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A NUMERICAL SOLUTION FOR THE EQUATION OF THE LIFTING LINE INCLUDING GROUND AND TUNNEL EFFECTS

Lazăr DRAGOŞ[*,**], Adrian CARABINEANU[*,**], Raluca DUMITRACHE[***]

[*] Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, 010014, Bucharest, Romania
[**] Institute of Mathematical Statistics and Applied Mathematics "Gh. Mihoc – C. Iacob" , Romanian Academy, Calea 13 Septembrie 13, 050711, Bucharest, Romania
[***] Faculty of Applied Sciences, Chair of Mathematics II, University Politehnica of Bucharest, 060042, Bucharest, Romania
Corresponding author: Adrian CARABINEANU, e-mail: acara@fmi.unibuc.ro

Using the Gauss-type quadrature formulas one discretizes the hyper singular integral equations of the lifting line with ground or tunnel effects. Numerical calculations are performed for the elliptical and rectangular flat wings. Ground and tunnel effects are pointed out.

Key words: lifting line; hyper singular integral equation; discretization; ground and tunnel effects.

1. INTRODUCTION

Prandtl's lifing line theory [1], [6], [14] is the first mathematical model for the three-dimensional wing (the finite span airfoil). It was elaborated in 1918 by L. Prandtl and it remained during the first half of the 20-th century the only theory for this wing. The German scientist, gifted with an extraordinary engineering intuition, guessed very well the simplifications which may be performed in order to obtain an integro-differential equation for the circulation in a cross section of the wing. Later, Homentcovschi [12], Dragoş [9] have shown that the one-dimensional lifting line equation may be obtained from the two-dimensional lifting surface equation by means of a technique which consists in fact in the asymptotic expansion of the kernel with respect to the aspect ratio. Lifting line equation, the equation of the jump of the pressure for the wings of low aspect ratio and the integral equations for the two-dimensional profiles are the one-dimensional integral equations of aerodynamics for which numerous numerical schemes were conceived (we mention for example [2], [3], [4], [5], but the list is much longer). Apart from the mathematical theory (where we have to take also into account the boundary layer and turbulence effects) many experiments are performed in the wind tunnels. For the interpretation of the experimental results one has to perform wall corrections [13]. For these corrections one has to conceive mathematical models and one of the models is presented herein. When one of the walls is far from the wing (or it is missing) instead of tunnel effects one encounters ground effects. The ground effect is understood as an increase in the lift-to-drag ratio of a wing moving close to the ground. In order to exploit the ground effect a wing-in-ground-effect vehicle has been conceived. It is the ekranoplan which can be defined as a vehicle with an engine and heavier than air that is designated to fly close to an underlying surface for efficient utilization of the ground effect [15]. In our paper we employ Gauss-type quadrature formulas for solving the lifting line equations in ground and tunnel effects. We employ the lifting line equation in ground effects given by Dragoş in [6]. For obtaining this equation, in order to satisfy the slipping condition on the ground, Dragoş placed in the free fluid stream another wing symmetric to the original one with respect to the ground-plane [6], [10]. This is in fact a variant of the well known image method. In our paper, in order to study the tunnel effects, we generalize this method by placing a grid of wings such that each wall of the tunnel should be a plane of symmetry for the wings from the cascade. Utilizing the integration by parts we transform Prandtl's integro-differential equation into an equation containing the finite part of a hyper singular integral with a generalized Cauchy kernel. For this integral we utilize the very efficient Gauss-type quadrature formulas given by Dragoş in [6], [7], [8]. In the case of elliptical flat wing we compare the numerical results with analytical ones and we notice a perfect agreement.

We also notice that in the presence of the walls of the tunnel, the circulation increases. We calculate the lift coefficient for the rectangular flat wing in ground effects and we notice that the coefficient increases as the wing comes near the ground.

