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ON WEIGHTED EQUILIBRIUM PROBLEMS

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We consider some systems of general vector equilibrium problems and weighted equilibrium problems and find equivalence conditions between them. Then we establish for these classes of problems a few existence results under different types of generalized weighted monotonicity assumptions.

Keywords: Weighted equilibrium problems, System of vector equilibrium problems, Generalized weighted monotonicity, Hemicontinuity, Convexity.

1. INTRODUCTION

The scalar equilibrium problem was first introduced and studied on real Hilbert spaces [10], and then on Hausdorff topological spaces [5]. Then, in [2] there were considered two classes of vector equilibrium problems, on a closed convex set of a Hausdorff space, and on a closed pointed convex cone. Also, the same authors introduced the set-valued equilibrium problem. In [8] is introduced and investigated a quasi-equilibrium problem on a Hilbert space. The invex equilibrium (or equilibrium - like) problem is defined on an invex subset of a Hilbert space in [9].

In [11], the general equilibrium problems on a real Banach space and then on the dual space is considered. The existence of solutions in set-valued cases was obtained [3] on reflexive and then on arbitrary Banach spaces, and also on their duals.

2. SOME PRELIMINARIES

For each given $m \in$ □ , we denote by $\mathbb{R}_+^m$ the nonnegative orthant of $\mathbb{R}^m$, i.e.,

$$\mathbb{R}_+^m = \{ u = (u_1,\ldots,u_m) \in \mathbb{R}^m \mid u_j \geq 0, \text{ for } j = 1,\ldots,m \}$$

and

$$\operatorname{int}\mathbb{R}_+^m = \{ u = (u_1,\ldots,u_m) \in \mathbb{R}^m \mid u_j \geq 0, \text{ for } j = 1,\ldots,m \}$$

its relative interior. Also, let

$$T_+^m = \{ u = (u_1,\ldots,u_m) \in \mathbb{R}^m \mid \sum_{j=1}^m u_j = 1 \}$$

be a simplex of $\mathbb{R}_+^m$ and

$$\operatorname{int} T_+^m = \{ u = (u_1,\dots,u_m) \in \operatorname{int}\mathbb{R}^m \mid \sum_{j=1}^m u_j = 1 \}$$

its relative interior.

Let $I = \{1,\dots,n\}$ be a finite index set and for each $i \in I$ let l_i be a positive integer. For each $i \in I$, let X_i be a real topological vector space (not necessarily Hausdorff), K_i a nonempty convex subset of X_i, and Y_i an arbitrary set.

We denote $X = \prod_{i\in I} X_i$, $K = \prod_{i\in I} K_i$ and $x = (x_i)_{i\in I}$. Also, we denote by $\mathcal{F}(K)$ the family of all nonempty finite subsets of K and by $\operatorname{co}A$ the convex hull of the set A.

For each $i \in I$, let $f_i : K \to Y_i$ and $\Psi_i : Y_i \times K_i \times K_i \to \mathbb{R}^{l_i}$ two maps and $\Psi = (\Psi_i)_{i\in I}$. We consider the systems of vector equilibrium problems

($\Psi-$SVEP): Find $\bar{x} \in K$ such that, for each $i \in I$,

$$\Psi_i(f_i(\bar{x}), \bar{x}_i; y_i) \notin R_+^{l_i} \setminus \{0\} \text{ for all } y_i \in K_i;$$

and $(\Psi-\text{SVEP})_w$: Find $\bar{x} \in K$ such that, for each $i \in I$

$$\Psi_i(f_i(\bar{x}), \bar{x}_i; y_i) \notin \operatorname{int}\mathbb{R}_+^{l_i} \text{ for all } y_i \in K_i.$$

