THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 10, Number 2/2009, pp. 000-000

A GENERAL COMPUTER-AIDED METHOD TO OBTAIN THE MATHEMATICAL MODEL FOR MULTI-BODY BRANCHED SYSTEMS

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This paper presents a general method to obtain the mathematical model for the elasto-dynamic behavior of the multi-body branched systems developed in the welcoming frame of the Computational Mechanics. This particular class of multi-body systems generally named branched systems refers to the mechanical systems with a number of branches activated and controlled by the driving links of a common driver, taking into consideration the flexibility of the elements (the "rigid body system" assumption does not hold). The method allows us to consider the essentially nonlinear position functions of the driver mechanisms (one in each branch) in their real form, without any linearization. So, the achieved mathematical model is a system of second order nonlinear differential equations with non-constant coefficients. As the mathematical model is achieved in a symbolic form, it serves all the targets as an analytic one, but it is obtained for any number of branches (and, by consequence, for any number of degrees of freedom).

Key words: multi-body systems, elasto-dynamics, Mathematica®.

1. INTRODUCTION

In modern technological processes, the increasing operating velocities of the equipment and the increasing dynamic loads on the mechanisms that accompany development and perfection of the machinery, accentuated essentially the role of the design for dynamics and, therefore, the roles of the elasto-dynamic behavior and the vibroactivity have thus emerged. For real multi-body systems requiring a large number of components and essentially nonlinear transfer functions, the study of the elasto-dynamic behavior highly grows in complexity. So, the demand for analyzing their dynamical behavior to a high accuracy in translational and rotational degrees of freedom taking into consideration the nonlinear kinematics became a challenging task. As this objective puts serious problems to the dynamicist, the theoretical mathematician and the numerical mathematician as well, the Computational Mechanics seems to be a suitable frame for it.

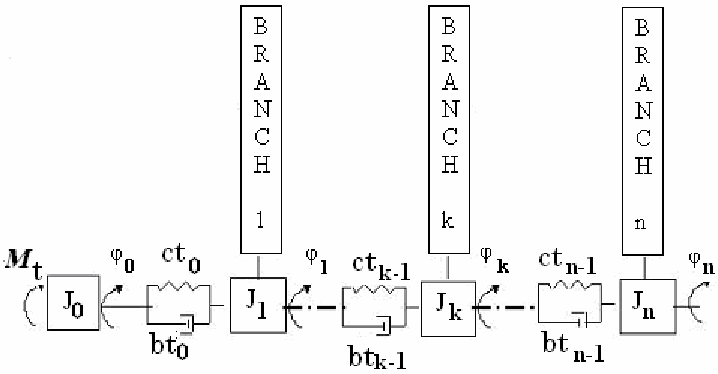


Fig. 1.1 A general scheme of an n-BM.

In this welcoming frame, we developed a new computer-aided (CA) way to analysis and synthesis of a particular (but, large) class of multi-body systems from the elasto-dynamic point of view. Basically, the method allows us to obtain the mathematical model (in symbolic or numeric form) for any mechanical system that can be described as a combination of branched systems with variable number of branches.

This particular class of multi-body systems can be named, generally, „*branched systems*" - where the term refers to mechanical systems with a number of branches activated and controlled by the driving links of a common driver, taking into consideration the flexibility of the elements (the "rigid body system" assumption does not hold); a general scheme is shown in Fig. 1.1. We'll use the notation "n-BM" to denote a multi-body branched system with n branches. It is obvious that an immediate application can be any system with a camshaft driver. For example, the camshaft from Fig.1.2 can be the driver for a 16-BM and it will be used in the study case from section 5.

The method has been developed in the general assumptions frame of flexible multi-body systems dynamics. In this paper we refer explicitly to the following two assumptions:

(i) The damping assumption: the friction is proportional to the relative speed between two neighboring inertial bodies; the energy dissipation in the elastic elements is proportional to the relative speed of the masses connected by the element. So, the damping coefficient is defined by $b = 2\zeta\sqrt{mc}$, when m,b,c are related by the classic linear motion equation $my'' + by' + cy = 0$ for one degree of freedom and ζ is the viscous damping factor.