2. THE STATEMENT OF THE PROBLEM

A uniform subsonic stream (Mach number *M<1*) is perturbed by a thin airfoil having the equation (in dimensional variables)

$$z = h(x, y), (x, y) \in D, |h| << 1, \left|h_x{}'\right| << 1. \tag{1}$$

Let

$$x = x_{\pm}(y), -b \le y \le b\,,\ \left|\frac{x_+ - x_-}{b}\right| << 1 \tag{2}$$

be the equations of the leading and trailing edges of the airfoil. For an airfoil placed in a free uniform flow we have Prandtl's lifting line equation [1], [6], [14]:

$$2\beta C(y) - a(y)\int_{-b}^{b}{}^{*}\frac{C(\eta)}{(\eta - y)^2}d\eta = 2\,j(y) \tag{3}$$

where the unknown is the circulation *C(y);*, $\beta = \sqrt{1 - M^2}$, $j(y) = -2\int_{x_-(y)}^{x_+(y)}\sqrt{\frac{x - x_-(y)}{x_+(y) - x}}h_x{}'(x, y)dx$ and $a(y) = \frac{x_+(y) - x_-(y)}{2}$. The star "*" stands for the finite part of the hyper singular integral.

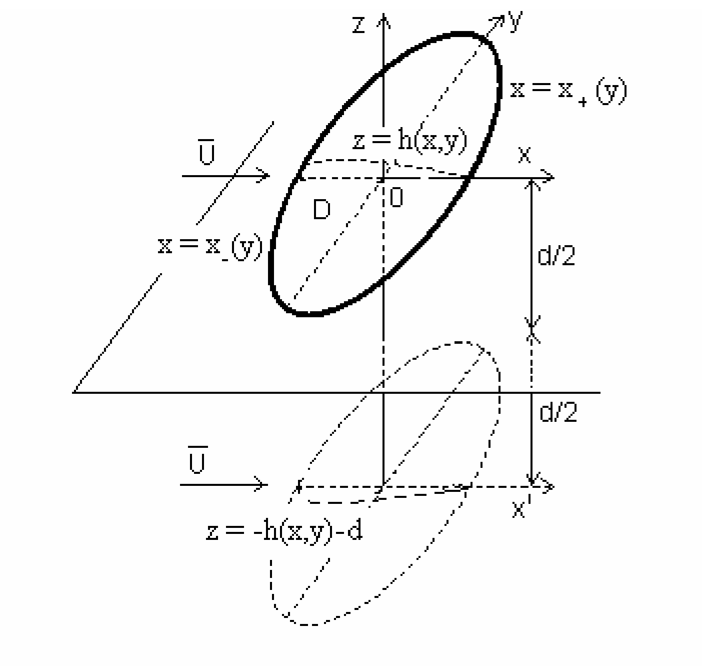

Figure 1.

According to Dragoş [6], if we want to take into account the ground effects (we consider that $z=-\frac{d}{2}$ is the equation of the solid plane representing the ground, where we impose the slipping condition which states that the normal component of the velocity must vanish) we have to modify Prandtl's equation which becomes

$$2\beta C(y)-a(y)\int_{-b}^{b}{}^{*}\frac{C(\eta)}{(\eta-y)^2}d\eta+\int_{-b}^{b}C(\eta)N_0(y,y_0)=2j(y). \tag{4}$$

We notice that a new kernel has been added

$$N_0(y,y_0,-d)=\frac{d^2-{y_0}^2}{(d^2+y^2)^2}\left[I_1(y,y_0)-a(y)\right]+\frac{d^2\beta^2}{d^2+{y_0}^2}I_3(y,y_0) \tag{5}$$

with $y_0=y-\eta$ and

$$I_\upsilon(y,y_0)=-\frac{1}{\pi}\int_{x_-(y)}^{x_+(y)}\sqrt{\frac{x-x_-(y)}{x_+(y)-x}}\frac{x}{\left(x^2+\beta^2({y_0}^2+d^2)\right)^{\frac{\upsilon}{2}}}dx,\upsilon=\overline{1,3} \tag{6}$$

In order to obtain eq. (4) one employed the image method (figure 1). According to this method the presence of an airfoil symmetric to the initial one with respect to the $z=-\frac{d}{2}$ plane ensures the achievement of the slipping condition on this plane..

In the sequel we shall consider a wing in the tunnel $-\frac{d}{2}<z<\frac{d_1}{2}$. The slipping conditions on the walls is achieved if instead of the original airfoil we consider an infinite grid of airfoils for which the walls of the tunnel represent planes of symmetry, i.e. we shall consider the airfoils

$$z=-h(x,y)-d+n(d_1+d),n=0,\pm1,\pm2,...,$$
$$z=h(x,y)+n(d_1+d),n=0,\pm1,\pm2,....$$

Indeed, the airfoils $z=-h(x,y)-d+n(d_1+d)$ are the symmetric of the airfoils $z=h(x,y)-n(d_1+d)$ with respect to the plane $z=-\frac{d}{2}$ and the airfoils $z=-h(x,y)-d-n(d_1+d)$ are the symmetric of the airfoils $z=h(x,y)+(n+1)(d_1+d)$ with respect to the plane $z=\frac{d_1}{2}$.

