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THE GENERALIZED SOLUTION OF THE BOUNDARY-VALUE PROBLEMS REGARDING THE BENDING OF ELASTIC RODS ON ELASTIC FOUNDATION. I. THE SYSTEM OF GENERALIZED EQUATIONS

Antonela TOMA

The "Politehnica" University of Bucharest
e-mail: antonela2222@yahoo.com

The complete system of equations which describes the bending of the elastic rods on elastic foundation, using the properties of the convolution algebra D'_+, is determined. The obtained equations contain the given and constrained loads as well as the boundary conditions of the problem.

Key words: rods theory, elastic foundation, jump discontinuities, distributions theory

1. INTRODUCTION

In solving the problem of the bending of the elastic rod on elastic foundation we come across difficulties owing to the following factors:

1. The rod is loaded with discontinuous loads;
2. The action of some concentrated loads and moments;
3. Discontinuities of the mechanical properties of the rod and of the elastic foundation.

As a result the classical mathematical analysis (with the classical definition of a derivative) can be applied only on the rod parts where the loads and the mechanical properties of the rod and foundation do not have discontinuities.

The classical method of solving the problems in which appear discontinuities is the partition of the rod into segments (which have distinct mechanical and geometrical properties). We obtain a system of boundary (the ends of the segments rod) value problems so that the solution of the problem on each rod segment is continuous. To solve the problem with discontinuities we must take into account the continuity conditions at the interface of the rod segments.

This method was used in [4] to obtain the displacement solution for steady-state longitudinal vibrations of an inhomogeneous elastic rod having n homogeneous elastic segments.

The general and unitary method to deal with the problems concerning external discontinuities (e.g. discontinuous loading) and internal discontinuities (e.g. owning to the mechanical properties) is the distribution theory.

In the framework of this theory we obtain a single equation which contains the boundary, initial and jump conditions.

The distribution theory was used for analyzing beams with internal and external discontinuities, [8], [9], [11], [12].

A bending problem with discontinuities in which the distribution theory is not systematically applied, being a combination between classical mathematical analysis and the distribution theory, is studied in [12].

The use of the distribution theory and of the convolution algebra $D'_+ \subset D'(\mathbb{R})$, is more efficient, because the unit element is the Dirac distribution $\delta(x) \in D'_+$ and on the other hand this algebra is without divisors of zero.

With the help of the fundamental solution in D'_+ of the operator which describes the bending of the elastic rods on elastic foundation, the general expression of deflection $\tilde{v}$ in the distribution space D'_+ is given for any kind of rod loading.

From the condition that the deflection $\tilde{v}$ should have the support in $[a,b]$, there are obtained four equations. These equations ensure the determination of all unknowns which appear such as: constraint concentrated loads and moments due to the fixing of the rod, the deflection jumps and their derivatives at the rod ends.

2. THE SYSTEM OF GENERALIZED EQUATIONS OF THE ELASTIC RODS BENDING ON ELASTIC FOUNDATION

Let us consider a straight homogeneous elastic rod of finite length, $x \in [a,b]$, and with constant section (Fig. 1.1), where Ox represents the rod axis. We shall denote by $v(x),\ q(x),\ T(x),\ M(x),$ $x \in [a,b]$ the deflection (displacement), the intensity of the distributed load, shearing force and bending moment, respectively.

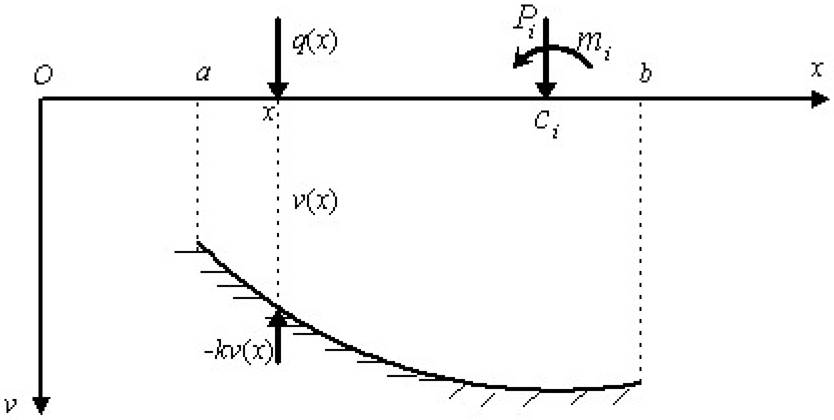

Fig. 1.1. Elastic rod on elastic foundation

We denote by $\tilde{\partial}_x = \frac{\tilde{d}}{dx}$, $\partial_x = \frac{d}{dx}$ the derivative in classic sense and the derivative in distribution sense, respectively. Then [6] the equations system of bending of elastic rods is

$$\tilde{\partial}_x T(x) + q(x) = 0, T(x) = \tilde{\partial}_x M(x), M(x) = -EI\tilde{\partial}_x^2 v(x), x \in \bigcup_{i=1}^{n-1}(c_i, c_{i+1}) \subset [a,b], c_1 = a, c_n = b, \tag{2.1}$$

where EI represents the bending stiffness.

