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NONDIFFERENTIABLE MATHEMATICAL PROGRAMS. OPTIMALITY AND HIGHER-ORDER DUALITY RESULTS

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For a general class of nondifferentiable mathematical programs, we give necessary optimality conditions and duality results for a higher-order dual of Mond-Weir type.

Key words: multiobjective programming, optimization.

1. INTRODUCTION

In this paper we consider a general class of nondifferentiable mathematical programming problems, namely,

$$\min f(x)+\sum_{j=1}^{s}(x^T B_j x)^{\frac{1}{2}} \qquad \text{subject to } x\in X_0 \qquad (P)$$

where $X_0=\{x\in R^n \mid g(x)\geq 0\}$, $f:R^n\to R$ and $g:R^n\to R^m$ are twice differentiable functions, and B_j, $j=\overline{1,s}$, are positive semi-definite (symmetric) $n\times n$ matrices. Let $g=(g_1,...,g_m)^T$. For $s=1$ see, for example Mond[4], Preda [6], Preda and Koller [7].

The study of higher-order duality is important due to the computational advantage over first-order duality as it provides better bounds for the value of the objective function when approximations are used (Mangasarian [2], Yang [8]).

In Section 2 we introduce a general Mond-Weir type [5] higher-order dual to problem *(P)* and give some definitions of higher-order ρ-invexity and generalized higher-order ρ-invexity. In Section 3, some necessary optimality conditions are given.

In Section 4, for the general higher-order dual of Mond-Weir type defined in Section 2, weak duality, strong duality, strict converse duality, and converse duality results are presented.

2. PRELIMINARIES AND SOME DEFINITIONS

Let $h, k_1, ..., k_m : R^n\times R^n\to R$ be differentiable functions with respect to each argument. In the following, the operator ∇ is taken relative to the first argument while the operator ∇_p is taken relative to the second one.

We put $k(u,p)=(k_1(u,p),...,k_m(u,p))^T$ where the symbol ^{T} denotes transpose. Also, let $I_\alpha\subseteq\{1,2,...,m\}$, $\alpha=\overline{0,r}$, with $\bigcup_{\alpha=0}^{r} I_\alpha=\{1,2,...,m\}$ and $I_\alpha\cap I_\beta=\varnothing$, for $\alpha\neq\beta$.

We introduce the following general Mond-Weir type [5] higher-order dual *(HGD)* with respect to *(P)*:

$$\max f(u)+h(u,p)+u\sum_{j=1}^{s}B_j w_j - p^T\nabla_p h(u,p)-\sum_{i\in I_0}y_i g_i(u)-\sum_{i\in I_0}y_i k_i(u,p)+p^T\nabla_p\left[\sum_{i\in I_0}y_i k_i(u,p)\right]$$

subject to

$$\nabla_p h(u,p)+\sum_{j=1}^{s}B_j w_j=\nabla_p(y^T k(u,p)), \tag{2.1}$$

$$\sum_{i\in I_\alpha}y_i g_i(u)+\sum_{i\in I_\alpha}y_i k_i(u,p)-p^T\nabla_p\left[\sum_{i\in I_\alpha}y_i k_i(u,p)\right]\le 0, \alpha=\overline{1,r}, \tag{2.2}$$

$$w_j^T B_j w_j \le 1, j\in\{1, 2, ...,s\}, \tag{2.3}$$

$$y\ge 0, \tag{2.4}$$

where $u, w_1, \dots, w_s, p\in R^n$ and $y\in R^m$.

Let $\rho\in R$, $\rho' = (\rho_1',\dots,\rho_m')\in R^m$ and $d: R^n\times R^n\to R_+$.

Definition 2.1. The objective function f and constraint functions g_i, $i=\overline{1,m}$, are said to be (ρ,ρ')-higher-order type I at u with respect to a function η if the inequalities

$$f(x)+x^T\sum_{j=1}^{s}B_j w_j - f(u)-u^T\sum_{j=1}^{s}B_j w_j \ge$$
$$\eta(x,u)^T\left[\nabla_p h(u,p)+\sum_{j=1}^{s}B_j w_j\right]+h(u,p)-p^T(\nabla_p h(u,p))+\rho d^2(x,u)$$

and

$$-g_i(u)\le\eta(x,u)^T\nabla_p k_i(u,p)+k_i(u,p)-p^T(\nabla_p k_i(u,p))-\rho_i' d^2(x,u)\ ;\ i=\overline{1,m}$$

hold for all x.