(ii) The bending assumption: the bending deflection is defined as the vertical (only) displacement of the neutral axis of the deformed beam [3].

Fig. 1.2 Camshaft for a 16-BM

The paper starts by a presentation of the main terms and definitions introduced in this work: vibratory element, dynamic model and descriptor matrix of an n-BM, followed by a concise description of the mathematical model for the elasto-dynamic behavior of the multi-body branched systems and a short presentation of the software algorithm. Since the mathematical model is achieved in a symbolic form, it serves to a very large suite of further important results in analysis and synthesis of the studied n-BM. The last section presents a 16-BM as a study case and exemplifies some results.

2. TERMS AND DEFINITIONS

In this work, new terms, definitions and concepts have been introduced. In the following subsections, the main terms are presented.

2.1 Vibratory element

The vibratory element [7, 4] is a generic name for a mechanical item that provides all the necessary information to describe the punctual dynamical behavior by three components (see, Fig. 2.1): a kinematical one - Π; an inertial one- J or m; a deformability one - c (optionally with damping, b). The type and dimension of the inertial and deformability components are to be chosen according to the motion (see, [4] section 3.2.2). The kinematical properties in a certain point are represented by the kinematical analog (often called the position function) that describes the kinematical ratio between adjacent elements if such a ratio differs from 1. After [1, 6], the following notations are used:

x - the input motion, which can be either linear or angular;

s - the output motion, which also can be either linear or angular;

Π - the position function that relates the velocities of the driving and driven members:

$$\frac{\partial \Pi}{\partial x} = F_{\text{input}} / F_{\text{output}} = \left(\frac{\partial s}{\partial t}\right) / \left(\frac{\partial x}{\partial t}\right).$$

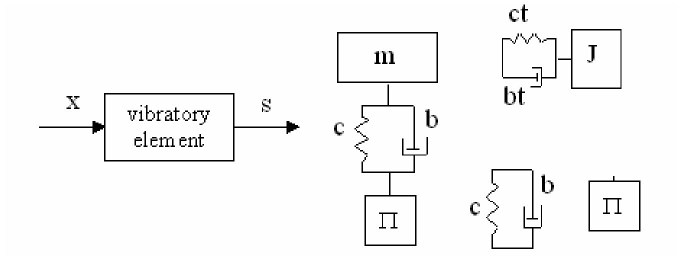


Fig. 2.1 Vibratory element's forms.

In our view, a vibratory element can contain only one of each of the inertial, elastic and kinematical components, but not necessarily all three of them. It means that the vibratory element remains defined (see, [5, 1, 4]) even in the absence of one or even two of its components. In Fig. 2.1, the vibratory element's forms are shown.

2.2 Dynamic Model

Generally, every specific system can be described either as a continuous model or as a lumped model with a single or a finite number of concentrated masses. In this work, the lumped model has been considered.

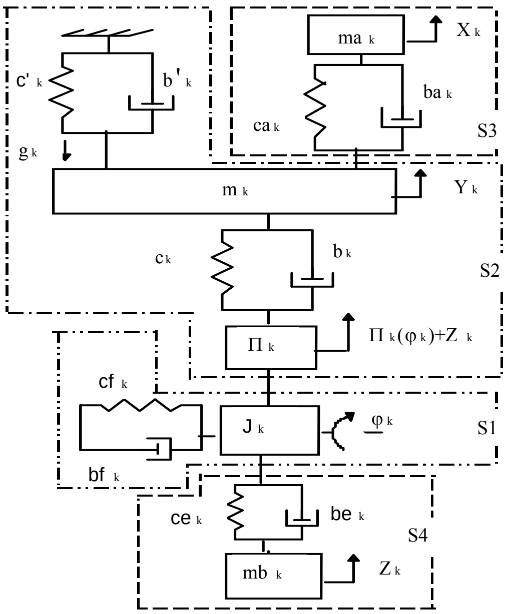


Fig. 2.2 Dynamic model of a typical branch.

