THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 9, Number 2/2008, pp. 000-000

THE SOLUTION TO W. MAY'S PROBLEM FOR ISOMORPHISM OF COMMUTATIVE GROUP ALGEBRAS OF MIXED GROUPS OVER FINITE FIELDS

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Suppose G is an abelian group whose p-component G_p is totally projective (or more generally, is an S-group) and F_p is the simple field of p-elements. It is shown that the F_p-isomorphism $F_pH \cong F_pG$ for any group H implies $H_p \cong G_p$. This almost completely solves a problem of Warren May (1988) from the *Proceedings of the American Mathematical Society* posed, however, for an arbitrary field F with char(F) = $p > 0$. In particular, $H \cong G$ provided G is a p-mixed Warfield group of torsion-free rank one. The last consequence enlarges in some aspect a result of May (1988) published in the same *Proceedings* for the case when G is a p-local Warfield group.

Key words: group algebras, isomorphisms, σ-summable and totally projective groups.
2000 *AMS Subject Classification*: Primary 20 C07; Secondary 20 K10, 20 K15, 20 K20, 20 K21.

1. INTRODUCTION

The determinative aim of this research paper is to check whether or not the total projectivity of the p-primary component of an arbitrary abelian group is an invariant property for the commutative modular group algebra of characteristic p, that is a well-known significant problem due to May [20]. In the sequel, we shall show that this question can be settled positively for group algebras over the finite modular field F_p of char(F_p) = p.

We remember that, as usual, G is an abelian group with p-torsion component G_p and socle $G[p]$, and F_pG is its group algebra over F_p. For such an algebra F_pG, the letter $S(F_pG)$ denotes the normalized Sylow p-group in F_pG and $I(F_pG; C)$ denotes the relative augmentation ideal of F_pG with respect to the subgroup C of G. To facilitate the expressions, we designate $1 + I(F_pG; C)$ as $S(F_pG; C)$ whenever C is a p-group. All other notations and the terminology not explicitly defined herein are standard and follow essentially the cited in the bibliography our articles and the classical books [13]. For instance, the symbol $\coprod$ will always denote a restricted (= bounded) direct product, which product is termed a coproduct too.

In [2]-[7], we have studied conjectures in this branch and also partially solved the aforementioned May's problem under some additional restrictions either on the coefficient field or on the group basis. More precisely, in [4], [9] and [12] we have used the technique of so-called *summable groups*. However, more helpful is the method by using σ- *summable groups* (see, for example, [2] and [7]) because of the fact that they are of arbitrary lengths co-final with ω while the summable groups have lengths not exceeding ω_1 (cf. [13]).

2. MAIN RESULTS

The following two lemmas are well-known in the abstract group theory but, for completeness of the exposition, we include them as a folklore (for instance, see [13]).

First Lemma. *Assume that A is an abelian p-group and N is its subgroup. Then,* $\forall n \geq 1$, $N \cap A^{pn} = N^{pn}$, *that is N is pure in A, if and only if* $N[p] \cap A^{pn} = N^{pn}[p]$.

Remark. In the general case, when N is not a neat subgroup of A, the intersection $N[p] \cap A^p = N^p[p]$ does not imply that $N \cap A^p = N^p$ as some simple examples show when $N^p \neq 1$.

Second Lemma. *Given that N is neat in the abelian p-group such that* $N[p] = A[p]$. *Then* $N = A$.

Main Lemma. *For any abelian group A the equalities* $A[p] = \prod_{i \in I} A_i [p]$ *and* $A^{pn}[p] = \prod_{i \in I} A_i^{pn} [p]$ *hold for all natural numbers* $n \geq 1$ *and some subgroups* $A_i \leq A_p \Leftrightarrow A_p = \prod_{i \in I} A_i$.

Proof. Write $A[p] = \prod_{i \in I} A_i [p] = (\prod_{i \in I} A_i)[p]$. We shall argue below that $\prod_{i \in I} A_i$ is pure in A. For this purpose, it is enough to see by the First Lemma that the following identities hold: $A^{pn} [p] = A[p] \cap A^{pn} = (\prod_{i \in I} A_i [p]) \cap A^{pn} = (\prod_{i \in I} A_i)^{pn}[p] = (\prod_{i \in I} A_i^{pn})[p] = \prod_{i \in I} A_i^{pn} [p]$. Thereby, we obtain precisely the hypothesis from the text. Finally, we apply the Second Lemma to get the desired equality.

