, \gamma', y, y', z, z', u) \in \mathbb{R}_+ \times E \times E \times \mathbb{R}^{4m} \times U$.

Occupation measures

One replaces φ with inf-convolutions of type $\varphi^{\varepsilon}(x) := \inf_{y \in \mathbb{R}^m}\left\{\frac{1}{2\varepsilon}|x - y|^2 + \varphi\left(y\right)\right\}$, for $x \in \mathbb{R}^m$ and $\varepsilon > 0$. To every such penalized solution corresponding to a predictable control, one can associate a measure taking into account all the components: time, accessible mode, occupation of the space (given by Y) and corrective term (Z) as well as the control. A further variable takes into account the gradient. These measures are shown (in the proof of Proposition 62) to satisfy convenient compactness criteria and exhibit a support condition related to the subgradient. To facilitate passage to the limit in penalizations, we give a (support) condition related to the distance to the lysogeny domains. The reader is invited to note that the solution $Y^{T,\xi,u}$ belongs to $\mathcal{O} := (\cup_{\gamma\in E}\mathcal{O}_\gamma)\cup\{\Delta\}$. Therefore, under the boundedness assumption on the domains, $Y_t^{T,\xi,u} \in \overline{\mathbb{B}}(0,C)\cup\{\Delta\}$, for some generic constant C. We introduce the following sets

$$
\begin{aligned}
&E^{\Delta,t,T} := [t,T]\times E^2\times\left(\left(\overline{\mathbb{B}}(0,C)\times\mathbb{R}^m\right)\cup\left(\Delta\times\{0\}\right)\right)\times\overline{\mathbb{B}}(0,2C+|\Delta|)\\
&\Theta_\varepsilon(t,T,\xi) :=\\
&\left\{
\begin{array}{l}
\mu\in\mathcal{P}(\overline{\mathbb{B}}(0,C)\cup\{\Delta\})\times\mathcal{P}(E^{\Delta,t,T}\times U),\\
\forall\phi\in C_b^{1,2}([t,T]\times\mathbb{R}^m),\\
\mathbb{E}^{\gamma_0}[\phi(T,\xi)] = \int_{\mathbb{R}^m}\phi(t,y)\,\mu^1(dy)\\
\quad+\int_{E^{\Delta,t,T}}\langle\nabla_y\phi(s,y_1),H(s,y_1)\,y_2\rangle\,\mu^2(dsd\gamma'd\gamma dy_1dy_2dz,U)\\
\quad-\int_{E^{\Delta,t,T}\times U}\langle\nabla_y\phi(s,y_1),f(s,\gamma',\gamma,y_1,z,u)\lambda(\gamma)\,Q(\gamma,\{\gamma'\})\rangle\,\mu^2(dsd\gamma'd\gamma dy_1dy_2dzdu)\\
\quad+\int_{E^{\Delta,t,T}}\left(\partial_t\phi(s,y_1)+\phi(s,y_1+z)-\phi(y_1)\right)\lambda(\gamma)\,Q(\gamma,\{\gamma'\})\,\mu^2(dsd\gamma'd\gamma dy_1dy_2dz,U),\\
\mathit{Supp}(\mu^2)\subset\left\{\begin{array}{l}(s,\gamma',\gamma,y_1,y_2,z,u):\forall a\in\mathbb{R}^m,\\ \langle a-y_1,y_2\rangle+\varphi^\varepsilon_{\mathcal{O}_\gamma}(y_1)\le\varphi^\varepsilon_{\mathcal{O}_\gamma}(a)+(\rho+\beta|y_2|)\,|a-y_1|^2\end{array}\right\}\\
\int_{E^{\Delta,t,T}}\left(d^2_{\mathcal{O}_\gamma}\wedge 1\right)(y_1)\,\mu^2(dsd\gamma'd\gamma dy_1dy_2dz,U)\le C\varepsilon\\
\int_{E^{\Delta,t,T}}|y_2|^2\,\mu^2(dsd\gamma'd\gamma dy_1dy_2dz,U)\le C.
\end{array}
\right\}
\end{aligned}
$$

The link between these sets and the actual solution of our initial problems will appear explicitly in the proofs. For now, all one needs to know is the following.

Proposition 62 *Let us fix $\varepsilon > 0$, the time horizon $T > 0$, $0 \le t \le T$ and the final data ξ. Then, the family $\Theta_\varepsilon(t,T,\xi)$ is non-empty, convex and compact (with respect to the usual topology on the space of probability measures).*

As explained before, we embed the solutions of our (approximating) BSDE into a measure. The linear restriction is a mere reformulation of Itô's formula. The support condition is linked to gradients. The distance to lysogeny domains follows from the estimates on approximating solutions as do the second order moments (guaranteeing compactness).