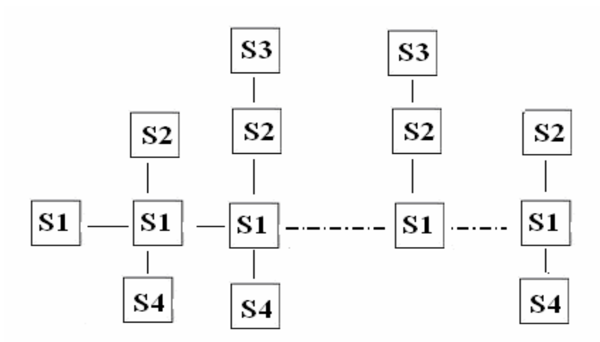


Fig. 2.3 A general scheme of the dynamic model of an n-BM that contains only vibratory elements from subsystems S1-S4.

Under the assumption that the n-MB develops only rotation and translational motion and the main shaft driver undergoes bending (according to assumption ii/ section 1), the dynamical model can be represented as a series and/or parallel connection of vibratory elements from four subsystems.

Important to observe that, in each subsystem, the same equation of motion is respected:

$$[\mathbf{M}_s]\ddot{\mathbf{q}}_s + [\mathbf{B}_s]\dot{\mathbf{q}}_s + [\mathbf{C}_s]\mathbf{q}_s = \mathbf{F}_s \tag{2.1}$$

where s=1,4 is the subsystem number.

The subsystems S1 and S4 deal with the driver (which serves as a programming tool for the whole system), which undergoes mainly torsion and bending deformations [1, 2] and vibrations. The subsystems S2 and S3 deal with the branches- each branch representing a chain of mechanical elements (for details, see [1, 4]).

A real system can lead to a dynamic model that contains more than one vibratory element in a branch, on one hand, and, on the other hand, not all the subsystems S1-S4 are necessarily present. Therefore, we introduced the term "***typical branch***" to indicate a branch composed by four vibratory elements - one from each subsystem S1-S4 (Fig. 2.2). Now, the dynamic model of an n-BM can be redefined as a series connection of typical and/or atypical branches. Fig. 2.3 presents a general scheme of the dynamic model of an n-BM that contains only vibratory elements from subsystems S1-S4.

2.3 Descriptor matrix

In order to automatically generate a matrix form of the equations of motion of an n-BM, we introduced a matrix description. Named the descriptor matrix of an n-BM, it describes in a symbolic way the dynamic model defined above: each column of the descriptor matrix describes a branch of the n-BM and each row, a subsystem. The component kj of the descriptor matrix represents the number of vibratory elements from the subsystem k that exist in the branch j-1. If each branch contains, at most, one vibratory element from each subsystem, then a single descriptor matrix, DE, is used: DE_{kj} is 1, if in the branch j-1 exists a vibratory element from subsystem k, and 0 if otherwise. The first column of DE always describes the motor as a vibratory element of subsystem S1. For computation reasons, if the subsystem S3 is not used in the dynamic model, the third row of DE must be a zero-row ($DE_{3j} = 0$, j=1, n+1; see, for example, the columns (a) and (b) in Table 1). In the case of a dynamic model with more than one main mass in at least one driven branch (for example, the column (e) in Table 1), a second descriptor matrix, MVE, is needed. The components of the second row of the descriptor matrix DE express, now, the number of main masses that exist in each branch. For example, see, the column (e) in Table 1 and application from section 5 (for details, see [4]).

Table 1 Descriptor matrix of a 2-BM.

	(a)	(b)	(c)	(d)	(e)	
additional elements	-	-	+	+	Two main masses in branch 2	
bending	-	+	-	+	*DE*	*MVE*
	$\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&2&1\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\0&1&0\\0&1&0\end{bmatrix}$

3 MATHEMATICAL MODEL

Let us remember two facts that have been already discussed.

1) Each one of the subsystems S1-S4 describes a simple loading case —torsion, bending or compression-extension; the mathematical model for each one of them has the same form written in the same matrix form: equation (2.1).

2) The dynamic model of an n-BM is, after section 2.2, a collection of vibratory elements from subsystems S1-S4.