Remark. If A and B are pure subgroups of the abelian p-group G, then $A \times B$ need not be pure in G. Moreover, $G^{p\alpha}[p]$ is not a fully invariant subgroup of $G[p]$ although $G^{p\alpha}$ is so in G for each ordinal $\alpha \geq 1$.

The following group-theoretic assertion is formulated and proved in [19].

Third Lemma. *Suppose M and N are subgroups of an abelian p-group A. Then M is N-high in A, that is M is maximal with respect to* $M \cap N = 1$, *if and only if M is neat in A and* $A[p] = M[p] \times N[p]$.

Before proving our central statement that motivates the present exploration, we need one crucial proposition.

Main Proposition. *Let* $G_p = \prod_{i \in I} G_i$. *Then* $F_pH \cong F_pG$ *as* F_p*-algebras for some group H yields that* $H_p = \prod_{i \in I} H_i$, *where* $length(G_i) = length(H_i)$.

Proof. Write down $G[p] = \prod_{i \in I} G_i [p]$. But, by a fundamental result of Beers-Richman-Walker ([1]), the F_p-isomorphism of group algebras F_pG and F_pH insures an isometry φ, which is a height-preserving isomorphism, between the socles $G[p]$ and $H[p]$ with heights as computed in G and H respectively. Hence we deduce that $H[p] = \prod_{i \in I} A_i [p]$, where A_i are subgroups of H_p naturally constructed in the following manner: For each $i \in I$, we set $A_i = < h_i \in H_p \mid 1 \neq \varphi^{-1}(h_i^{ps}) \in G_i [p]$ for some $s \in N \cup \{0\}$ and $< h_i^p > \cap (\prod_{h'_i} < h'^p_i >) = 1$ running over each other generating $h'_i \neq h_i >$. Henceforth, $\varphi^{-1}(h_i^{ps}) \neq \varphi^{-1}(h'^{pt}_i)$ since otherwise $h_i^{ps} = h'^{pt}_i$ implies $h_i^{ps} = 1 = h'^{pt}_i$, whence $1 = \varphi^{-1}(h_i^{ps}) = \varphi^{-1}(h'^{pt}_i)$ which is against our choice. So, in more precise words, A_i is generated via such distinct elements of H_p whose nonidentity images under the action of φ^{-1} lie in $G_i[p]$ and such that the images of p-degrees of arbitrary products between the different generating members are different as well. For instance, $u_{i1}^{\varepsilon 1} \ldots u_{it}^{\varepsilon t} \neq 1$ and $v_{j1}^{\delta 1} \ldots v_{jm}^{\delta m} \neq 1$ must have different nonidentity images in $G_i[p]$ by the action of the isomorphism φ^{-1}. Thereby, $u_{i1}^{\varepsilon 1} \ldots u_{it}^{\varepsilon t} = v_{j1}^{\delta 1} \ldots v_{jm}^{\delta m}$ implies that $u_{i1}^{\varepsilon 1} = \ldots = u_{it}^{\varepsilon t} = v_{j1}^{\delta 1} = \ldots = v_{jm}^{\delta m} = 1$.

Thus we observe that $A_i \subseteq H_p$ and $A_i [p] = \{a_i \in A_i \mid \varphi^{-1}(a_i) \in G_i [p]\}$. In fact, if $a_i \in A_i[p]$ it follows that $a_i = h_{i1}^{\varepsilon 1} \ldots h_{ik}^{\varepsilon k}$ where $0 < \varepsilon_m < order(h_{im})$; $1 \leq m \leq k$. Since $(h_{i1}^{\varepsilon 1})^p \ldots (h_{ik}^{\varepsilon k})^p = 1$, in accordance with our construction we find $(h_{i1}^{\varepsilon 1})^p = \ldots = (h_{ik}^{\varepsilon k})^p = 1$. But by definition $\varphi^{-1}(h_{i1}^{\varepsilon 1}) \in G_i [p], \ldots, \varphi^{-1}(h_{ik}^{\varepsilon k}) \in G_i [p]$, so $\varphi^{-1}(a_i) = \varphi^{-1}(h_{i1}^{\varepsilon 1}) \ldots \varphi^{-1}(h_{ik}^{\varepsilon k}) \in G_i [p]$.