Adding for each additional airfoil of the grid a kernel of form (5) into Prandt'l equation we obtain the following equation for the circulation in a cross section of the lifting line in a tunnel:

$$\begin{aligned}&2\beta C(y)-a(y)\int_{-b}^{b}{}^{*}\frac{C(\eta)}{(\eta-y)^2}d\eta+\sum_{n=-\infty}^{\infty}\int_{-b}^{b}C(\eta)N_0(y,y_{0,}-d+n(d_1+d))d\eta-\\&\sum_{n=-\infty,n\neq0}^{\infty}\int_{-b}^{b}C(\eta)N_0(y,y_0,n(d_1+d))d\eta=2j(y)\end{aligned} \tag{7}$$

Performing the change of variable

$$x=s(y)+a(y)t,\ \ s(y)=\frac{x_+(y)+x_-(y)}{2}, \tag{8}$$

we get for the integrals I_υ the expression :

$$I_\upsilon(y,y_0)=-\frac{1}{\pi}a(y)s(y)\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{dt}{\left(\sqrt{P(t)}\right)^\upsilon}-\frac{1}{\pi}a^2(y)\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{t}{\left(\sqrt{P(t)}\right)^\upsilon}dt \quad ; \quad \upsilon=1,3 \tag{9}$$

$$P(t)=a^2(y)t^2+2a(y)s(y)t+s^2(y)+\beta^2({y_0}^2+D^2)\,. \tag{10}$$

3.THE DISCRETIZATION OF THE LIFTING LINE EQUATION

Performing the change of variables $y=by',\eta=b\eta',y_0=\eta y_0'$ and denoting again $\eta,y,y_0,C(\eta),j(y),a(y)$ instead of $\eta',y',y_0',C(b\eta'),j(by'),a(by')$, equation (7) becomes:

$$\begin{aligned}&2b\beta C(y)-a(y)\int_{-1}^{1}{}^{*}\frac{C(\eta)}{(\eta-y)^2}d\eta+b^2\sum_{n=-\infty}^{\infty}\int_{-1}^{1}C(\eta)N_0(by,by_0,-d+n(d_1+d))d\eta-\\&-b^2\sum_{n=-\infty}^{\infty}\int_{-1}^{1}C(\eta)N_0(by,by_0,n(d_1+d))d\eta=2bj(y)\end{aligned} \tag{11}$$

Since $C(\pm1)=0$, we shall seek for a solution of equation (11) having the form $C(y)=\sqrt{1-y^2}c(y)$ where $c(\pm1)$ is finite.

For calculating the integrals from (9) and (11) we employ the Gauss quadrature formulas given in [6] and [8]. Denoting $c_j=c(y_j),a_j=a(y_j),j_j=j(y_j)$ for $j=1,...,m$, we have:

$$\int_{-1}^{1}{}^{*}\sqrt{1-\eta^2}\frac{c(\eta)}{(\eta-y_j)^2}d\eta=\frac{\pi}{m+1}\sum_{k=1,k\neq j}^{m}\left[1-(-1)^{k+j}\right]\frac{1-{y_k}^2}{(y_k-y_j)^2}c_k-(m+1)\frac{\pi}{2}c_j \tag{12}$$

$$\int_{-1}^{1}\sqrt{1-\eta^2}c(\eta)N_0(by_j,by_j-b\eta,e)d\eta=\frac{\pi}{m+1}\sum_{k=1}^{m}\left(1-{y_k}^2\right)N_0(by_j,by_j-b\eta,e)c_k \tag{13}$$

with

$$y_j=\cos\left(\frac{j\pi}{m+1}\right),\ j=1,...,m. \tag{14}$$

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{1}{\left(\sqrt{P(t)}\right)^\upsilon}dt=\frac{2\pi}{2m+1}\sum_{\alpha=1}^{m}(1+t_\alpha)\frac{1}{\left(\sqrt{P_{jk}(t_\alpha)}\right)^\upsilon},\upsilon=1,2,3 \tag{15}$$

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{t}{\left(\sqrt{P(t)}\right)^\upsilon}dt=\frac{2\pi}{2m+1}\sum_{\alpha=1}^{m}(1+t_\alpha)\frac{t_\alpha}{\left(\sqrt{P_{jk}(t_\alpha)}\right)^\upsilon},\upsilon=1,2,3 \tag{16}$$

where $t_\alpha=\cos\left(\frac{2\alpha-1}{2m+1}\right)\pi,\alpha=1,...,m$.