At the points $c_1 = a,\ c_2,\ c_3,\ ...,\ c_n = b,$ can act given or constraint concentrated loads i.e. concentrated forces P_i and moments m_i, $i = \overline{1,n}$ (Fig. 1.1), so that

$$v(x) \in C^1([a,b]) \cap C^4(J), q(x) \in C^0(J), T(x) \in C^1(J), M(x) \in C^2(J), \tag{2.2}$$

where $J = \bigcup_{i=1}^{n-1}(c_i, c_{i+1})$, $c_1 = a,\ c_n = b$.

The rod is acted upon by external loads, being thus deformed, and lies on an elastic foundation, interacting with it. The foundation acts over the rod with reactions distributed all over its length, by opposing to it. The simplest model for an elastic rod on elastic foundation is the Winkler one [3]. For a Winkler model, it is assumed that the reaction of the elastic foundation $q_e(x), x \in [a,b]$ exerted on the rod is proportional to the deflection of it at that point and is independent of the deflection of other parts of the foundation hence

$$q_e(x) = -kv(x),\ x \in [a,b], \tag{2.3}$$

where k is called the rigidity coefficient of the elastic foundation.

The reaction of the elastic foundation must be considered as an additional load with respect to the given ones, working normal on the rod axis.

Consequently, taking into account (2.1) the equations system of the bending of elastic rods on elastic foundation is

$$\tilde{\partial}_x T(x) + q(x) - kv(x) = 0, T(x) = \tilde{\partial}_x M(x), M(x) = -EI\tilde{\partial}_x^2 v(x), \tag{2.4}$$

where $x \in \bigcup_{i=1}^{n-1}(c_i, c_{i+1}) \subset [a,b],\ c_1 = a,\ c_n = b$.

To rewrite the system (2.4) in the distribution space $D'(\mathbb{R})$ the functions $v,\ q,\ T,\ M,$ will be analytically prolonged with null values out of the interval $\left[a,b\right]$ and we shall denote them by $\tilde{v},\ \tilde{q},\ \tilde{T},\ \tilde{M},\ x \in \mathbb{R}$, respectively.

Thus we have

$$\tilde{v}(x) = \begin{cases} v(x), & x \in \left[a,b\right] \\ 0, & x \notin \left[a,b\right] \end{cases}, \quad \tilde{q}(x) = \begin{cases} q(x), & x \in \left[a,b\right] \\ 0, & x \notin \left[a,b\right] \end{cases}$$
$$\tilde{T}(x) = \begin{cases} T(x), & x \in \left[a,b\right] \\ 0, & x \notin \left[a,b\right] \end{cases}, \quad \tilde{M}(x) = \begin{cases} M(x), & x \in \left[a,b\right] \\ 0, & x \notin \left[a,b\right] \end{cases}, \tag{2.5}$$

and the equations (2.4) becomes

$$\tilde{\partial}_x \tilde{T}(x) + \tilde{q}(x) - k\tilde{v}(x) = 0, \tilde{T}(x) = \tilde{\partial}_x \tilde{M}(x), \tilde{M}(x) = -EI\tilde{\partial}_x^2 \tilde{v}(x), x \in \mathbb{R} - \left\{c_1, c_2, c_3, ..., c_n\right\}. \tag{2.6}$$

We mention that the shearing force $T(x),$ at the point $x \in \left[a,b\right]$ of the rod represents the resultant of all given and constraint normal forces from the interval $(x,b]$.

Analogously, the bending moment $M(x),$ at the point $x \in \left[a,b\right],$ represents the sum of the given and constraint moments from the interval $(x,b]$ with respect to x.

The moment of a force with respect to the points x as well as a concentrated moment are considered positive if they determine a clockwise rotation.

The concentrated and the distributed loads are considered positive if they act in the sense of the Ov axis.

We shall denote by $D'(\mathbb{R})$ the distribution (continuous linear functional) defined on the test functions space $D(\mathbb{R})$, which are indefinite derivable functions with compact support.

We denote by $D'_+ \subset D'(\mathbb{R})$ the distributions from $D'(\mathbb{R})$ having the supports on $[0,\infty)$. We mention that the distributions from D'_+ represent a convolution algebra without divisors of zero. Consequently, if $f, g \in D'_+$, hence $\operatorname{supp} f, \operatorname{supp} g \subset [0,\infty)$ and $f * g = 0$, then $f = 0$ or $g = 0,$ where $f * g$ is defined as follows

$$\left(f * g, \varphi\right) = \left(f(x) \times g(y), \varphi(x+y)\right) = \left(f(x), \left(g(y), \varphi(x+y)\right)\right),\ \varphi \in D(\mathbb{R}). \tag{2.7}$$

We observe that mechanical quantities $\tilde{v},\ \tilde{q},\ \tilde{T},\ \tilde{M},$ defined by (2.5) represent function type distributions from D'_+, because their supports are in $[a,b] \subset [0,\infty)$.