Moreover, $A_i^{pn} = < h_i^{pn} \in H_p^{pn} \mid 1 \neq \varphi^{-1}(h_i^{pn+s}) \in G_i [p]$ for some $s \in N \cup \{0\}$ and $< h_i^{pn+1} > \cap (\prod_{h'_i}^{pn} < h'^{pn+1}_i >) = 1$ running over each other generating $h_i^{pn} \neq h'^{pn}_i >$, hence as in the preceding situation $A_i^{pn} [p] = \{c_i \in A_i^{pn} \mid \varphi^{-1}(c_i) \in G_i^{pn}[p]\}$ for every natural number n.

Furthermore, take $g_i \in G_i$. Then we have $g_i^{pti} = 1$ for some natural t_i, hence $g_i^{pti-1} \in G_i [p]$. But $\varphi(g_i^{pti-1}) = h_i^{pti-1}$ for some $h_i \in H_p$ because φ as an isometric map preserves the heights. That is why $h_i^{pti-1} \in A_i [p]$. Consequently, $\varphi(G_i [p]) = A_i [p]$ and $\varphi(G_i^{pn} [p]) = \varphi(G_i[p] \cap G^{pn}[p]) = \varphi(G_i[p]) \cap \varphi(G^{pn}[p]) = A_i[p] \cap H^{pn}[p] = (A_i \cap H^{pn})[p]$. By what we have shown above, by using the definition, $\varphi(G_i^{pn}[p]) = A_i^{pn}[p]$ for all $n \in N$. Thus all A_i are pure in H_p.

Besides, the groups A_i are independent for every subscript $i\in I$, namely $A_i \cap (\prod_{j\neq i} A_j) = 1$. To confirm this, we observe that $G_i \cap (\prod_{j\neq i} G_j) = 1 \Leftrightarrow G_i[p] \cap (\prod_{j\neq i} G_j[p]) = 1$ and taking in both sides the homomorphism φ, we are done.

Another approach to select certain pure subgroups is like this.

We claim that $H[p] = \prod_{i\in I} H_i[p]$, where every subgroup $H_i \leq H_p$ is pure in H. In fact, by what we have just argued above, $H[p] = A_1[p] \times \prod_{i\in I\setminus\{1\}} A_i[p]$, where $A_1 \cap \prod_{i\in I\setminus\{1\}} A_i = 1$. But $A_1 \subseteq H_1$, where $H_1 \leq H_p$ is maximal with respect to the last intersection property. Therefore $H[p] = H_1[p] \times \prod_{i\in I\setminus\{1\}} A_i[p]$, whence $A_1[p] = H_1[p]$. By the same token, $A_2 \cap \prod_{i\in I\setminus\{2\}} A_i = 1$ hence it is routine to verify that $A_2 \cap (\prod_{i\in I\setminus\{1,2\}} A_i \times H_1) = 1$. Thus $A_2 \subseteq H_2$ with $A_2[p] = H_2[p]$, where $H_2 \leq H_p$ is maximal with respect to this intersection, etc. The demonstrated procedure combined with the fact from the Third Lemma that the N-high subgroups for any group N are neat, hence pure in our situation because of the construction of A_i, are sufficient to derive the claim.

We wish to indicate as in the previous approach that $\varphi(G_i^{p^n}[p]) = H_i^{p^n}[p]$ for each $n \in N_0 = N \cup \{0\}$ such that $G_i[p] \neq G[p]$ and $H_i[p] \neq H[p]$, since otherwise the Second Lemma will assure that $G_i = G$ and $H_i = H$, a contradiction.

Now, we shall prove that $\prod_{i\in I} A_i = \prod_{i\in I} H_i$ will generate H_p. Indeed, $G[p] = \prod_{i\in I} G_i[p]$ and $G^{p^n}[p] = \prod_{i\in I} G_i^{p^n}[p]$ for each positive integer n. Thus, by what we have already shown above, $\varphi(G[p]) = H[p] = \prod_{i\in I} H_i[p] = \prod_{i\in I} \varphi(G_i[p]) = \varphi(\prod_{i\in I} G_i[p])$, and moreover $H^{p^n}[p] = \varphi(G^{p^n}[p]) = \varphi(\prod_{i\in I} G_i^{p^n}[p]) = \prod_{i\in I} \varphi(G_i^{p^n}[p]) = \prod_{i\in I} H_i^{p^n}[p]$ because φ maps $G_i^{p^n}[p] = G_i[p] \cap G^{p^n}[p]$ onto $H_i[p] \cap H^{p^n}[p] = H_i^{p^n}[p]$. We therefore have the necessary relations to arrive at the Main Lemma that $H_p = \prod_{i\in I} H_i$. The proof is complete.