Second, following the approximating construction of solution to the initial problem (93) (see proof of Theorem 61), one considers the lower limit of sets

$$\Theta_0(t,T,\xi) = \liminf_{\varepsilon\to 0}\Theta_\varepsilon(t,T,\xi).$$

Admit, for the time being (the actual proof is given afterwards) that the solutions to the initial BSVI (with control) can be seen as elements of the limit set $\Theta_0(t,T,\xi)$. Then one is entitled to ask oneself if these solutions also enjoy similar properties (regularity, support and linear-type restriction). This is, indeed, the case as summarized by the following result.

Theorem 63 *(i) (**convexity and compactness**) The set $\Theta_0(t,T,\xi)$ is a non-empty, convex and compact subset of $\mathcal{P}(\overline{\mathbb{B}}(0,C)\cup\{\Delta\})\times\mathcal{P}(E^{\Delta,t,T}\times U)$ and, for every $\mu=(\mu^1,\mu^2)\in\Theta_0(t,T,\xi)$,*

$$\int_{E^{\Delta,t,T}\times U}|y_2|^2\,\mu^2(dsd\gamma'd\gamma dy_1dy_2dzdu)\le C.$$

*(ii) (**support and subdifferential**) Every measure $\mu = (\mu^1, \mu^2) \in \Theta_0(t, T, \xi)$ satisfies the support condition*

$$Supp\left(\mu^2\right) \subset \left\{(s, \gamma', \gamma, y_1, y_2, z, u) : y_1 \in \mathcal{O}_\gamma,\ y_2 \in \partial^- \varphi_{\mathcal{O}_\gamma}(y_1)\right\}.$$

*(iii) **(linear constraint)** Every limit measure $(\mu^1, \mu^2) \in \Theta_0(t, T, \xi)$ satisfies*

$$\begin{aligned}
&\mathbb{E}^{\gamma_0}\left[\phi(T, \xi)\right] \\
&\quad \in \int_{\mathbb{R}^m} \phi(t, y)\, \mu^1(dy) \\
&\qquad + \liminf_{\varepsilon \to 0} \left\{ \int_{E^{\Delta,t,T} \times U} \langle \nabla_y \phi(s, y_1), H(s, y_1)\, y_2 \rangle\, \eta^2 (ds d\gamma' d\gamma dy_1 dy_2 dz du) : \eta \in \Theta_\varepsilon(t, T, \xi) \right\} \\
&\qquad - \int_{E^{\Delta,t,T} \times U} \left\langle \textstyle\int_E f(s, \gamma', \gamma, y_1, z, u) \lambda(\gamma)\, Q(\gamma, d\gamma') \right\rangle \mu^2 (ds d\gamma' d\gamma dy_1 dy_2 dz du) \\
&\qquad + \int_{E^{\Delta,t,T} \times U} \left(\partial_t \phi(s, y_1) + \phi(s, y_1 + z) - \phi(y_1)\right) \lambda(\gamma)\, Q(\gamma, \{\gamma'\})\, \mu^2 (ds d\gamma' d\gamma dy_1 dy_2 dz du).
\end{aligned}$$

3.4.6 Proofs of the results in Sections 3.4.4 and 3.4.5

This section gathers all the proofs of the results in Sections 3.4.4 and 3.4.5.

Proof of Proposition 59 We provide the equivalence between the BSVI (93) and the system of ordinary differential inclusions (105). The proof strongly relies on the structure properties mentioned in Subsection 3.4.3. The idea is to associate a specific form to the jump component $Z^{T,\xi}$ and plug it into the driver written as in Subsection 3.4.3.

(Elements of) Proof of Theorem 61 To prove Theorem 61, one uses a penalizing approach similar to the classical Moreau-Yosida-Brézis one for the convex context. More precisely, we situate the problem in the nonconvex setup introduced by Răşcanu, Rotenstein [147]. The equations that make the object of our framework cannot be tackled by the semigroup operators theory because of the particular structure of the multivalued term.

