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SOME SKEW-LAPLACE DISTRIBUTIONS

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We present a new family of asymmetric distributions, the skew-Laplace (SL) distribution, including basic properties such as a stochastic representation and moment estimators. The class is termed the *GSL* class of distributions. The flexibility of the class in terms of accommodating more general types of skewness than the *SL* distribution is illustrated by computing moments and, in particular, asymmetry and kurtosis parameters.

Key words: Kurtosis; Skewness; Skew-Laplace distribution.

1. INTRODUCTION

Let X be a random variable with $\text{Laplace}(\mu,b)$ distribution (also known as the double exponential distribution, because it can be thought of as two exponential distributions, with an additional location parameter, spliced together back-to-back), whose probability density function is

$$f(x\mid\mu,b)=\frac{1}{2b}e^{-\frac{|x-\mu|}{b}},\ x\in R, \tag{1}$$

where μ is a location parameter and $b>0$ a scale parameter.

The cumulative density function is

$$F(x\mid\mu,b)=\begin{cases}\frac{1}{2}e^{-\frac{|x-\mu|}{b}} & \text{when } x\le\mu\\ 1-\frac{1}{2}e^{-\frac{|\mu-x|}{b}} & \text{when } x>\mu\end{cases}, \tag{2}$$

The expected value of a Laplace distribution is $E(x)=\mu$. As in the case of other symmetrical distributions, such as the Normal and the logistic distributions, Laplace's location is the same as its mean, median, and mode. The variance is $\text{Var}(x)=2b^2$

In this paper, we assume that $\mu=0$ and $b=1$. In this case, the probability density function is

$$h(x)=\frac{1}{2}e^{-|x|},\ x\in R, \tag{3}$$

and the cumulative distribution function is

$$H(x)=\begin{cases}\frac{e^{x}}{2} & \text{if}\quad x\le 0\\ 1-\frac{e^{-x}}{2} & \text{if}\quad x>0\end{cases} \tag{4}$$

We have $E[X]=0$ and $\mathrm{Var}[X]=2$.

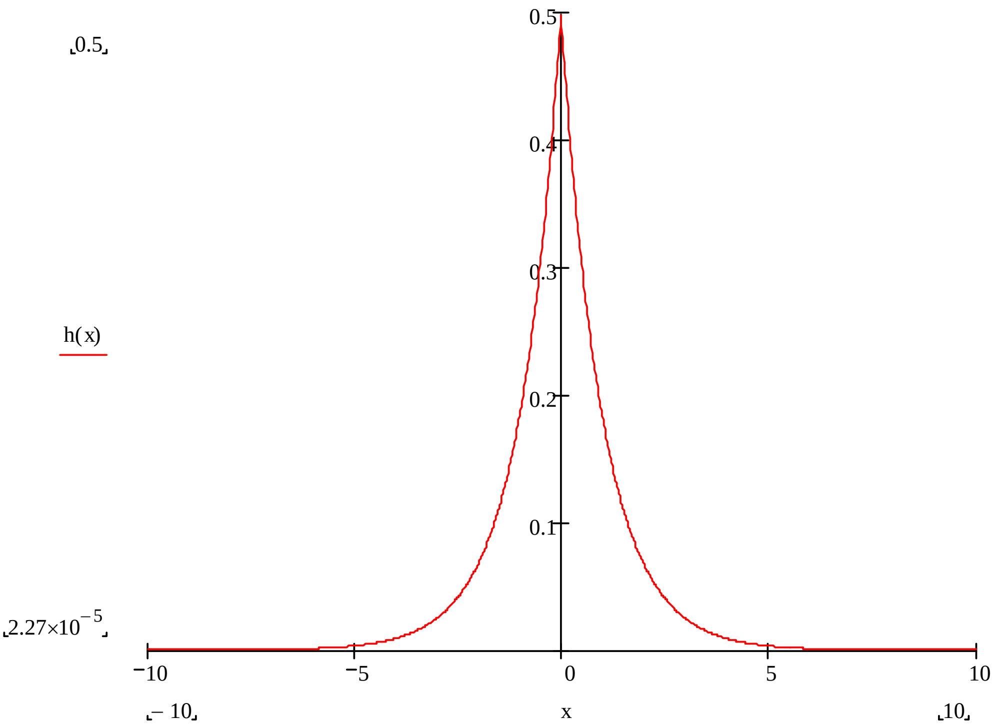


Fig. 1 Laplace probability density function

We denote the standard Laplace distribution with asymmetry parameter α by $\{SL(\alpha), \alpha \in R\}$, with density function

$$f(x\,|\,\alpha) = 2h(x)H(\alpha x), \quad x, \alpha \in R \tag{5}$$

where $h(\cdot)$ and $H(\cdot)$ are the density and distribution functions of the standard Laplace distribution, respectively.