Definition 2.2. The objective function f and constraint functions g_i, $i=\overline{1,m}$, are said to be (ρ,ρ')-higher-order pseudo-quasi type I at u with respect to a function η if the implications:

$$\eta(x,u)^T\left[\nabla_p h(u,p)+\sum_{j=1}^{s}B_j w_j\right]\ge -\rho d^2(x,u)\Rightarrow$$
$$\Rightarrow f(x)+x^T\sum_{j=1}^{s}B_j w_j - f(u)-h(u,p)-u^T\sum_{j=1}^{s}B_j w_j + p^T(\nabla_p h(u,p))\ge 0$$

and

$$-g_i(u)\ge k_i(u,p)-p^T(\nabla_p k_i(u,p))\Rightarrow$$
$$\Rightarrow \eta(x,u)^T\nabla_p k_i(u,p)\ge \rho_i' d^2(x,u), i=\overline{1,m}$$

hold for all x.

3. NECESSARY OPTIMALITY CONDITIONS

Let x^0 be a feasible solution for *(P)*. We define the sets $\mathbf{S} = \{1,2,...,s\}$, $\mathbf{B}(x^0) = \{j \in \mathbf{S} \mid x^{0T} B_j x^0 > 0\}$, $\overline{\mathbf{B}}(x^0) = \{j \in \mathbf{S} \mid x^{0T} B_j x^0 = 0\}$, $\mathbf{Z}(x^0) = \{z \in R^n \mid z^T \nabla g_i(x^0) \geq 0, \forall i \in M(x^0)$ and

$$z^T \nabla f(x^0) + \sum_{j \in \mathbf{B}(x^0)} \frac{z^T B_j x^0}{(x^{0T} B_j x^0)^{\frac{1}{2}}} + \sum_{j \in \overline{\mathbf{B}}(x^0)} (z^T B_j z^0)^{\frac{1}{2}} < 0\},$$

where $M(x^0) = \{i \mid 1 \leq i \leq m, g_i(x^0) = 0\}$.

Now necessary optimality conditions for x^0 to be an optimal solution for *(P)* are as following.

Theorem 3.1 *If x^0 is an optimal solution for (P) and $\mathbf{Z}(x^0) = \varnothing$, then there exist $y \in R^m$, $y \geq 0$ and $w_j \in R^n$, $j \in \mathbf{S}$ such that $y^T g(x^0) = 0$, $\nabla y^T g(x^0) = \nabla f(x^0) + \sum_{j=1}^{s} B_j w_j$; $w_j^T B_j w_j \leq 1$ and $(x^{0T} B_j x^0)^{\frac{1}{2}} = x^{0T} B_j w_j$ for $j \in \mathbf{S}$.*

4. DUALITY RESULTS FOR *(P)* AND *(HGD)*

In this section, for *(P)* and *(HGD)*, we consider weak duality, strong duality, strict converse duality, and converse duality results.

Theorem 4.1 (Weak duality). *Let $\eta : R^n \times R^n \to R^n$ such that for all $x \in X_0$ and a feasible solution $(u, y, w_1, ..., w_s, p)$ for (HGD) we have*

$$\eta(x,u)^T \left[\nabla_p h(u,p) + \sum_{j=1}^{s} B_j w_j - \nabla_p \left(\sum_{i \in I_0} y_i k_i(u,p) \right) \right] \geq -\rho d^2(x,u) \Rightarrow$$

$$\Rightarrow f(x) + x^T \sum_{j=1}^{s} B_j w_j - \left(f(u) + u^T \sum_{j=1}^{s} B_j w_j - \sum_{i \in I_0} y_i g_i(u) \right) -$$

$$-\left(h(u,p) - \sum_{i \in I_0} y_i k_i(u,p) \right) + p^T \left[\nabla_p h(u,p) - \nabla_p \left(\sum_{i \in I_0} y_i k_i(u,p) \right) \right] \geq 0, \tag{4.1}$$

$$\begin{gathered} -\sum_{i \in I_\alpha} y_i g_i(u) - \sum_{i \in I_\alpha} y_i k_i(u,p) + p^T \left[\nabla_p \left(\sum_{i \in I_\alpha} k_i(u,p) \right) \right] \geq 0 \Rightarrow \\ \Rightarrow \eta(x,u)^T \left[\nabla_p \left(\sum_{i \in I_\alpha} y_i k_i(u,p) \right) \right] \geq -\rho_\alpha d^2(x,u), \alpha = \overline{1,r} \end{gathered} \tag{4.2}$$

and

$$\rho + \sum_{\alpha=1}^{r} \rho_\alpha \geq 0. \tag{4.3}$$

Then inf (P)≥sup (HGD).