Thus, it is natural to look for the same matrix form for the mathematical model of the entire n-BM. So,

$$[\mathbf{M}]\ddot{\mathbf{q}} + [\mathbf{B}]\dot{\mathbf{q}} + [\mathbf{C}]\mathbf{q} = \mathbf{F} \tag{3.1}$$

is the matrix form for the mathematical model of an n-BM and Ndof is the number of DOF; Ndof $=n_1+ n_2+ n_3+ n_4$, where n_k is the number of vibratory elements from subsystem Sk; $\mathbf{q}$ is the generalized coordinates vector; $\mathbf{F}$ is the generalized forces vector; $\mathbf{M}$ is the global inertia matrix; $\mathbf{C}$ is the global stiffness matrix; $\mathbf{B}$ is the global damping matrix that is deduced from $\mathbf{C}$ according to assumption *i*/section 1. The dimension of the global vectors and the global square matrices is Ndof.

Due to the nonlinearities of the kinematical analog (often called the position function) of the driver mechanisms (one in each branch), the global generalized forces vector contains the generalized coordinates and/or their first and second derivatives. Moving all these terms on the same side of (3.1), the matrix form of the mathematical model is

$$\mathbf{[Mj]\ddot{q} + [Mb]\dot{q} + [Mc]q = Feq} \tag{3.2}$$

Now, the global matrices $\mathbf{Mj}, \mathbf{Mb}$ and $\mathbf{Mc}$ are not symmetrical. The force vector $\mathbf{Feq}$ is:

$$\begin{aligned}
&\mathbf{F}\mathrm{eq}_1 = M_{0t} && \\
&\mathbf{F}\mathrm{eq}_k = \lambda_k \dot{\Pi}_k && 2 \le k \le n_1+1 \\
&\mathbf{F}\mathrm{eq}_{n_1+1+k} = -(m_k \ddot{\Pi}_k + b'_k \dot{\Pi}_k + c'_k \Pi_k + g_k) && 1 \le k \le n_2 \\
&\mathbf{F}\mathrm{eq}_{n1+n2+k+1} = -ma_k \ddot{\Pi}_k && 1 \le k \le n_3 \\
&\mathbf{F}\mathrm{eq}_{n_1+n_2+n_3+1+k} = 0 && 1 \le k \le n_4
\end{aligned} \tag{3.3}$$

where M_{0t} represents the motor torque.

4. SHORT PRESENTATION OF THE SOFTWARE ALGORITHM

In order to study the elasto-dynamic behavior of an n-BM using our method, we developed a complex and performing software [1, 4, 5, 6] able to generate the mathematical model for elasto-dynamic behavior of multi-body branched systems. Since, at that moment, only Mathematica® [8] was able to develop symbolic calculus, we developed a Mathematica-based software for this method. A general flow-chart of it can be seen in Fig. 4.1. According to the descriptor matrix DE (and MVE, if it is needed) received as the input data, the program proceeds as follows:

- computes the current number of generalized coordinates of each vibratory element in the dynamic model;
- organizes the generalized coordinates vector, the global force vector and the global matrices $\mathbf{q}_s, \mathbf{F}_s, \mathbf{M}_s, \mathbf{B}_s, \mathbf{C}_s$ for each subsystem S1-S4 that appear in Eq. (2.1);
- organizes the global generalized coordinates vector, the global force vector and the global matrices $\mathbf{q}, \mathbf{F}, \mathbf{M}, \mathbf{B}, \mathbf{C}$ that appear in Eq. (3.1);
- computes the global vector $\mathbf{Feq}$ after Eq. (3.3) and the global matrices $\mathbf{Mj}, \mathbf{Mb}, \mathbf{Mc}$ that appear in Eq. (3.2) representing the matrix form of the mathematical model of the studied branched system;
- stores a symbolic form of the mathematical model.