Definition 1 We say that X is distributed according to the skew-Laplace distribution with parameter α and denote it by $X \to SL(\alpha)$, if its probability density is given by (5), where $\alpha \in R$, $h(\cdot)$ and $H(\cdot)$, as defined above.

Now, using (3) and (4) in (5), we have

$$f(x\,|\,\alpha) = \begin{cases} e^{x} - \frac{1}{2}e^{x(1-\alpha)} & \text{for} \quad x \le 0 \quad \alpha \le 0 \\ \frac{1}{2}e^{x(1+\alpha)} & \text{for} \quad x \le 0 \quad \alpha > 0 \\ \frac{1}{2}e^{x(\alpha-1)} & \text{for} \quad x > 0 \quad \alpha \le 0 \\ e^{-x} - \frac{1}{2}e^{-x(1+\alpha)} & \text{for} \quad x > 0 \quad \alpha > 0 \end{cases} \tag{6}$$

Next, we will consider the family of asymmetric distributions which depends on two parameters namely α and β, that in the special case $\beta = 0$ reduces to the skew-Laplace (SL) distribution.

Figures 2 show graphs of the density (5) for different values of $\alpha \in R$.

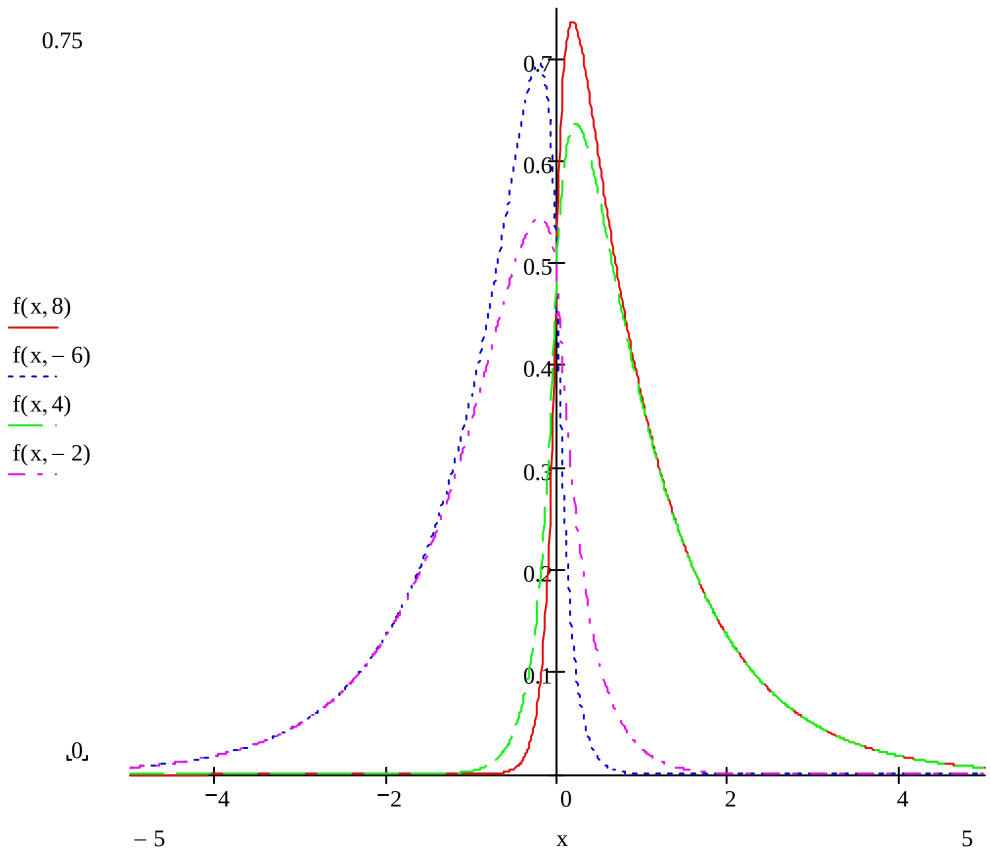


Fig. 2 Skew-Laplace (SL) probability density function

2. SOME PROPERTIES

We shall now compute some moments of a random variable $X \to SL(\alpha)$:
– the expected value

$$E[X] = \int_{-\infty}^{\infty} x f(x \mid \alpha) dx \tag{7}$$

$$E[X] = \begin{cases} \dfrac{1}{(\alpha-1)^2} - 1 & \text{for} \quad \alpha \le 0 \\ 1 - \dfrac{1}{(1+\alpha)^2} & \text{for} \quad \alpha > 0 \end{cases} \tag{8}$$