Example: The conditions on the intersection cannot be removed, since otherwise for some fixed $h_i \in A_i$ of order(h_i) $> p$ and for each $h\in H[p]$ it follows that $hh_i \in A_i$, hence $h \in A_i$ that gives $H[p] \subseteq A_i$ and which is impossible.

Moreover, $\langle h_i\rangle \cap \prod_{h'_i \neq h_i} \langle h'_i\rangle = \langle h_i\rangle \cap \langle h'_i, \dots\rangle \neq 1$ in general, since otherwise we should have a direct sum of cyclic groups. For example, if $p=5$, $h_i^2 = h'^3_i \neq 1$ may occur.

Now we are ready to attack the main attainment.

Main Theorem. *Suppose G is an abelian group for which G_p is an A-group (in particular, is an S-group or an N-group). Then $F_pH \cong F_pG$ as F_p-algebras for any group H implies $H_p \cong G_p$.*

Proof. Complying with the Main Proposition in [2] along with [21], we consider G_p as a reduced group.

First of all, we assume that G_p is totally projective. We note that it suffices to establish only that H_p is totally projective because, by virtue of [21], the Ulm-Kaplansky invariants of G_p and H_p can be determined from the group algebra $F_pG = F_pH$. Whence, owing to [13], $G_p \cong H_p$ as stated. In order to do this, we will further distinguish the following two basic cases:

Case1 – length(G_p) is limit. Thus it follows from [13] that $G_p = \prod_{i\in I} G_i$, where *length*(G_i) $<$ *length*(G_p) $= \lambda$ and all G_i are totally projective. Applying the Main Proposition, we infer that $H_p = \prod_{i\in I} H_i$ where length(H_i) = length(G_i) < length(G_p) = length(H_p) = λ. Next, we shall detect that all H_i are totally projective by the usage of standard transfinite induction on the lengths and more especially on λ. Toward this end, we may assume that the theorem holds true for all ordinals $\alpha < \lambda$. Without loss of generality, we can presume also that all length(G_i) are limit, consequently so are the lengths like length(H_i). In fact, this is possible because if for some abelian p-group A the ratio $A^{p^{\alpha+n}} = 1$ is fulfilled then $(A^{p^\alpha})^{p^n} = 1$, and thus from [23] it follows obviously that A is totally projective if and only if the same is A/A^{p^α}, where, of course, α is limit. Moreover, we may restrict our attention on cofinal with ω ordinals since if β is an arbitrary ordinal, then $\beta+\omega$ is always cofinal with ω. Besides, we assume that length(H_i) + $\omega \leq \lambda$ (notice that if length(H_i) = length(G_i) are not cofinal with ω, they however are limit and because G_i are totally projective then $G_i = \prod_{j\in J} A_{ij}$, where eventually length(A_{ij}) are cofinal with ω etc.). And so, such a component H_i will be totally projective by a criterion due to Linton-Megibben (e.g. [2] or [3]) if and only if $H_i/H_i^{p^\gamma}$ is σ-summable for all cofinal with ω ordinals $\gamma \le \lambda$ and $H_i/H_i^{p^\gamma}$ is totally projective for all other $\gamma < \lambda$. But by application of the Main Proposition in [], $F_pH \cong F_pG$ guarantees that $F_p(G/G_p^{p^\gamma}) \cong F_p(H/H_p^{p^\gamma})$. Thus since $G_p/G_p^{p^\gamma} = (G/G_p^{p^\gamma})_p$ is totally projective, by virtue of the induction hypothesis $H_p/H_p^{p^\gamma} = (H/H_p^{p^\gamma})_p$ is totally projective as well. Therefore the same conclusion is valid by [13] even for $H_i/H_i^{p^\gamma}$ as its direct factor because of the canonical isomorphism $H_p/H_p^{p^\gamma} \cong \prod_{i\in I} H_i/H_i^{p^\gamma}$. In this way, since $G_p/G_p^{p^\gamma}$ is totally projective hence σ-summable for each cofinal with ω ordinal $\gamma \le \lambda$, the σ-summability for $H_p/H_p^{p^\gamma}$ holds referring to [2]. But, for each $i \in I$, the factor-groups $H_i/H_i^{p^\gamma}$ may be regarded as subgroups of $H_p/H_p^{p^\gamma}$ with equal lengths whenever $\gamma \le$ length$(H_i) < \lambda =$ length(H_p), whence from [2] they are trivially σ-summable groups. Finally, by what we have foregoing illustrated, we conclude that all H_i are totally projective and thus their coproduct H_p must be itself totally projective too by exploiting [13]. So, the first half is completed.

