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DOMINATION NUMBERS OF THE COMPLETE GRID GRAPHS $P_{13} \times P_n$

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This paper concerns the domination numbers $\gamma(P_k \times P_n)$ for the complete $P_k \times P_n$ grid graphs for $k = 13$ and $n \geq 1$. These numbers were previously established for $1 \leq k \leq 12$ and $n \geq 1$ [3], [4]. The dominating set decision problem is NP-Complete [2]. There is no formula for the domination number of a graph. In this paper, we introduce the concept of transforming the domination from a vertex in a dominating set D of a graph $G = (V, E)$ to a vertex in $V - D$, where G is a simple connected graph. We give an algorithm using this transformation to obtain a dominating set of a graph G.

Key words: Dominating set; Domination numbers; Transformation of a dominating set; Cartesian product of two paths.

1. INTRODUCTION

A graph $G = (V, E)$ is a mathematical structure which consists of two sets *V* and *E*, where *V* is a finite and nonempty, and every element of *E* is an unordered pair $\{u, v\}$ of distinct elements of *V*; we simply write uv instead of $\{u, v\}$.

The elements of *V* are called vertices, while the elements of *E* are called edges. The order of *G* is the cardinality $|V|$ of its vertex-set, the size of *G* is the cardinality $|E|$ of its edges-set [1].

Two vertices *u* and *v* of a graph *G* are said to be adjacent if $uv \in E$. For a vertex *v* of *G*, the neighborhood of *v* is the set of all vertices of *G* which are adjacent to *v*; the neighborhood of *v*is denoted by $N(v)$. The closed neighborhood of *v* is $\bar{N}(v)$, $\bar{N}(v) = N(v) \cup \{v\}$.

If *D* is a set of vertices of *G*, then the neighborhood of *D* is $N(D) = \bigcup_{v \in D} N(v)$, and $\bar{N}(D) = \bigcup_{v \in D} \bar{N}(v)$. The degree of a vertex *v* is $d(v) = |N(v)|$.

Let $G = (V, E)$ be a graph; a set $D \subseteq V$ is called a dominating set of *G* if every vertex in $V - D$ is adjacent to at least one vertex of *D*; *i.e.* if $\bar{N}(D) = V$.

A dominating set *D* of *G* is said to be a minimum dominating set of *G* if $|D| \leq |D_1|$ for any dominating set D_1 of *G*.

A minimal dominating set in a graph *G* is a dominating set that contains no dominating set as a proper subset. The cardinality of a minimum dominating set of *G* is known as the domination number of *G*, and is denoted by $\gamma(G)$.

2. DEFINITIONS

Let D be a dominating set of a graph $G=(V,E)$.

1. We define the function C_D, which we call the weight function, as follows: $C_D : v \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, $C_D(v)=\left|\tilde{N}(v)\right|$, where $\tilde{N}(v)=\{w\in D : vw\in E \text{ or } w=v\}$, *i.e.* the weight of v is the number of vertices in D which dominate v.

2. We say that $v\in D$ has a moving domination if there exists a vertex $w\in N(v)-D$ such that $wu\in E$ for every vertex $u\in\{x\in N(v) : C_D(x)=1\}$.

3. We say that a vertex $v\in D$ is a redundant vertex of D if $C_D(w)\geq 2$ for every vertex $w\in\bar{N}(v)$.

4. If $v\in D$ has a moving domination, we say that v is inefficient if transforming the domination from v to any vertex in $N(v)$ would not produce any redundant vertex.

3. COMPLETE GRID GRAPH $P_k\times P_n$

For two vertices v_0 and v_n of a graph G, a v_0-v_n walk is an alternating sequence of vertices and edges $v_0,e_1,v_1,\dots,e_n,v_n$ such that consecutive vertices and edges are incident.

A path is a walk in which no vertex is repeated. A path with n vertices is denoted by P_n, it has $n-1$ edges; the length of P_n is $n-1$; the Cartesian product $P_k\times P_n$ of two paths is the complete grid graph with vertex set $V=\{(i,j):1\leq i\leq k, 1\leq j\leq n\}$, where $(u_1,u_2)(v_1,v_2)$ is an edge of $P_k\times P_n$ if $|u_1-v_1|+|u_2-v_2|=1$ [4].

If D is a dominating set of $P_k\times P_n$ which has no redundant vertex, then a vertex $v\in D$ has a moving domination if and only if one of the following two cases occurs:

Case 1: for every vertex $w\in N(v)$, we have $C_D(w)\geq 2$. In this case the domination of v can be transformed to any vertex in $N(v)-D$.

Case 2: there exists exactly one vertex $u\in N(v)$ such that $C_D(u)=1$. In this case, the domination of v can be transformed only to u.