– the variance

$$\mathrm{Var}(X) = M\left[X^2\right] - (M[X])^2 \tag{9}$$

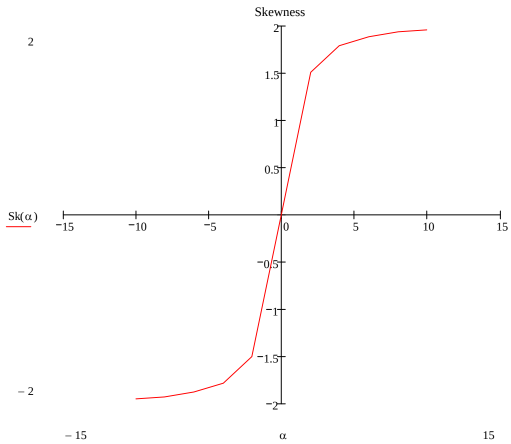
$$\mathrm{Var}(X) = \begin{cases} 1 + \dfrac{2}{(\alpha-1)^2} - \dfrac{1}{(\alpha-1)^4} & \text{for} \quad \alpha \le 0 \\ 1 + \dfrac{2}{(\alpha+1)^2} - \dfrac{1}{(\alpha+1)^4} & \text{for} \quad \alpha > 0 \end{cases} \tag{10}$$

– asymmetry and kurtosis

$$\sqrt{\beta_1} = \frac{m_3}{\sigma^3},\ \beta_2 = \frac{m_4}{\sigma^4}$$

$$\sqrt{\beta_1} = \begin{cases} \dfrac{2\left(\dfrac{1}{(\alpha-1)^6} - 1\right)}{\left(1 + \dfrac{2}{(\alpha-1)^2} - \dfrac{1}{(\alpha-1)^4}\right)^{\frac{3}{2}}} & \text{for} \quad \alpha \le 0 \\ \dfrac{2\left(1 - \dfrac{1}{(\alpha+1)^6}\right)}{\left(1 + \dfrac{2}{(\alpha+1)^2} - \dfrac{1}{(\alpha+1)^4}\right)^{\frac{3}{2}}} & \text{for} \quad \alpha > 0 \end{cases} \tag{11}$$

$$\beta_2 = \begin{cases} -\dfrac{3}{(\alpha-1)^8} - \dfrac{12}{(\alpha-1)^6} + \dfrac{18}{(\alpha-1)^4} + \dfrac{12}{(\alpha-1)^2} + 9 & \text{for} \quad \alpha \le 0 \\ -\dfrac{3}{(\alpha+1)^8} - \dfrac{12}{(\alpha+1)^6} + \dfrac{18}{(\alpha+1)^4} + \dfrac{12}{(\alpha+1)^2} + 9 & \text{for} \quad \alpha > 0 \end{cases}$$



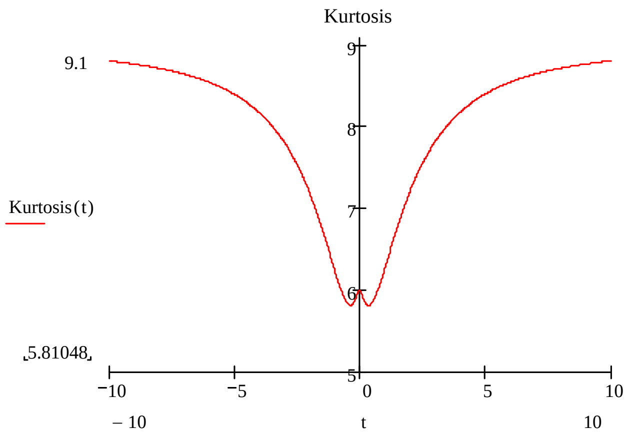


Fig. 3 The skew-ness and kurtosis

From equation (11), admissible intervals for the asymmetry and kurtosis are

$$-2 < \sqrt{\beta_1} < 2 \tag{12}$$

$$5.8105 \le \beta_2 \le 8.8080 \tag{13}$$

A formula for computing the k th moments of the SL distribution, is available namely,

$$d_{2k+1} = \begin{cases} (2k+1)!\left(\dfrac{1}{(\alpha-1)^{2k+2}} - 1\right) & \text{for } \alpha \le 0 \\ (2k+1)!\left(1 - \dfrac{1}{(\alpha+1)^{2k+2}}\right) & \text{for } \alpha > 0 \end{cases} \tag{14}$$

$$d_{2k} = (2k)! \quad \text{for } \alpha \in R$$

In this article, an extension of the SL distribution is considered, which makes the skew parameter of the density more flexible in the sense that it can take values in a wider range than the skew parameter of the ordinary SL distribution. Hence, it may be able to capture higher degrees of asymmetry than the SL distribution. Another interesting aspect of the distribution is its stochastic representation, which can be used in the derivation of its moments.