For these functions and their derivatives, the points $c_i, i = \overline{1,n}$ generally represent points of discontinuities of the first kind. Thus, the action point c_i of the concentrated force P_i represents a discontinuity point of the first kind for the shearing force $\tilde{T}(x),\ x\in\mathbb{R}$ and the derivative $\tilde{\partial}_x\tilde{M}(x)$ of the bending moment $\tilde{M}(x),\ x\in\mathbb{R}$.

Also, the action point c_i of the concentrated moment of intensity m_i (which determines the clockwise rotation) is the discontinuity point of first kind for the bending moment $\tilde{M}(x),\ x\in\mathbb{R}$ and the ordinary point for the shearing force $\tilde{T}(x),\ x\in\mathbb{R}$.

Consequently, a point $c_i,\ i=\overline{1,n}$ can be a discontinuity point of the first kind both for $\tilde{T}(x)$ and $\tilde{M}(x)$.

Taking into account the definition of the shearing force and of the bending moment we state [7]:

The jump of the shearing force $\tilde{T}(x),\ x\in\mathbb{R}$ at a point $c_i,\ i=\overline{1,n}$ denoted by $\left[\tilde{T}(x)\right]_{c_i}$, has the expression

$$\left[\tilde{T}(x)\right]_{c_i}=-P_i=\tilde{T}(c_i+0)-\tilde{T}(c_i-0), \tag{2.8}$$

where P_i represents the intensity of the concentrated force applied at the point c_i .

The jump of the bending moment $\tilde{M}(x),\ x\in\mathbb{R}$ at a point $c_i,\ i=\overline{1,n}$ denoted by $\left[\tilde{M}(x)\right]_{c_i}$, has the expression

$$\left[\tilde{M}(x)\right]_{c_i}=-m_i=\tilde{M}(c_i+0)-\tilde{M}(c_i-0), \tag{2.9}$$

where m_i represents the intensity of the concentrated moment applied at c_i , having the direct rotation sense shown in Fig. 1.1.

According to [5], we state

Let be f a real-value function of class $C^1(\mathbb{R})$ excepting the points $c_i,\ i=\overline{1,n}$, where it has discontinuities of first kind with the jump $\left[f(x)\right]_{c_i}=f(c_i+0)-f(c_i-0)$; then

$$\partial_x f(x)=\tilde{\partial}_x f(x)+\sum_{i=1}^{n}\left[f(x)\right]_{c_i}\delta(x-c_i), \tag{2.10}$$

where $\delta(x-c_i)\in D'(\mathbb{R})$ represents the Dirac distribution concentrated at the point c_i .

Using the formula (2.9), we state

The function type distributions $\tilde{v}(x),\tilde{q}(x),\tilde{T}(x),\tilde{m}(x)\in D'_+$ defined by (2.5) satisfy in D'_+ the following equations

$$\partial_x\tilde{T}(x)+\tilde{q}(x)-k\tilde{v}(x)=-\sum_{i=1}^{n}P_i\delta(x-c_i), \tag{2.11}$$

$$\partial_x\tilde{M}(x)-\tilde{T}(x)=-\sum_{i=1}^{n}m_i\delta(x-c_i), \tag{2.12}$$

$$EI\partial_x^2\tilde{v}(x)+\tilde{M}(x)=EI\Big(\left[\tilde{v}\right]_a\delta'(x-a)+\left[\tilde{v}\right]_b\delta'(x-b)+\left[\tilde{\partial}_x\tilde{v}\right]_a\delta(x-a)+\left[\tilde{\partial}_x\tilde{v}\right]_b\delta(x-b)\Big), \tag{2.13}$$

$$EI\partial_x^4\tilde{v}(x)+k\tilde{v}(x)=q_1+EI\Big(\left[\tilde{v}\right]_a\delta'''(x-a)+\left[\tilde{v}\right]_b\delta'''(x-b)+ \\ +\left[\tilde{\partial}_x\tilde{v}\right]_a\delta''(x-a)+\left[\tilde{\partial}_x\tilde{v}\right]_b\delta''(x-b)\Big), \tag{2.14}$$

where the distribution $q_1(x)\in D'_+$ has the expression

$$q_1(x)=\tilde{q}(x)+\sum_{i=1}^{n}P_i\delta(x-c_i)+\sum_{i=1}^{n}m_i\delta'(x-c_i). \tag{2.15}$$