Proofs of the Results of Section 3.4.5 We give the proof of the linear formulations associated to the ε-approximating problems. We begin with proving that this set is non-empty. For simplicity reasons, we assume that the domain $\mathcal{O}$ is switch-invariant (i.e. $\mathcal{O}$ does not depend on $\gamma \in E$). The general result follows similar patterns and relies on the solution of the approximating problem

$$\text{(107)} \qquad \begin{cases} -dY_t^{\varepsilon,T,\xi,u} + H(t, Y_t^{\varepsilon,T,\xi,u}) \nabla \varphi^\varepsilon_{\mathcal{O}_{\Gamma_{t-}}}(Y_t^{\varepsilon,T,\xi,u}) dt \\ \qquad = \displaystyle\int_E f(t, \gamma', \Gamma_{t-}, Y_{t-}^{\varepsilon,T,\xi,u}, Z_t^{\varepsilon,T,\xi,u}(\gamma), u_t)\, \widehat{q}(dt, d\gamma') - \int_E Z_t^{\varepsilon,T,\xi,u}(\gamma') q(dt, d\gamma'), \\ Y_T^{\varepsilon,T,\xi,u} = \xi \in \mathbb{L}^0\left(\Omega, \mathcal{F}_T, \mathbb{P}^{\gamma_0}; \mathbb{R}^m\right). \end{cases}$$

To end this section, we give the proof of Theorem 63 characterizing the (relaxed) occupation measures associated to the BSVI (93). Before giving the proof of this theorem, we invite the reader to note that if one passes to the limit as $\varepsilon \to 0$ and looks at the proof of Theorem 61, one gets a solution of (106). Then, one obtains the solution of (93). Therefore, the limits of the occupation measures introduced before characterize (but may not be limited to) all the controlled solutions of (93). This justifies our interest in the properties of such $\Theta_0(t, T, \xi)$.

3.5 Approximate and approximate null-controllability for a class of piecewise linear Markov switch systems

3.5.1 The control system and main results

The space framework is similar to the one found in the previous section. We consider now a switch system given by a process $(X(t), \Gamma(t))$ on the state space $\mathbb{R}^N \times E$, for some $N \geq 1$ and the family of modes E. The control state space is assumed to be some Euclidian space $\mathbb{R}^d$, $d \geq 1$. The component $X(t)$ follows a controlled differential system depending on the hidden variable γ. We will deal with the following model.

$$dX_s^{x,u} = [A(\Gamma_s) X_s^{x,u} + Bu_s]\, ds + \int_E C(\Gamma_{s-}, \theta) X_{s-}^{x,u} \widetilde{q}(ds, d\theta), \ s \geq 0, \ X_0^{x,u} = x. \tag{108}$$

The operators $A(\gamma) \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times d}$ and $C(\gamma, \theta) \in \mathbb{R}^{N \times N}$, for all $\gamma, \theta \in E$. For linear operators, we denote by $\ker$ their kernel and by Im the image (or range) spaces.

Moreover, the control process $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is an $\mathbb{R}^d$-valued, $\mathbb{F}^0-$ progressively measurable, locally square integrable process. The space of all such processes will be denoted by $\mathcal{U}_{ad}$ and referred to as the family of admissible control processes. The explicit structure of such processes can be found in Jacobsen [97, Proposition 4.2.1], for instance. Since the control process does not (directly) intervene in the noise term, the solution of the above system can be explicitly computed with $\mathcal{U}_{ad}$ processes instead of the (more usual) predictable processes.

The duality abstract characterization of approximate null-controllability

We begin with recalling the following approximate controllability concepts.

Definition 64 *The system (108) is approximately controllable in time $T > 0$ starting from the initial mode $\gamma_0 \in E$, if, for every $\mathcal{F}_{[0,T]}$-measurable, square integrable random variable $\xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_{[0,T]}, \mathbb{P}^{0,\gamma_0}; \mathbb{R}^N\right)$, every initial condition $x \in \mathbb{R}^N$ and every $\varepsilon > 0$, there exists some admissible control process $u \in \mathcal{U}_{ad}$ such that $\mathbb{E}^{0,\gamma_0}\left[\left|X_T^{x,u} - \xi\right|^2\right] \leq \varepsilon$. The system (108) is said to be approximately null-controllable in time $T > 0$ if the previous condition holds for $\xi = 0$ ($\mathbb{P}^{0,\gamma_0}$-a.s.).*

At this point, let us consider the backward (linear) differential equation

$$\left\{\begin{array}{l} dY_t^{T,\xi} = \left[-A^*(\Gamma_t) Y_t^{T,\xi} - \displaystyle\int_E (C^*(\Gamma_t, \theta) + I) Z_t^{T,\xi}(\theta) \lambda(\Gamma_t) Q(\Gamma_t, d\theta)\right] dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_E Z_t^{T,\xi}(\theta)\, q(dt, d\theta), \\ Y_T^{T,\xi} = \xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_{[0,T]}, \mathbb{P}^{0,\gamma_0}; \mathbb{R}^N\right). \end{array}\right. \tag{109}$$

Classical arguments on the controllability operators and the duality between the concepts of controllability and observability lead to the following characterization (cf. Goreac, Martinez [83, Theorem 1]).