Employing the previous quadrature formulas we calculate N_0 and the integrals I_υ as follows:

$$N_0(by_j, by_j - by_k, e) =$$
$$= \frac{e^2\beta^2}{e^2 + \left(y_j - y_k\right)^2} I_3\left(by_j, by_j - b_k\right) + \frac{e^2 - \left(by_j - by_k\right)^2}{e^2 + \left(by_j - by_k\right)^2}\left[-a_j + I_1(by_j, by_j - by_k)\right] \tag{17}$$

with

$$I_\upsilon(by_j, by_j - by_k) = -\frac{2}{2m+1} a_j s_j \sum_{\alpha=1}^{m} \frac{1+t_\alpha}{\left(\sqrt{P_{jk}(t_\alpha)}\right)^\upsilon} - \frac{2}{2m+1} a_j{}^2 \sum_{\alpha=1}^{m} \frac{(1+t_\alpha)t_\alpha}{\left(\sqrt{P_{jk}(t_\alpha)}\right)^\upsilon} \quad , \quad \upsilon = 1,3 \tag{18}$$

$$P_{jk}(t_\alpha) = \left(a_j t_\alpha + s_j\right)^2 + \beta^2\left[D^2 + \left(by_j - by_k\right)^2\right] \tag{19}$$

$$a_j = a(by_j), s_j = s(by_j) \tag{20}$$

By utilizing the quadrature formulas we discretize the lifting line equation and we obtain the linear algebraic system:

$$\sum_{k=1}^{m} A_{jk} c_k = 2bj_j, \; j = 1, \ldots, m. \tag{21}$$

where we denoted:

$$A_{jj} = 2b\beta\sqrt{1 - y_j{}^2} + a_j(m+1)\frac{\pi}{2} + b^2 \frac{\pi}{m+1} \sum_{n=-\infty}^{\infty} N_0\left(by_j, 0, -d + n(d_1 + d)\right)\left(1 - y_j{}^2\right) -$$
$$-b^2 \frac{\pi}{m+1} \sum_{n=-\infty, n\neq 0}^{\infty} N_0\left(by_j, 0, n(d_1 + d)\right)\left(1 - y_j{}^2\right) \tag{22}$$

$$A_{jk} = -a_j \frac{\pi}{m+1}\left[1 - (-1)^{k+j}\right] \frac{1 - y_k{}^2}{\left(y_k - y_j\right)^2} + b^2 \frac{\pi}{m+1} \sum_{n=-\infty}^{\infty} N_0\left(by_j, by_j - by_k, -d + n(d_1 + d)\right)\left(1 - y_k{}^2\right) -$$
$$-b^2 \frac{\pi}{m+1} \sum_{n=-\infty}^{\infty} N_0\left(by_j, by_j - by_k, n(d_1 + d)\right)\left(1 - y_k{}^2\right), k \neq j. \tag{23}$$

4. LIFT COEFFICIENT. NUMERICAL RESULTS

We shall study the flat elliptical airfoil with angle of attack ε i.e.

$$x_\pm = \pm\sqrt{1 - \frac{y^2}{b^2}}, h(x, y) = -\varepsilon x \tag{24}$$

For the airfoil in a free stream ($d \to \infty, d_1 \to \infty$) the exact analytical solution is known [6]:

$$c(y) = \frac{2\pi\varepsilon}{\beta + \frac{\pi}{2b}}. \tag{25}$$

Numerical results for the circulation *C(y)* have been obtained for $b=1$, $\varepsilon=0.1$, $m=10$, $\beta=1$, $n\in[-20,20]$ and for various values of d and d_1 in figure 2. We notice that for great values of d and d_1 the exact and the numeric results are very close.