This distribution represents the resultant of the densities of the distributed loads $\tilde{q}(x)$, of the concentrated forces $\sum_{i=1}^{n}P_i\delta(x-c_i)$ and of the concentrated moments $\sum_{i=1}^{n}m_i\delta'(x-c_i)$ given and constraint. The symbol $[\;\;]_{c_i}$ represents the jump of a certain value at the point $x=c_i$, and P_i, m_i represent the intensity of the concentrated forces and the intensity of the concentrated moment at the point $c_i,\, i=\overline{1,n}$, respectively

Indeed, because the shearing force $\tilde{T}$ is of class $C^1(\mathbb{R})$ excepting the discontinuity points of first kind $c_i,\, i=\overline{1,n}$; on the basis of the formulas (2.7) and (2.10) we have

$$\partial_x\tilde{T}(x)=\tilde{\partial}_x\tilde{T}(x)+\sum_{i=1}^{n}\left[\tilde{T}\right]_{c_i}\delta(x-c_i)=\tilde{\partial}_x\tilde{T}(x)-\sum_{i=1}^{n}P_i\delta(x-c_i). \tag{2.16}$$

Using the first equation from (2.6) we obtain

$$\partial_x\tilde{T}(x)+\tilde{q}(x)-k\tilde{v}(x)=-\sum_{i=1}^{n}P_i\delta(x-c_i), \tag{2.17}$$

namely the equation (2.11). Proceeding analogously and taking into account that the bending moment $\tilde{M}$ is of class $C^2(\mathbb{R})$ excepting the discontinuity point of the first kind $c_i,\, i=\overline{1,n}$, we can write

$$\partial_x\tilde{M}(x)=\tilde{\partial}_x\tilde{M}(x)+\sum_{i=1}^{n}\left[\tilde{M}\right]_{c_i}\delta(x-c_i). \tag{2.18}$$

Taking into account the formulas (2.9) and (2.6), we obtain

$$\partial_x\tilde{M}(x)-\tilde{T}(x)=-\sum_{i=1}^{n}m_i\delta(x-c_i), \tag{2.19}$$

namely the relation (2.12).

From (2.2) and (2.5), it results that the deflection $\tilde{v}\in C^1(\mathbb{R})$ excepting the discontinuity points of the first kind $c_1=a$ and $c_2=b$.

Consequently, we have

$$\partial_x\tilde{v}(x)=\tilde{\partial}_x\tilde{v}(x)+\left[\tilde{v}\right]_a\delta(x-a)+\left[\tilde{v}\right]_b\delta(x-b). \tag{2.20}$$

The differentiation of the above relation in the distribution sense yields

$$\partial_x\tilde{v}(x)=\tilde{\partial}_x\tilde{v}(x)+\left[\tilde{v}\right]_a\delta(x-a)+\left[\tilde{v}\right]_b\delta(x-b). \tag{2.21}$$

On the basis of the last relation from (2.6), we obtain

$$EI\partial_x^2\tilde{v}(x) = EI\tilde{\partial}_x^2\tilde{v}(x) + EI\left(\left[\tilde{v}\right]_a \delta'(x-a) + \left[\tilde{v}\right]_b \delta'(x-b) + \left[\tilde{\partial}_x\tilde{v}\right]_a \delta(x-a) + \left[\tilde{\partial}_x\tilde{v}\right]_b \delta(x-b)\right)$$
$$-\tilde{M} + EI\left(\left[\tilde{v}\right]_a \delta'(x-a) + \left[\tilde{v}\right]_b \delta'(x-b) + \left[\tilde{\partial}_x\tilde{v}\right]_a \delta(x-a) + \left[\tilde{\partial}_x\tilde{v}\right]_b \delta(x-b)\right), \quad (2.22)$$

namely the relation (2.13).

As far as the equation (2.14) is concerned, this is obtained by the elimination of the values $\tilde{T}$ and $\tilde{M}$ from the equation (2.11) and by the equations obtained from the first derivative of the equation (2.12) and the second derivative of the equation (2.13).

We remark that the equations (2.11), (2.12) and (2.13) represent the complete system of equations of bending the elastic rod of on an elastic foundation in the distributions space $D'_+ \subset D'(\mathbb{R}),$ with respect to the mechanical quantities $\tilde{v}, \tilde{T},$ and $\tilde{M}$.

The fourth order differential equation (2.14) represents the bending equation of elastic rods on an elastic foundation and it is written only with respect to the deflection $\tilde{v}$ of the rod in the distributions space D'_+ .

3. CONCLUSION

The obtained system of equations in the distribution space D'_+ displays the following advantages:

1. it incorporates all the continuous and discrete loads;
2. it contains the boundary conditions as well as the constrained loads;
3. due to the writing of the equations in D'_+ we can determine the expressions of all unknowns of the problem under a unitary and general form.

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