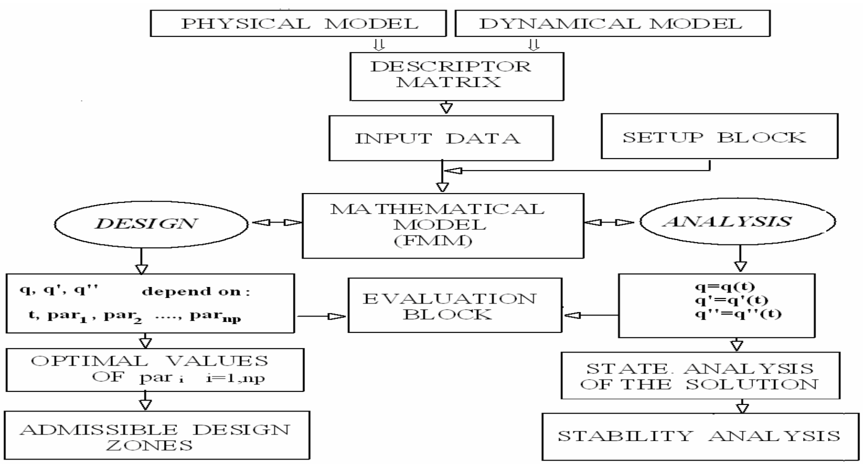


Fig. 4.1 The flow-chart.

If the analysis of the studied n-BM is the target, the numerical requested data are introduced and the obtained mathematical model is solved. Once the mathematical model is solved, a very rich set of results can be obtained:

- the dynamic perturbations (the dynamically perturbed speeds and also the dynamically perturbed accelerations) for each inertial point in numeric and/or graphic form (see, figs. 5.2-5.5);

- the main characteristics of the solution (that are established on the beginning);
- the main frequency;
- the stability information.

The performance conditions can be checked, as well.

If the synthesis of the studied n-BM is the target, the saved symbolic matrix form of the mathematical model serves as a start point to another Mathematica® based software developed to do an optimal synthesis based on “the State Matrix Strategy- a quasi-optimization tool” [1, 2, 3, 5].

5. STUDY CASE

Let us consider, as an example, the 16-BM with the main shaft driver shown in Fig. 1.2; all the branches are typical (see, Fig. 2.2). The descriptor matrix DE is a 17x17 matrix with unit components except for the first column that represents the motor ($DE_{ij} = 0, i = 2-17, j = 1$ and $DE_{ij} = 1$ in rest). In order to take into consideration the bending of the main shaft, the influence coefficients are to be known. We obtain the (16x16) influence coefficients matrix using our general computer-aided method especially developed to reach the influence coefficients for any statically determined or undetermined straight beams of a constant cross section, modeled as a p-lumped beam for all the combinations of loading and boundary conditions, unrelated of how big p is (see, [3]), modeling the shaft driver as a simply supported 16 masses lumped beam.

According to DE, our software generates the 65 components of the global vectors and the (65x65) components of each global matrix and it composes, after (3.2) and (3.3), the symbolic form of the mathematical model of this16-BM. The computed mathematical model for this study case is a system of 65 second order nonlinear differential equations with some non-constant coefficients. At user’s request, it is stored in a special file. For this study case, the complete symbolic mathematical model is stored in a 234KB file. Our software generates, also in a symbolic form, the dynamic matrix, the eigenvalues and the eigenvectors. These symbolic expressions allow us to study how the elasto-dynamic behavior of the system is influenced by any parameter or combination of parameters. A very rich data base can be achieved by repeated running for a large number of variants of the model (the variants differ by design or value of one parameter at least). So, the increased capability to obtain, to save and to compare the results from a large number of cases synergistically combined with the Mathematica®[8] software symbolic calculus capabilities and the statistical representation of the results, allowed us to developed a tool for design optimization [2] that permits choosing the optimal combination of parameters for any predefined criteria.

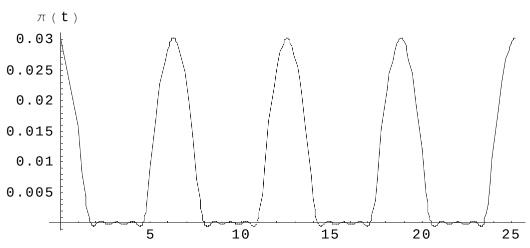


Fig. 5.1 The main shaft at torsion (S1 in Fig. 2.2)

To any set of numerical input data, the mathematical model is solved and the solution can be analyzed by numerical, statistical or graphical tools.