Case2 – length(G_p) is nonlimit. Write length$(G_p) = \lambda + n$, where λ is limit and n is natural. Hence $(G_p^{p^\lambda})^{p^n} = 1$, and on the other hand as we have seen above $F_pG^{p^\lambda} \cong F_pH^{p^\lambda}$ and $F_p(G/G_p^{p^\lambda}) \cong F_p(H/H_p^{p^\lambda})$ where $G_p^{p^\lambda}$ and $G_p/G_p^{p^\lambda}$ are both totally projective. Invoking standard facts and Case1, we infer that $H_p^{p^\lambda}$ and $H_p/H_p^{p^\lambda}$ are also totally projective, hence so is H_p (see [23] or [13]), thus completing the second part as well.

Let us now choose G_p to be a μ - elementary A-group for some arbitrary but fixed ordinal μ (see, for instance, [15] or [16]) so that G_p is not totally projective. Suppose G_p is an isotype and almost nice subgroup (hence an almost balanced subgroup) of the totally projective p-group T. We next consider the outer direct products $G \times T$ and $H \times T$. Thus $F_p(G \times T) = (F_pG)T = (F_pH)T = F_p(H \times T)$, whence $S(F_pG) = S(F_pH)$ and $S(F_p(G\times T); T) = S(F_p(H\times T); T)$. Consulting with [D1] we have $S(F_pH) \cap S^{p^\alpha}(F_p(H\times T); T) = S(F_pG) \cap S^{p^\alpha}(F_p(G\times T); T) = S(F_pG) \cap S(F_p(G^{p^\alpha}\times T^{p^\alpha}); T^{p^\alpha}) = S(F_pG^{p^\alpha}; G_p^{p^\alpha}) = S^{p^\alpha}(F_pG; G_p) = S^{p^\alpha}(F_pG) = S^{p^\alpha}(F_pH)$ for each ordinal α. Consequently, we deduce that $S(F_pH)$ is isotype in $S(F_p(H\times T); T)$. Moreover, G_p is almost balanced in T and thus, employing [7], we obtain that T is balanced in $S(F_p(G\times T); T)$. Hence from [13] it follows at once that G_p is almost balanced in $S(F_p(G\times T); T)$ and so $S(F_pG) = S(F_pG; G_p)$ (cf. [2]) is almost balanced in $S(F_p(G\times T); T)$. Therefore $S(F_pH)$ is almost balanced in $S(F_p(H\times T); T)$. But it is well-known that H_p is balanced in $S(F_pH)$, so H_p is almost balanced in $S(F_p(H\times T); T)$ in view of [13]. On the other hand, $S(F_p(G\times T); T)$ is totally projective by virtue of [3]. Consequently $(G_p; S(F_p(G\times T); T))$ and $(H_p; S(F_p(H\times T); T))$ are pairs of almost balanced subgroups of totally projective p-groups. Denote $A = S(F_p(G\times T); T) = S(F_p(H\times T); T)$. After this, we examine the outer direct products $G\times A$ and $H\times A$. Clearly, $F_p(G\times A) = (F_pG)A = (F_pH)A = F_p(H\times A)$. Besides, conforming with [D1], $I(F_p(G\times A); G_p) = F_p(G\times A).I(F_pG; G_p) = F_p(G\times A).I_p(FG; G) = F_p(H\times A).I_p(FH; H) = F_p(H\times A).I(F_pH; H_p) = I(F_p(H\times A); H_p)$. But then $I(F_p(G\times A); G_p \times G_p) = I(F_p(H\times A); H_p \times H_p)$. Thus we conclude that $F_p(G\times A)/I(F_p(G\times A); G_p \times G_p) \cong F_p(G/G_p \times A/G_p) \cong F_p(H/H_p \times A/H_p) \cong F_p(H\times A)/I(F_p(H\times A); H_p \times H_p)$. By exploiting [3] we write $A = T \times A/T$, hence $A/G_p \cong T/G_p \times A/T$ is totally projective. That is why, $G/G_p \times A/G_p$ is a group with totally projective p-primary component isomorphic to A/G_p. So, by what we have shown in the totally projective step, we derive $A/G_p \cong A/H_p$. Finally, [15] and [16] ensure that $G_p \cong H_p$, as desired. Thus, we extract that H_p is also a μ-elementary A-group.