3. A GENERALIZED SL DISTRIBUTION

Our main objective is to define a family of skew distributions with more flexibility in its shape by changing the asymmetry and kurtosis of the model.

Definition 2 We say that X is distributed according to the GSL distribution with parameters α and β and denote it by $X \to GSL(\alpha, \beta)$, if its probability density is given by

$$f(x \mid \alpha, \beta) = \begin{cases} 2h\left(\dfrac{x}{1+\beta}\right)\left[\dfrac{\beta}{1+\beta} + \dfrac{(1-\beta)}{1+\beta} H\left(\dfrac{\alpha x}{1+\beta}\right)\right] & \text{if } x < 0 \\ 2h\left(\dfrac{x}{1-\beta}\right) H\left(\dfrac{\alpha x}{1-\beta}\right) & \text{if } x \geq 0 \end{cases} \tag{15}$$

where $\alpha \in R$, $\beta \in [0,1)$, and $h(\cdot)$ and $H(\cdot)$ are defined as above.

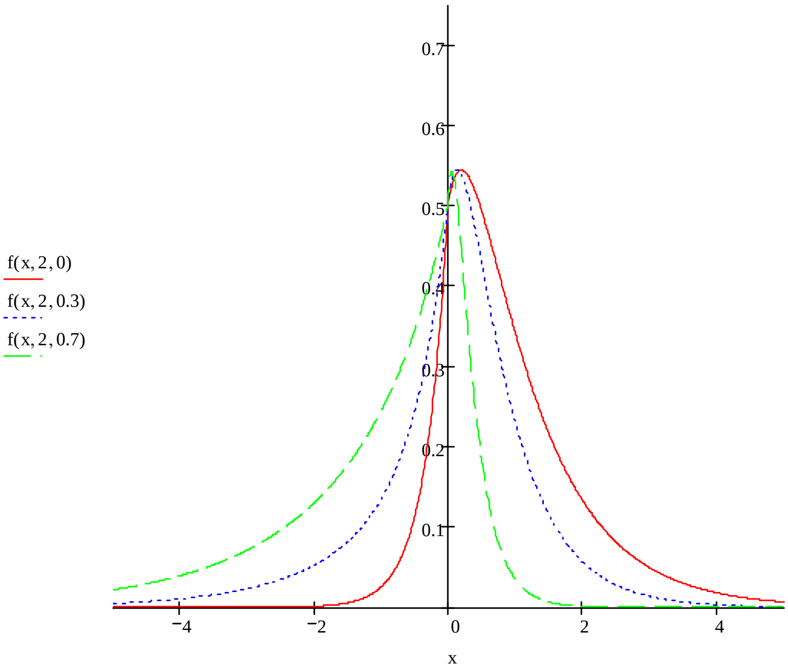


Figure 4. Examples of the $GSL(2, \beta)$ density for $\beta = 0$ (solid line), $\beta = 0.3$ (dotted line) and $\beta = 0.7$ (dashed line).