For example, for the position function of the driver mechanism shown in Fig. 5.1, the normalized dynamic perturbations and perturbed speeds obtained in each inertial point of the driven branch are shown in Figs. 5.2- 5.5; the notations from Fig. 2.2 are used here.

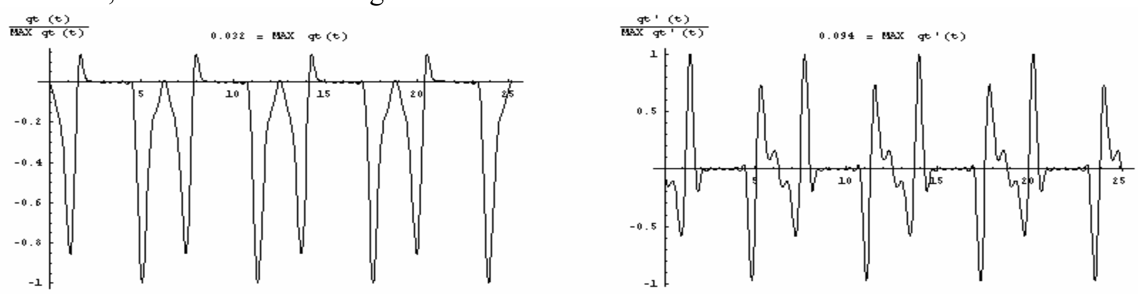


Fig. 5.2 The main shaft at torsion (S1 in Fig. 2.2)

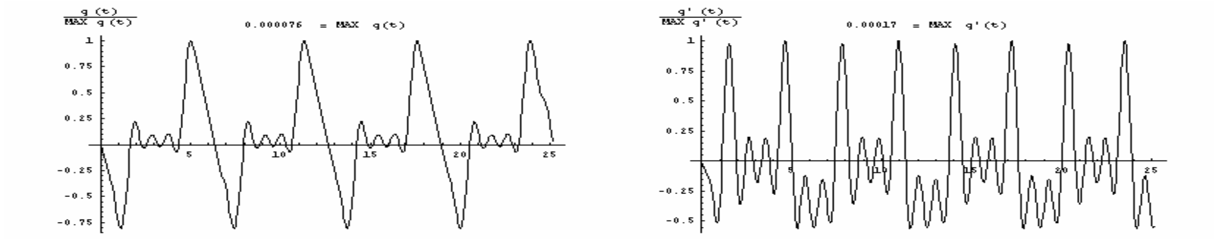


Fig. 5.3 The main mass (S2 in Fig. 2.2)

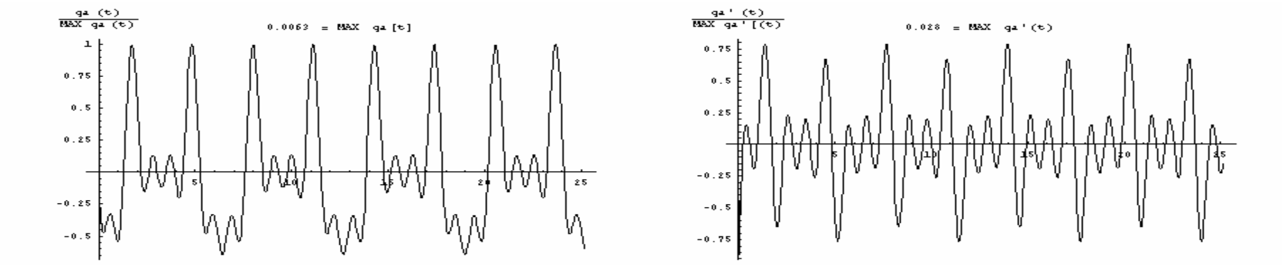


Fig. 5.4 The additional mass (S3 in Fig. 2.2)