Relative to problems ($\Psi-$SVEP) and $(\Psi-\text{SVEP})_w$ we introduce the weighted general equilibrium problem over product sets ($\Psi-$WEPP): Find $\bar{x} \in K$ with respect to the weight vector $W = (W_1,\dots,W_n) \in \prod_{i=1}^n (R_+^{l_i} \setminus \{0\})$ such that

$$\sum_{i\in I} W_i \cdot \Psi_i\left(f_i(\bar{x}), \bar{x}_i; y_i\right) \le 0 \text{ for all } y_i \in K_i,\ i \in I;$$

and the system ($\Psi-$SWEP): Find $\bar{x} \in K$ with respect to the weight vector $W = (W_1,\dots,W_n)$ such that, for each $i \in I, W_i \in R_+^{l_i} \setminus \{0\}$ and

$$\mathrm{W}_i \cdot \Psi_i\left(\mathrm{f}_i(\bar{\mathrm{x}}), \bar{\mathrm{x}}; \mathrm{y}_i\right) \le 0 \text{ for all } y_i \in K_i.$$

We denote by K^w (respectively K_s^w) the solution set of ($\Psi-$WEPP) (respectively, ($\Psi-$SWEP)) and by K_n^w (respectively, K_{sn}^w) the normalized solution set of ($\Psi-$WEPP) (respectively, ($\Psi-$SWEP)).

The following lemma shows that the solution sets of ($\Psi-$WEPP) and ($\Psi-$SWEP) coincide.

Lemma 2.1. *Let* $W = (W_1,\dots,W_n) \in \prod_{i=1}^n (R_+^{l_i} \setminus \{0\})$ (*respectively*, $W = (W_1,\dots,W_n) \in \prod_{i=1}^n T_+^{l_i}$) *be a weight vector*. *Suppose that* $\Psi_i\left(f_i(x), x_i; x\right) = 0$ *for any* $i \in I$ *and* $x \in K_i$. *Then* $K^w = K_s^w$ (*respectively*, $K_n^w = K_{sn}^w$).

The next result shows that ($\Psi-$SVEP) or $(\Psi-\text{SVEP})_w$ can be solved using ($\Psi-$SWEP).

Lemma 2.2. *Each normalized solution* $\bar{x} \in K$ *with vector* $W \in \prod_{i=1}^n T_+^{l_i}$ (*respectively* $W \in \prod_{i=1}^n \operatorname{int} T_+^{l_i}$) *of* ($\Psi-$SWEP) *is a solution of* $(\Psi-\text{SVEP})_w$ (*respectively*, ($\Psi-$SVEP)).

Remark 2.1. This type of equivalence results were obtained for different classes of equilibrium problems or variational inequalities by M. A. Noor (see, for example [9] and some of the references therein).

From Lemmas 2.1 and 2.2, the next result follows.

Lemma 2.3. *Each normalized solution* $\overline{x} \in K$ *with weight vector* $W \in \prod_{i=1}^{n} T_{+}^{l_i}$ (*respectively* $W \in \prod_{i=1}^{n} \operatorname{int} T_{+}^{l_i}$) *of* ($\Psi -$WEPP) *is a solution of* $\left(\Psi - \mathrm{SVEP}\right)_w$ (*respectively*, ($\Psi -$SVEP)).

3. EXISTENCE RESULTS

In this section we consider three classes of generalized weighted monotone mappings . Then we establish some existence results for a solution of ($\Psi -$WEPP).

Definition 3.1. *A family* $\left(f_i\right)_{i\in I}$ *of functions is said to be*:

(i) *weighted monotone wrt* $\left(W,\Psi\right)$ *if for all* $x, y \in K$ *we have*

$$\sum_{i\in I} W_i \cdot \left(\Psi_i\left(f_i\left(y\right), x_i; y_i\right) - \Psi_i\left(f_i\left(x\right), x_i; y_i\right)\right) \le 0 ,$$

and weighted strictly monotone wrt $\left(W,\Psi\right)$ *if the inequality is strict for all* $x \ne y$;

(ii) *weighted pseudomonotone wrt* $\left(W,\Psi\right)$ *if for all* $x, y \in K$ *we have*

$$\sum_{i\in I} W_i \cdot \left(\Psi_i\left(f_i\left(x\right), x_i; y_i\right)\right) \le 0 \Rightarrow \sum_{i\in I} W_i \cdot \left(\Psi_i\left(f_i\left(y\right), x_i; y_i\right)\right) \le 0 ,$$