For the next duality results we suppose that h and k satisfy some "initial" conditions (defined by (4.4)) below considered in Zhang [9] and Mishra and Rueda [3].

Theorem 4.2 (Strong duality). *Let x^0 be a local or global optimal solution of (P) with $\mathbf{Z}(x^0)=\varnothing$ and assume that*

$$h(x^0,0)=0\,,\ k(x^0,0)=0\,,\ \nabla_p h(x^0,0)=\nabla f(x^0)\,,\ \nabla_p k(x^0,0)=\nabla g(x^0) \tag{4.4}$$

Then there exist $y\in R^m$ and $w_1,\dots,w_s\in R^n$ such that $(x^0,y,w_1,\dots,w_s,p=0)$ is a feasible solution for (HGD) and the corresponding values of (P) and (HGD) are equal. If the weak duality Theorem 4.1 also holds, then $(x^0,y,w_1,\dots,w_s,p=0)$ is an optimal solution for (HGD).

Theorem 4.3 (Strict converse duality). *Let x^0 be an optimal solution of (P) with $\mathbf{Z}(x^0)=\varnothing$ and assume (4.4) holds. Assume also that the hypotheses of the weak duality Theorem 4.1 are satisfied. If $(\bar{x},\bar{y},\bar{w}_1,\dots,\bar{w}_s,\bar{p})$ is an optimal solution of (HGD) and if*

$$\eta(x,\bar{x})^T\left[\nabla_p h(\bar{x},\bar{p})+\sum_{j=1}^{s}B_j\bar{w}_j-\nabla_p\left(\sum_{i\in I_0}\bar{y}_i k_i(\bar{x},\bar{p})\right)\right]\geq-\rho d^2(x,\bar{x})\Rightarrow$$
$$\Rightarrow f(x)+x^T\sum_{j=1}^{s}B_j\bar{w}_j-\left(f(\bar{x})+\bar{x}^T\sum_{j=1}^{s}B_j\bar{w}_j-\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)-$$
$$-\left(h(\bar{x},\bar{p})-\sum_{i\in I_0}\bar{y}_i k_i(\bar{x},\bar{p})\right)+\bar{p}^T\left[\nabla_p h(\bar{x},\bar{p})-\nabla_p(\sum_{i\in I_0}\bar{y}_i k_i(\bar{x},\bar{p}))\right]>0,$$

for any $x\neq\bar{x}$, then $x^0=\bar{x}$, i.e., $\bar{x}$ is an optimal solution of (P) and the optimal values of the objective functions of (P) and (HGD) are equal.

The proof is along the usual lines of those of similar theorems (see, for example, Preda [6]).

Suppose now that h and $k_1,k_2,\dots,k_m$ are twice differentiable with respect to the second argument and differentiable with respect to the first one.

Theorem 4.4 (Converse duality). *Let $(x^0,y^0,w_1^0,\dots,w_s^0,p^0)$ be an optimal solution of (HGD) such that (4.4) holds and assume that*

(i1) the set of vectors

$$\left\{[\nabla_p^2 h(x^0,p^0)-\nabla_p^2(\sum_{i\in I_0}y_i^0 k_i(x^0,p^0))]_j,[\nabla_p^2(\sum_{i\in I_\alpha}y_i^0 k_i(x^0,p^0))]_j,\alpha=\overline{1,r},\,j=\overline{1,n}\right\}$$

is linear independent, where $[\nabla_p^2 h-\nabla_p^2(\sum_{i\in I_0}y_i^0 k_i)]_j$ is the j^{th} row of $\nabla_p^2 h-\nabla_p^2(\sum_{i\in I_0}y_i^0 k_i)$ and $[\nabla_p^2(\sum_{i\in I_\alpha}y_i^0 k_i)]_j$ is the j^{th} row of $\nabla_p^2(\sum_{i\in I_\alpha}y_i^0 k_i)$;

(i2) the matrix aa^T is positive or negative definite, where a is the vector

$$\nabla f(x^0)+\nabla h(x^0,p^0)-\nabla(y^{0T}g(x^0))-\nabla(y^{0T}k(x^0,p^0))-\nabla_p(\sum_{i\in I_0}y_i^0 k_i(x^0,p^0))+\nabla_p h(x^0,p^0).$$

Then x^0 is a feasible solution to (P) and the corresponding values of the objective functions of (P) and (HGD) are equal. Further, if the hypotheses of the weak duality Theorem 4.1 hold, then x^0 is an optimal solution to (P).

Remark: Recently some optimality results for nondifferentiable programming problems have been given in [1]. It is interesting to compare these results and those presented above.

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