To end the proof, let G_p be an arbitrary A-group. Then, by definition (cf. [15] or [16]), G_p is a direct sum of μ-elementary A-groups. Furthermore, the Main Proposition plus the ideas of the above steps and some well-known standard group-theoretic arguments from [15] lead us to H_p is a direct sum of μ-elementary A-groups, hence it is an A-group. On the other hand, we wish to employ ([15] and [16]) or ([1], Theorems 4.2 and 4.3) or [14] to infer that G_p and H_p must be isomorphic, so finishing this point. The proof is concluded in all generality.

Main Corollary. *Suppose G is a global Warfield group (in particular, is a mixed simply presented group). Then the F_p – isomorphism $F_pH \cong F_pG$ for some another group H yields that $H_p \cong G_p$.*

Proof. Since by making use of the technicalities in [17] and [18] the group G_p is an S-group, the statement follows according to the main attainment above. ♠

The following is important.

Main Remark. The formulated main assertion almost settles a question due to May raised in his paper [20] appeared in Proceedings of the AMS on 1988. The restriction is only on the coefficient field to be of finite power, but it is necessary for the proof of the Main Proposition about the decomposition and cannot be ignored yet. The case for an arbitrary cardinality of the field is examined in [6]. In this way, we formulate an actual problem.

Main Question. Does there exist a simple field F_p of characteristic $p \neq 0$ such that for any field K with characteristic $p > 0$ and any two arbitrary abelian groups G and H, the K - isomorphism $KG \cong KH$ forces the F_p - isomorphism $F_pG \cong F_pH$? If yes, the May's problem will be completely resolved.

Commentary. Adapting in our variant an analogous technique to that described in [25] and its generalization due to [26]-[17], together with some other standard group-theoretic arguments, one can conclude that the result of the type of Wallace-Warfield-Hunter may be extended for the classes of *S-groups* and *N-groups* (or more generally for *A-groups*), but this is a theme of some new research investigation.

The other valuable statement, which is an immediate consequence of the Main Theorem, states as follows:

Theorem. *Let G be a p-mixed abelian group of torsion-free rank one whose G_p is an A-group (in particular, is an S-group or an N-group). Then $F_pH \cong F_pG$ as F_p – algebras for some group H if and only if $H \cong G$.*

Proof. It follows automatically combining the Main Theorem and the method illustrated by us in [3], by utilizing the central result of [25] plus the above commentary as well. The proof is finished.

First Corollary. *Let G be a p-mixed Warfield group (or, in particular, a p-mixed simply presented group) of torsion-free rank one. Then $F_pH \cong F_pG$ are F_p-isomorphic algebras for some group H if and only if $H \cong G$.*

Proof. Invoking the methods in [17] and [18] we deduce that G_p is an S-group, whence the last affirmation is applicable, and we are done.

Second Corollary. *Let G be a p-primary A-group (in particular, an S-group or an N-group). Then $F_pH \cong F_pG$ as F_p-algebras for any group H if and only if $H \cong G$.*

Remark. The first consequence may be improved by dropping off the conditions on the torsion-free rank of G as well as on the simplicity of the field and by using a more direct approach in the proof (e.g. [10] and [11]). The second consequence strengthens a similar result of Beers-Richman-Walker ([1], Theorem 4.3) when G and H are both S-groups. Other result in this aspect for the p-torsion case was obtained also in [24].

CONCLUDING DISCUSSION

The main observations that immediately arise are that the Main Theorem will be valid for an arbitrary coefficient field K of characteristic $p \neq 0$ if the Main Proposition holds for such a ring, and as well that the same will be true for the other results of this type listed above.

Besides, whether adopting the methods described in this paper we may generalize the results obtained by us for direct sums of mixed groups of the present sorts each of which has torsion-free rank one. For example, every mixed simply presented group is the direct sum of simply presented subgroups of torsion-free rank one.