*and weighted strictly pseudomonotone wr*t $\left(W,\Psi\right)$ *if the second inequality is strict for all* $x \ne y$;

(iii) *weighted maximal pseudomonotone wrt* $\left(W,\Psi\right)$ *if it is weighted pseudomonotone wrt* $\left(W,\Psi\right)$ *and for all* $x, y \in K$ *we have*

$$\sum_{i\in I} W_i \cdot \left(\Psi_i\left(f_i\left(z\right), x_i; y_i\right)\right) \le 0 \forall z \in (x, y] \Rightarrow \sum_{i\in I} W_i \cdot \left(\Psi_i\left(f_i\left(x\right), x_i; y_i\right)\right) \le 0 \tag{3.1}$$

where $\left(x, y\right] = \prod_{i\in I} (x_i, y_i]$, *and weighted maximal strictly pseudomonotone wrt* $\left(W,\Psi\right)$ *if it is weighted strictly pseudomonotone wrt* $\left(W,\Psi\right)$ *and* (3.1) *holds.*

Definition 3.2. *A family* $\left(f_i\right)_{i\in I}$ *of functions is said to be weighted hemicontinuous wrt* $\left(W,\Psi\right)$ *if for all* $x, y \in K$ *and* $\lambda \in \left[0,1\right]$ *the mapping* $\lambda \mapsto \sum_{i\in I} W_i \cdot \Psi_i\left(f_i\left(x + \lambda\left(y - x\right)\right), x_i; y_i\right)$ *is continuous.*

Proposition 3.1. *We suppose that the family* $\left(f_i\right)_{i\in I}$ *of functions satisfies the following conditions*:

i) *it is weighted hemicontinuous and weighted pseudomonotone wrt* $\left(W,\Psi\right)$;

ii) *for any* $i \in I$ *and* $\lambda \in \left[0,1\right]$,

$$\Psi_i\left(f_i\left(x + \lambda\left(y - x\right)\right), x_i; x_i + \lambda(y_i - x_i)\right) = \lambda^{\tau} \Psi_i\left(f_i\left(x + \lambda\left(y - x\right)\right), x_i; y_i\right),$$

where $\tau > 0$ *is a fixed real constant.*

Then it is weighted maximal pseudomonotone wrt $\left(W,\Psi\right)$.

Theorem 3.1. *Assume that*

(i_1) *the family* $\left(f_i\right)_{i\in I}$ *is weighted maximal pseudomonotone wrt* $\left(W,\Psi\right)$;

(i_2) *there exists a nonempty closed and compact subset* D *of* K *and* $\tilde{y} \in D$ *such that, for all* $x \in K \setminus D$,

$$\sum_{i \in I} W_i \cdot \left(\Psi_i \left(f_i(x), x_i; \tilde{y}_i \right) \right) \le 0 ;$$

(i_3) *the mapping* $y \to \sum_{i \in I} W_i \cdot \left(\Psi_i \left(f_i(x), x_i; y_i \right) \right)$ *is convex on* K;

(i_4) $\sum_{i \in I} W_i \cdot \left(\Psi_i \left(f_i(x), x_i; x_i \right) \right) = 0$ *for all* $x \in K$;

(i_5) *for any* $A \in \mathcal{F}(K)$, *and* $x, y \in \mathrm{co}A$, *and every net* $\{x^\alpha\}_{\alpha \in \Gamma}$ *in* K *converging to* x, *we have*

$$\liminf_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i \left(f_i(x^\alpha), x_i^\alpha; y_i \right) = \sum_{i \in I} \Psi_i \left(f_i(x), x_i; y_i \right).$$

Then there exists a solution $x \in K$ of (Ψ − WEPP), *hence of* (Ψ − SWEP). *Furthermore, if* $W \in \prod_{i=1}^{n} T_+^{l_i}$ *, then there exists a normalized solution* $x \in K$ *of* (Ψ − WEPP), *hence of* $(\Psi - \mathrm{SVEP})_w$. *Also, if* $W \in \prod_{i=1}^{n} \mathrm{int}\, T_+^{l_i}$ *then* $x \in K$ *is a solution of* (Ψ − SVEP).