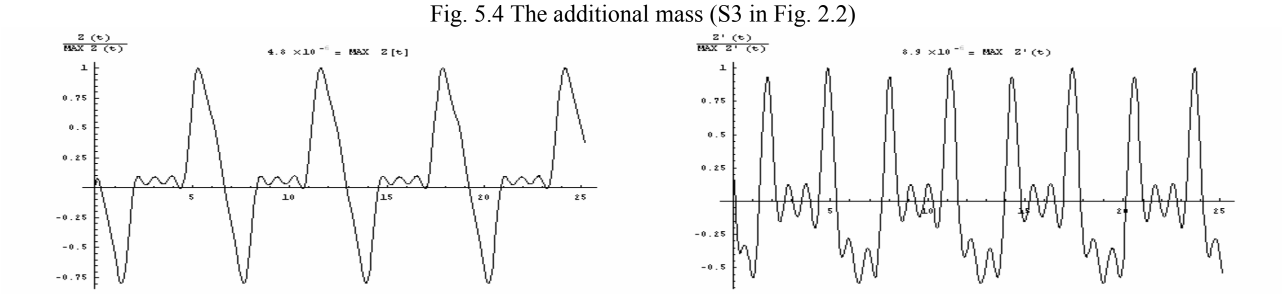
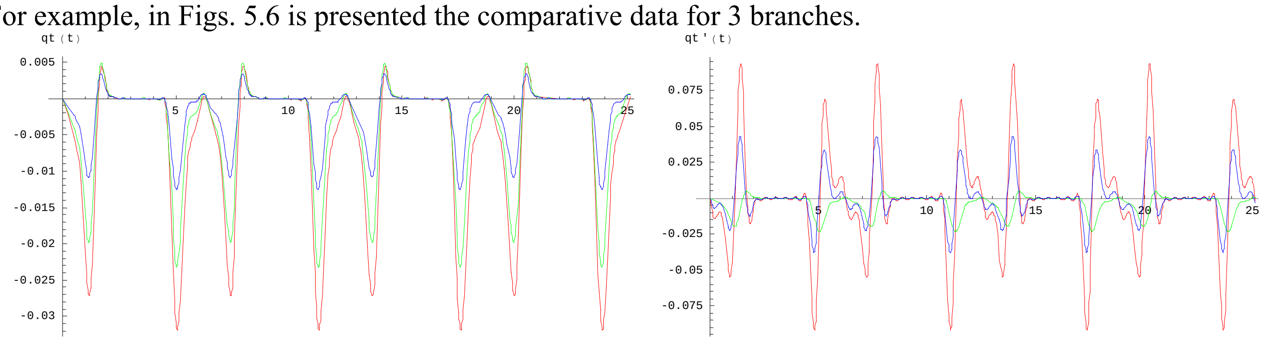
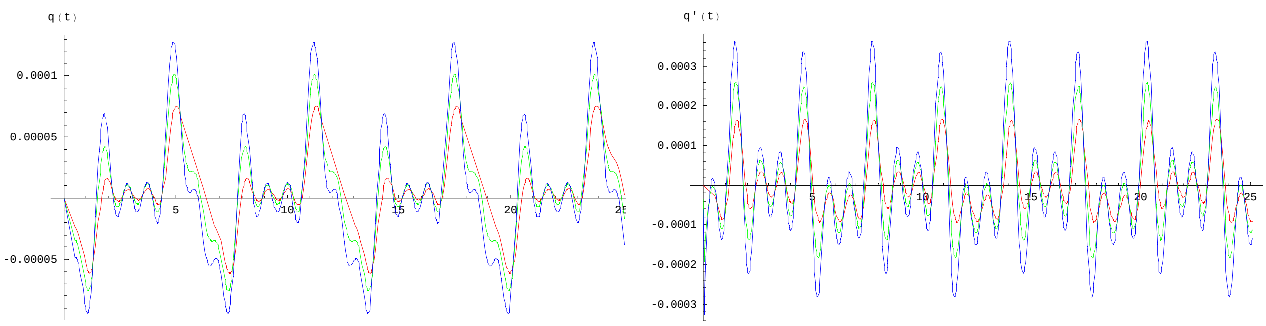


Fig. 5.5 The main shaft at bending (S4 in Fig. 2.2)