Theorem 65 ([83, Theorem 1]) *The necessary and sufficient condition for approximate null-controllability (resp. approximate controllability) of (108) with initial mode $\gamma_0 \in E$ is that any solution $\left(Y_t^{T,\xi}, Z_t^{T,\xi}(\cdot)\right)$ of the dual system (109) for which $Y_t^{T,\xi} \in \ker B^*$, $\mathbb{P}^{0,\gamma_0} \otimes Leb$ almost everywhere on $\Omega \times [0, T]$ should equally satisfy $Y_0^{T,\xi} = 0$, $\mathbb{P}^{0,\gamma_0}-$almost surely (resp. $Y_t^{T,\xi} = 0$, $\mathbb{P}^{0,\gamma_0} \otimes Leb - a.s.$).*

Main Result : An Iterative Invariance Criterion

Before stating the main result of our study, we need the following invariance concepts (cf. Curtain [57], Schmidt, Stern [158]).

Definition 66 *We consider a linear operator* $\mathcal{A} \in \mathbb{R}^{N\times N}$ *and a family* $\mathcal{C} = (\mathcal{C}_i)_{1\leq i\leq k} \subset \mathbb{R}^{N\times N}$.
(i) A set $V \subset \mathbb{R}^N$ *is said to be* $\mathcal{A}$*- invariant if* $\mathcal{A}V \subset V$.
(ii) A set $V \subset \mathbb{R}^N$ *is said to be* $(\mathcal{A};\mathcal{C})$*- invariant if* $\mathcal{A}V \subset V + \sum_{i=1}^{k} \operatorname{Im} \mathcal{C}_i$.

Let us construct now a mode-indexed family of linear subspaces of $\mathbb{R}^N$ denoted by $(V_\gamma^{M,n})_{0\leq n\leq M,\, \gamma\in E}$ by setting

$$\mathcal{A}^*(\gamma) := A^*(\gamma) - \int_E (C^*(\gamma,\theta) + I)\,\lambda(\gamma)Q(\gamma,d\theta) \text{ and } V_\gamma^{M,M} = \ker B^*, \tag{110}$$

for all $\gamma \in E$, and computing, for every $0 \leq n \leq M-1$,

$$V_\gamma^{M,n} \text{ the largest } \left(\mathcal{A}^*(\gamma)\,;\left[(C^*(\gamma,\theta)+I)\,\Pi_{V_\theta^{M,n+1}} : \theta\in E,\ Q(\gamma,\theta)>0\right]\right) \text{−invariant subspace of } \ker B^*. \tag{111}$$

Here, Π_V denotes the orthogonal projection operator onto the linear space $V \subset \mathbb{R}^N$. Whenever there is no confusion at risk, having fixed the maximal number of jumps $M \geq 1$, we drop the dependency on M (i.e. we write V_γ^n instead of $V_\gamma^{M,n}$ for all $0 \leq n \leq M$).

Remark 67 *(i) A simple recurrence argument shows that* $V_\gamma^{M,n} \subset V_\gamma^{M,m}$*, for every* $0 \leq n \leq m \leq M$ *and* $V_\gamma^{M,n} \subset V_\gamma^{M',n}$*, for all* $0 \leq n \leq M \leq M'$*. Moreover, since the dimension of* $\ker B^*$ *cannot exceed* N, $V_\gamma^{M,0} = V_\gamma^{\min(M,N^p),0}$.
(ii) This spaces do not depend on the choice of the controllability horizon $T > 0$*. Therefore, if the approximate (null-)controllability is described by these sets, it is independent of the time horizon.*

The main result of the paper is the following.

Theorem 68 *The switch system (108) is approximately null-controllable (in time* $T > 0$*) with* γ_0 *as initial mode, if and only if the generated set* $V_{\gamma_0}^0$ *reduces to* $\{0\}$.