For calculating the lift coefficient

$$C_L=\frac{2}{A}\int_{-b}^{b}C(y)dy$$

(where *A* stands for the area of the wing) one may employ the formula

$$C_L=\frac{2\pi b}{(n+1)}\sum_{i=1}^{m}\left(1-x_i^2\right)c_i. \tag{26}$$

The second wing that we are considering is the rectangular flat wing

$$x_{\pm}(y)=\pm1, h(x,y)=-\varepsilon x. \tag{27}$$

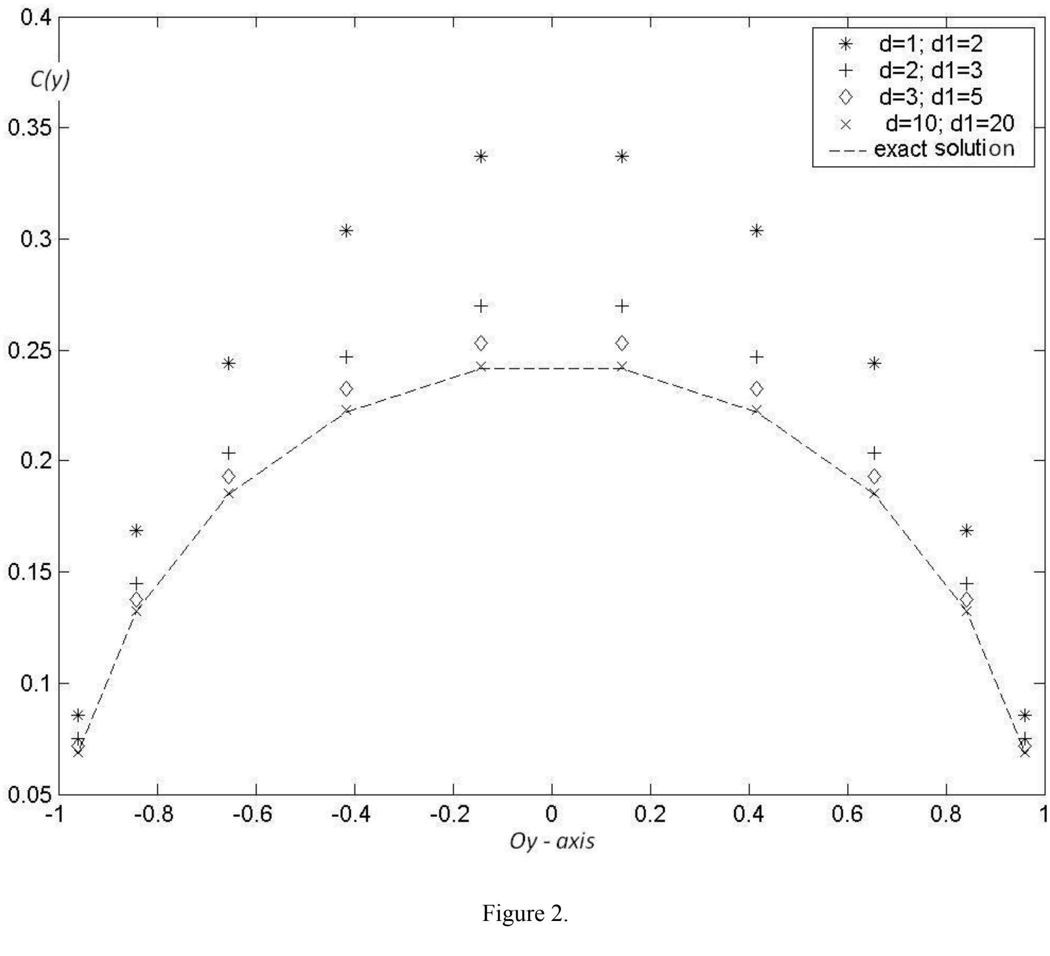

Figure 2.

We calculate for $\beta=1, b=10, d=100, d_1=100$ the coefficient $k_L=\dfrac{AC_L}{b\varepsilon}$ and we find for $m=20, n=20$ the value $k_L=0.511$. The same value was obtained in [11] for the rectangular wing in a free uniform stream (no ground or tunnel effects).

In table 1 we consider $d_1 = 100$ and we present the coefficient k_L versus d in order to investigate the ground effects. We notice that the lift coefficient increases as the wing comes near the ground.

Table 1

d	2	3	4	5	6	7	8	9	10	100
k_L	0.660	0.618	0.596	0.585	0.574	0.566	0.559	0.554	0.549	0.511

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