Remark 3.1. In the proof of Theorem 3.1, Theorem 2.2 from [6] is used.

From Theorem 3.1.we obtain

Theorem 3.2. *Assume that*

(j_1) *the family* $(f_i)_{i \in I}$ *is weighted maximal strictly pseudomonotone wrt* (W, Ψ);

(j_2) *there exists a nonempty closed and compact subset* D *of* K *and* $\tilde{y} \in D$ *such that, for all* $x \in K \setminus D$,

$$\sum_{i \in I} W_i \cdot \left(\Psi_i \left(f_i(x), x_i; \tilde{y}_i \right) \right) > 0 ;$$

(j_3) $\sum_{i \in I} W_i \cdot \left(\Psi_i \left(f_i(x), x_i; y_i \right) + \Psi_i \left(f_i(x), y_i; x_i \right) \right) = 0$ *for all* $x, y \in K$.

Then there exists an unique solution of (Ψ − WEPP), *hence it is the unique solution of* (Ψ − SWEP). *Moreover, if* $W \in \prod_{i=1}^{n} T_+^{l_i}$ *then there exists an unique normalized solution* $x \in K$ *of* (Ψ − WEPP) *which is also the unique solution of* $(\Psi - \mathrm{SVEP})_w$. *For* $W \in \prod_{i=1}^{n} \mathrm{int} T_+^{l_i}$, $x \in K$ *is the unique solution of* (Ψ − SVEP).

Remark 3.2. For the case of variational inequalities, see [1].

Definition 3.1. *We say that* f *is weighted* B-*pseudomonotone wrt* (W, Ψ) *if for each* $x \in K$ *and every net* $\{x^\alpha\}_{\alpha \in \Gamma}$ *in* K *converging to* x *with* $\limsup_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i \left(f_i(x^\alpha), x_i^\alpha; x_i \right) \ge 0$, *we have*

$$\limsup_{\alpha \in \Gamma} \sum_{i \in I} \Psi_i \left(f_i(x^\alpha), x_i^\alpha; y_i \right) \le \sum_{i \in I} \Psi_i \left(f_i(x), x_i; y_i \right) \text{ for all } y \in K.$$

Theorem 3.3. *Assume that*

(k_1) *the family* $(f_i)_{i \in I}$ *is weighted* B-*pseudomonotone wrt* (W, Ψ) *such that, for each* $A \in \mathcal{F}(K)$, *the mapping* $x \to \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i)$ *is lower semicontinuous on* $\mathrm{co}A$;

(k_2) *there exists a nonempty closed compact subset* D *of* K *and* $\tilde{y} \in \mathrm{D}$ *such that*

$$\sum_{i \in I} \Psi_i\left(f_i(x), x_i; \tilde{y}_i\right) < 0 \text{ for all } x \in K \setminus D;$$

(k_3) $\sum_{i \in I} \Psi_i\left(f_i(x), x_i; x_i\right) = 0$, *for all* $x \in K$.

Then there exists a solution $x \in K$ *of* ($\Psi -$WEPP), *hence a solution of* ($\Psi -$SWEP). *Furthermore, if* $W \in \prod_{i=1}^{n} T_+^{l_i}$ *then there exists a normalized solution* $x \in K$ *of* ($\Psi -$WEPP) *which is also a solution of* $(\Psi - \mathrm{SVEP})_w$, *and for* $W \in \prod_{i=1}^{n} \operatorname{int} T_+^{l_i}$, $x \in K$ *is a solution of* ($\Psi -$SVEP).

Remark 3.3. In order to prove the above result, we used a fixed point theorem from [7].

Remark 3.4. The case of relatively B-pseudomonotonicity is studied in [4].

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