The proof is postponed to Section 3.4.6. This proof uses the reduction of backward equations with respect to Marked point processes to a system of ordinary differential equations given in Confortola, Fuhrman, Jacod [49]. In order to formulate this system (see Proposition 73), we need to explain some concepts and notations. To prove necessity of the condition, one uses convenient feedback controls and the equivalence between invariance and the concept of feedback invariance (see Proposition 74). Sufficiency (given by Proposition 75) follows from (time-) invariance of convenient linear subspaces with respect to ordinary differential dynamics.

Comparison with [83]

We begin with giving a different (and simpler) proof of (some of) the results in [83]. Besides the general (abstract) characterization of approximate and approximate null-controllability, explicit invariance criteria were given in two specific settings.
(i) In **the case without multiplicative noise** $C = 0$, one notes that the subspaces V_γ^n (for $0 \leq n < M$) do not depend on n. They reduce, in fact, to the largest $\mathcal{A}^*(\gamma)$-invariant subspace of $\ker B^*$. Moreover, in this framework, $\mathcal{A}^*(\gamma)$-invariance and $A^*(\gamma)$-invariance coincide and Theorem 68 yields the following.

Criterion 69 ([83, Criterion 4]) *The system (108) is approximately null-controllable (with initial mode $\gamma_0 \in E$) if and only if the largest subspace of $\ker B^*$ which is $A^*(\gamma_0)$ - invariant is reduced to the trivial subspace $\{0\}$ for all $\gamma_0 \in E$.*

(ii) In **the case of Poisson-driven systems with mode-independent coefficients** A and C, one works with the mode-independent operator $\mathcal{A}^* := A^* - \int_E (C^*(\theta) + I)\,\lambda Q(d\theta)$. The reader familiar with Goreac, Martinez [83, Criterion 3] will note that the necessary and sufficient criterion concerns a notion of strict invariance. We get the same condition provided the system has the possibility to stabilize (the maximal number of jumps $M \geq N+1$ is allowed to exceed the dimension of the state space). Moreover, without loss of generality, one assumes that E is the support of Q.

Criterion 70 ([83, Criterion 3]) *Let us assume that $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times d}$ are fixed and $C(\theta) \in \mathbb{R}^{N\times N}$, for all $\theta \in E$ and that $\lambda(\gamma)\, Q(\gamma, d\theta)$ is independent of $\gamma \in E$. Moreover, we assume that $M \geq N+1$. Then the associated system is approximately null-controllable if and only if the largest subspace $V_0 \subset \ker B^*$ which is $(A^*; [C^*(\theta)\,\Pi_{V_0} : \theta \in E])$-invariant is reduced to $\{0\}$.*

Approximate or approximate null-controllability

Using Riccati techniques, one proves (see Goreac, Martinez [83, Criterion 3]) that, for Poisson-driven systems with mode-independent coefficients, approximate controllability and approximate null-controllability properties coincide. However, in the case of actual switching systems, the two notions have no reason to and do not coincide. This is illustrated by an explicit example. In fact, the reader may note that the null-controllability property strongly depends on the initial mode (through the computation of $V_{\gamma_0}^0$ as last step). A sufficient criterion (already available in Goreac, Martinez [83, Criterion 3]) is that the largest subspace of $\ker B^*$ which is $(\mathcal{A}^*(\gamma_0); [(C^*(\gamma_0, \theta) + I)\,\Pi_{\ker B^*} : \theta \in E, Q(\gamma_0, \theta) > 0])$ −invariant should be reduced to $\{0\}$. It turns out that asking this condition to hold true for all $\gamma_0 \in E$ actually implies approximate controllability. (The proof is postponed to Section 3.4.6.)

Condition 71 *Let us assume that the largest $(\mathcal{A}^*(\gamma); [(C^*(\gamma, \theta) + I)\,\Pi_{\ker B^*} : Q(\gamma, \theta) > 0])$-invariant subspace of $\ker B^*$ is reduced to $\{0\}$, for every $\gamma \in E$. Then, for every $T > 0$ and every $\gamma_0 \in E$, the system (108) is approximately controllable in time $T > 0$.*

Remark 72 *Please note that the notion of $(\mathcal{A}^*(\gamma); [(C^*(\gamma, \theta) + I)\,\Pi_{\ker B^*} : Q(\gamma, \theta) > 0])$-invariance and that of $(A^*(\gamma); [(C^*(\gamma, \theta) + I)\,\Pi_{\ker B^*} : Q(\gamma, \theta) > 0])$ -invariance coincide for subspaces of $\ker B^*$. Second, according to [83, Criterion 3], the notions of approximate and approximate null-controllability coincide in the context of Poisson-driven systems with mode-independent coefficients. Then, a careful look at [83, Criterion 3] provides an example of system which is approximately controllable without satisfying the sufficient condition given before.*

3.5.2 Proof of the results

Before giving the reduction of our backward stochastic equation to a system of ODE, we invite the reader to recall the stochastic structure of several concepts : final data, predictable and càdlàg adapted processes and compensator of the initial random measure. For doing this one can consult again the ordinary differential approach from Confortola, Fuhrman, Jacod [49].