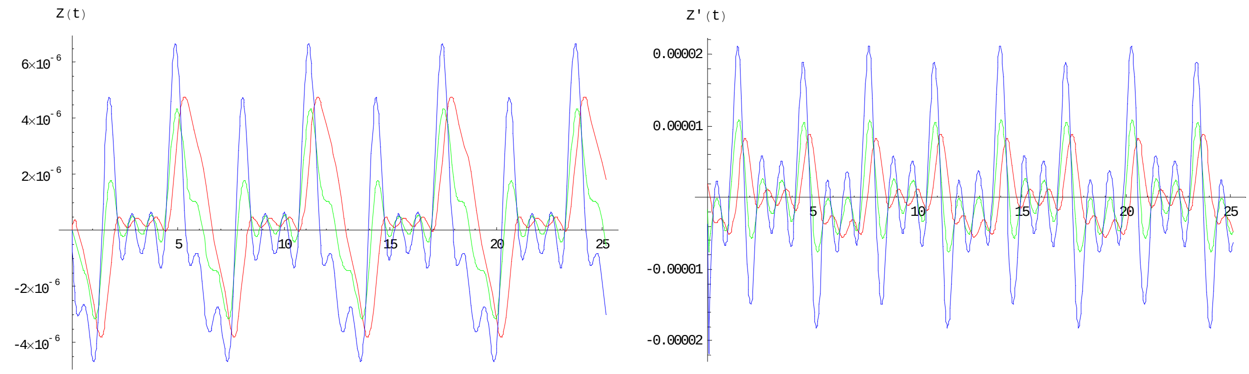
For each one of the 16 branches of the studied case, a set of information like Figs. 5.1-5.5 is available. For example, in Figs. 5.6 is presented the comparative data for 3 branches.



(a) The main shaft at torsion (S1 in Fig. 2.2)



(b) The main mass (S2 in Fig. 2.2)



(c) The main shaft at bending (S4 in Fig. 2.2)

Fig. 5.6 Comparative dynamic perturbations and perturbed speeds for 3 branches.

CONCLUDING REMARKS

In this work is presented a new CA method to study the elasto-dynamic behavior of the multi-bodies branched systems able to synthesize the matrix form of the mathematical model for a special kind of branched systems (n-BM) that can be represented by a special type of the dynamic model (section 2.2 and its descriptor matrix (section 2.3). The method allow us to consider the essentially nonlinear position functions of the driver mechanisms (one in each branch) in their real form, without any linearization unrelated of how big n is.

Since the mathematical model is achieve in a symbolic form it serve to a very large suite of further important results in analysis and synthesis of the studied branched system, as well.

Some results obtained for a 16-BM are presented in the last section.

REFERENCES

1. EISINGER BORCIA, C. E., Computer-Aided Elasto-Dynamic Analysis of Branched Systems Containing Mechanisms with Nonlinear Kinematics, Ben-Gurion University of the Negev, Israel, 1996.
2. EISINGER BORCIA, C. E., *A computer-aided optimization way of bending beams*, New Aspects of Engineering Mechanics Structures, Geology, Mathematics and Computers in Science Engineering. A Series of Reference Books and Textbooks, International Conference of Engineering Mechanics, Structures, Engineering Geology (EMESEG '09) Heraklion, WSEAS Press, pp.364-369, 2008, ISSN 1790-2769, ISBN 978-960-6766-88-6.
3. EISINGER BORCIA, C. E., *A general computer-aided method of obtaining the influence coefficients matrix*, Proc. of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science, vol. 9, nr.1, 2008
4. EISINGER BORCIA, C.E. , *Computational Methods for Branched Systems with Nonlinear Kinematics*, Research Trends in Mechanics- ch.2, vol.2, Ed. Academiei, 2008.
5. EISINGER BORCIA, C. E., SANDLER, B.Z., *Computer aided analysis and design of branched mechanisms* , Proceedings of EUROSIM'95 Simulation Congress, 663-668, 1995.
6. EISINGER BORCIA, C. E., SANDLER, B.Z., *Dynamic vibrations of branched mechanisms - analysis & synthesis*, Proceedings of IFToMM Ninth World Congress on the Theory of Machines and Mechanisms, 2, 1081-1085, 1995.
7. VULFSON, I. I., Vibroactivity of Branched and Ring Structured Mechanical Drivers, Hemisphere Publ. Corporation, 1989.
8. Wolfram Research Europe Ltd., MATHEMATICA for Microsoft Windows.

Received January 7, 2009