A question of some importance is of whether the major statement remains true for more general classes of abelian p-groups such as: the class of all *IT-groups* introduced by Hill-Megibben in [HM] or the class of all *totally Zippin p-groups* explored in [Me]. Following our scheme of proof, this is probably so.

R E F E R E N C E S

1. BEERS D., RICHMAN F., Walker E., *Group algebras of abelian groups,* Rend. Sem. Mat. Univ. Padova **69** (1983), 41-50.
2. DANCHEV Peter, *Commutative group algebras of σ - summable abelian groups,* Proc. Amer. Math. Soc. (9) **125** (1997), 2559-2564.
3. DANCHEV Peter, *Isomorphism of modular group algebras of totally projective abelian groups,* Comm. Algebra (5) **28** (2000), 2521-2531.
4. DANCHEV Peter, *On a question of W. L. May concerning the isomorphism of modular group algebras,* Comm. Algebra (5) **29** (2001), 1953-1958.
5. DANCHEV Peter, *Modular group algebras of coproducts of countable abelian groups,* Hokkaido Math. J. (2) **29** (2000), 255-262.
6. DANCHEV Peter, *Commutative group algebras of abelian Σ-groups,* Math. J. Okayama Univ. (2) **40** (1998), 77-90.
7. DANCHEV Peter, *Commutative group algebras of direct sums of σ-summable abelian p-groups,* Math. J. Okayama Univ. (2) **45** (2003), 1-15.
8. DANCHEV Peter, *Commutative group algebras of direct sums of countable abelian groups,* Kyungpook Math. J. (1) **44** (2004), 21-29.
9. DANCHEV Peter, *A note on the isomorphic modular group algebras of abelian groups with simply presented p-components,* Compt. rend. Acad. bulg. Sci. (12) **57** (2004), 13-14.
10. DANCHEV Peter, *A note on the isomorphism of modular group algebras of global Warfield abelian groups,* Bol. Soc. Mat. Mexicana (1) **10** (2004), 49-51.
11. DANCHEV Peter, *Isomorphic commutative group algebras of p-mixed Warfield groups,* Acta Math. Sinica (5) **21** (2005), 913-916.
12. DANCHEV Peter, *Commutative group algebras of summable p-groups,* Comm. Algebra (4) **35** (2007), 1275-1289.
13. FUCHS L., Infinite Abelian Groups, **I** and **II**, Mir, Moscow, 1974 and 1977. (In Russian).
14. HILL P., *The classification of N-groups,* Houston J. Math. **10** (1984), 43-55.
15. HILL P., *On the structure of abelian p-groups,* Trans. Amer. Math. Soc. **288** (1985), 505-525.
16. HILL P., MEGIBBEN C., *On the theory and classification of abelian p-groups,* Math. Z. **190** (1985), 17-38.
17. HUNTER R., *Balanced subgroups of abelian groups,* Trans. Amer. Math. Soc. **215** (1976), 81-98.
18. HUNTER R., WALKER E., *S-groups revisited,* Proc. Amer. Math. Soc. (1) **82** (1981), 13-18.
19. IRWIN J. and WALKER E., *On N-high subgroups of abelian groups,* Pac. J. Math. **11** (1961), 1363-1374.
20. MAY W., *Modular group algebras of simply presented abelian groups,* Proc. Amer. Math. Soc. (2) **104** (1988), 403-409.
21. MAY W., *Commutative group algebras,* Trans. Amer. Math. Soc. **136** (1969), 139-149.
22. MEGIBBEN Ch., *Totally Zippin p-groups,* Proc. Amer. Math. Soc. (1) **91** (1984), 15-18.
23. NUNKE R. J., *Homology and direct sums of countable abelian groups,* Math. Z. (3) **101** (1967), 182-212.
24. ULLERY W., *An isomorphism theorem for commutative modular group algebras,* Proc. Amer. Math. Soc. (2) **110** (1990), 287-292.
25. WALLACE K., *On mixed groups of torsion-free rank one with totally projective primary components,* J. Algebra **17** (1971), 482-488.
26. WARFIELD R. B., Jr., *Simply presented groups,* Proc. Sem. Abelian Group Theory, Univ. Arizona Lect. Notes, 1972.

Received: January 27, 2007