Reduction to a System of Linear ODEs We consider the family of (ordinary) differential equations

$$(112)\qquad \begin{cases} y^M\left(e_M(\omega),\cdot\right)=\xi^M\left(e_M(\omega)\right).\ \text{For } n\le M-1,\ y^n\left(e_n(\omega),T\right)=\xi^n\left(e_n(\omega)\right),\\ dy^n\left(e_n(\omega),t\right)=-A^*\left(\gamma\left|_{e_n(\omega)}\right|\right)y^n\left(e_n(\omega),t\right)dt\\ \qquad -\int_E\left(C^*\left(\gamma\left|_{e_n(\omega)}\right|,\theta\right)+I\right)\left(\widehat{y}^{n+1}\left(e_n(\omega),t,\theta\right)-y^n\left(e_n(\omega),t\right)\right)\widehat{q}^n_{e_n(\omega)}(dt,d\theta)\\ \left(=-\mathcal{A}^*\left(\gamma\left|_{e_n(\omega)}\right|\right)y^n\left(e_n(\omega),t\right)dt\right.\\ \left.-\sum_{\theta\in E}\lambda(\gamma_{|e_n(\omega)|})Q(\gamma_{|e_n(\omega)|},\theta)\left(C^*\left(\gamma\left|_{e_n(\omega)}\right|,\theta\right)+I\right)y^{n+1}\left(e_n(\omega)\oplus(t,\theta),t\right)dt\right),\end{cases}$$

The following result adapts [49, Lemma 7] to our case.

Proposition 73 *A càdlàg adapted process Y given by a family of functions (y^n) as in (98) is solution to (109) if and only if, for $\mathbb{P}$-almost all ω and all $0\le n\le M$, it satisfies the system (112).*

The proof is quasi-identical to the one of [49]. The only difference in our case is the presence of the term $-A^*\left(\gamma\left|_{e_n(\omega)}\right|\right)y^n\left(e_n(\omega),t\right)dt$ which is, of course, classical. The results of [49] apply directly if one assumes that $\lambda(\gamma)>0$ for all $\gamma\in E$ (that is if there exists no absorbing state). Otherwise, we actually get an ODE of type $dy^n\left(e_n(\omega),t\right)=-A^*\left(\gamma\left|_{e_n(\omega)}\right|\right)y^n\left(e_n(\omega),t\right)dt.$

An Iterative Invariance-Based Criterion (Proof of Theorem 68)

As already hinted in Goreac, Martinez [83], the (approximate) controllability properties can be expressed with respect to invariance conditions. The equivalence between the dual (backward) stochastic equation (109) and the (backward) ordinary differential system (105) yields the following approximate controllability criterion.

Proposition 74 *If the system (108) is approximately null-controllable with γ_0 as initial mode, then the generated set $V^0_{\gamma_0}$ reduces to $\{0\}$.*

At this point, the reader may want to note that these considerations involve one equation at the time. The invariant space obtained is then employed for the next equation and gives a coherent character to the system. The basic idea is to provide some kind of local in time invariance of the sets concerned. In Goreac, Martinez [83], this is done using Riccati techniques. But, except for special cases, the solvability of these stochastic schemes is far from obvious. Due to the ordinary differential structure of the equivalent system (112), we are able to elude these techniques and work directly on the deterministic systems.

Proposition 75 *Conversely, if the generated set $V^0_{\gamma_0}$ reduces to $\{0\}$, then the system (108) is approximately null-controllable with γ_0 as initial mode.*

Proof of Sufficiency Condition 71 for Approximate Controllability

Proof of Condition 71. In light of the Theorem [83, Theorem 1] and Proposition 73, one only needs to show that the only solution of (112) remaining in $\ker B^*$ is constant 0. ■

4 Future research milestones

In this chapter we point out some open problems and research directions on which the study will focus in the future. We present seven issues, each one having multiple open questions, aiming both forward and backward stochastic differential equations.

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