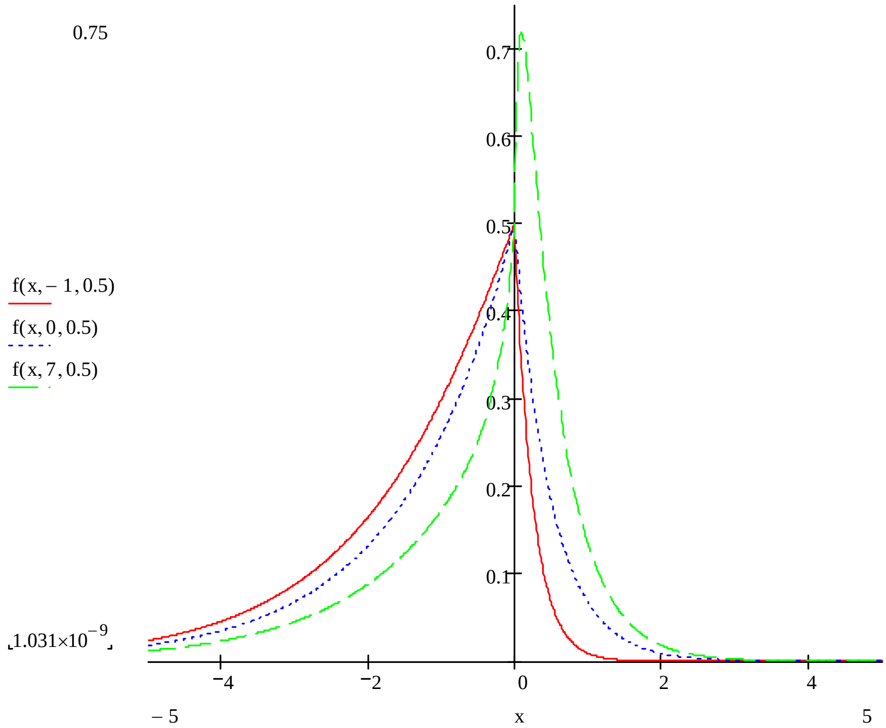


Figure 5. Examples of the $GSL(\alpha,0.5)$ density for $\alpha=-1$ (solid line), $\alpha=0$ (dotted line) and $\alpha=7$ (dashed line).

Figures 4 and 5 depict graphs of the density (15) for different values of $\alpha\in R$ and $\beta\in[0,1)$. The SL is plotted with the same asymmetry parameter $\alpha\in R$.

The following properties follow as consequences of Definition 2.

Proposition 1 The following properties hold:

(1) $f(x\,|\,\alpha=0,\beta=0)=h(x)$

(2) $f(x\,|\,\alpha,\beta=0)=2h(x)H(\alpha x)$

(3) $f(x\,|\,\alpha=0,\beta)=\begin{cases} h(x\,|\,(1+\beta)) & if\ \ x<0 \\ h(x\,|\,(1-\beta)) & if\ \ x\geq 0\end{cases}$

(4) If $X\rightarrow GSL(\alpha,\beta)$ then $\dfrac{P(X\geq 0\,|\,\alpha,\beta)}{P(X<0\,|\,\alpha,\beta)}=\dfrac{(1-\beta)d_0(\alpha)}{1-(1-\beta)d_0(\alpha)}$

(5) $\lim\limits_{\beta\to 1} f(x\,|\,\alpha,\beta)=h\left(\dfrac{x}{2}\right)I\{x<0\}$

(6) $\lim\limits_{\alpha\to+\infty} f(x\,|\,\alpha,\beta)=\left(\dfrac{2\beta}{1+\beta}\right)h\left(\dfrac{x}{1+\beta}\right)I\{x<0\}+2h\left(\dfrac{x}{1-\beta}\right)I\{x\geq 0\}$

(7) $\lim\limits_{\alpha\to-\infty} f(x\,|\,\alpha,\beta)=\left(\dfrac{2}{1+\beta}\right)h\left(\dfrac{x}{1+\beta}\right)I\{x<0\}$

(8) $\lim\limits_{\beta\to 1} f(x\,|\,\alpha,\beta)=\lim\limits_{\beta\to 1}\left(\lim\limits_{\alpha\to+\infty} f(x\,|\,\alpha,\beta)\right)=\lim\limits_{\beta\to 1}\left(\lim\limits_{\alpha\to-\infty} f(x\,|\,\alpha,\beta)\right)$

Results (1-3) of Proposition 1 establish that the class of the GSL distribution contains the Laplace distribution, the SL distribution and the class of the epsilon-SL distributions (take $\alpha\in R$, $\beta\in[0,1)$).

The result given next presents a stochastic representation for the distribution considered earlier. The main idea is to notice that if $X\rightarrow GSL(\alpha,\beta)$, then X can be represented as the product of two dependent random variables.

Proposition 2 Let $\alpha \in R$, $\beta \in [0,1)$ and $Y \to \text{Laplace}(0,1)$. If $V = |Y|$ and

$$S_v = \begin{cases} 1-\beta & \text{with probability } (1-\beta)H(\alpha v) \\ -(1+\beta) & \text{with probability } 1-(1-\beta)H(\alpha v) \end{cases} \tag{16}$$

where H is the distribution function of Y, then $Z = S_v V$ is distributed according to the probability density (15).

4. CONCLUSIONS

The flexibility of the class in terms of accommodating more general types of skewness than the SL distribution is illustrated by computing moments and, in particular, asymmetry and kurtosis parameters.

The skew-Laplace distribution is frequently used to fit the logarithm of particle sizes and it is also used in Economics, Engineering (reliability), Finance and Biology.

Greater generality in the family of the SL distributions is obtained by considering the form $2h(x)H(w(x))$, where $w(\cdot)$ is an odd function. Extensions to this family and some comparative study between skew-Laplace and skew-normal are currently under investigation.

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