4. AN ALGORITHM FOR FINDING A DOMINATING SET OF A GRAPH $P_k\times P_n$ USING A TRANSFORMATION OF DOMINATION OF VERTICES

1 – Let $P_k\times P_n=(V,E)$ be a graph of order greater than 1; $|V|=m$.

2 – Let $D=V$ be a dominating set of $P_k\times P_n$. Then, for any vertex $v\in D$ we have $C_D(v)=d(v)+1\geq 2$.

3 – Pick a vertex v_1 of D, and delete from D all vertices w, $w\in N(v_1)$. Then, for $1<n<\frac{m}{2}$, pick a vertex $v_n\in D-\bigcup_{i=1}^{n-1}\bar{N}(v_i)$ and delete from D all vertices w, $w\in N(v_n)-\bigcup_{i=1}^{n-1}\bar{N}(v_i)$.

4 – If D contains a redundant vertex, then delete it. Repeat this process until D has no redundant vertex.

5 – Transform domination from vertices of D which have moving domination to vertices in $V-D$ to obtain redundant vertices and go to step 4. If no redundant vertex can be obtained by a transformation of domination of vertices of D, then stop, and the obtained dominating set D satisfies: for every $v\in D$, $\exists w\in N(v)$ such that $C_D(w)=1$.

Example:

1 – Let (k,n) be the vertex in the k^{th} row and in the n^{th} column of the graph $G = P_{13} \times P_{12}$; $|V| = 156$.

2 – Let $D = V$, dominating set of *G*.

3 – Pick a vertex $v_1 = (1,2) \in D$, and delete from *D* all vertices w , $w \in N(v_1)$, then, for $1 < n < \frac{156}{2}$, pick a vertex v_n , $v_n \in D - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$, and delete from *D* all vertices w , $w \in N(v_n) - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$. We obtain the dominating set *D* (black circles) in Fig. 1.

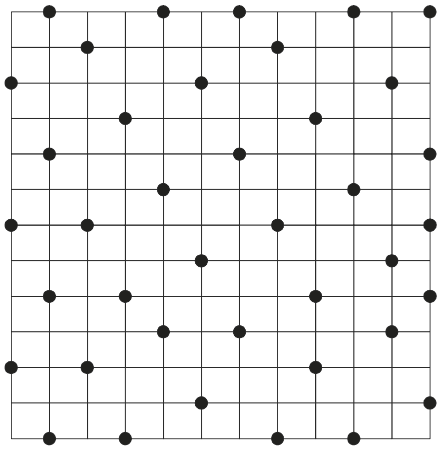


Fig. 1

4 – Since for every vertex $v \in D$, $\exists w \in \overline{N}(v)$ such that $C_D(w) = 1$, *D* has no redundant vertices.

5 – Transform the domination from the vertex $(7,12)$ to the vertex $(8,12)$ and delete, from *D*, the resulting redundant vertex $(9,12)$. Note that the vertices $(1,2),(1,7),(1,12),(9,4),(10,5)$ are the only vertices of *D* with moving domination, but inefficient.

Therefore, the set *D* indicated in Fig. 2 (black circles) is a dominating set of $G = P_{13} \times P_{12}$. Note that *D* is a minimum dominating set (see [3]), $\gamma(P_{13} \times P_{12}) = 38$.

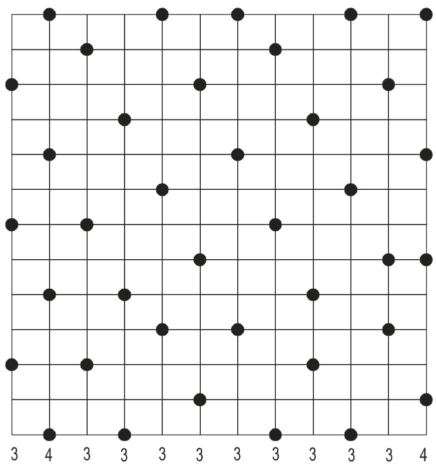


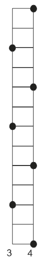
Fig. 2 – $n = 12$; $\gamma(P_{13} \times P_{12}) = 3 \times 10 + 8 = 3(n-2) + 8 = 3n + 2$.

And so, we gradually get domination numbers of $P_{13} \times P_n$, $1 \le n \le 16$.

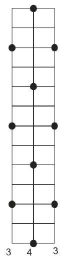
Table 1

Domination numbers $\gamma\left(P_{13} \times P_n\right)$ for $1 \le n \le 16$

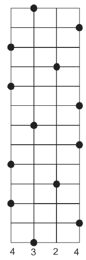
$\gamma(P_{13} \times P_1) = 5$



$\gamma(P_{13} \times P_2) = 7$

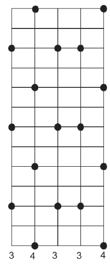


$\gamma(P_{13} \times P_3) = 10$



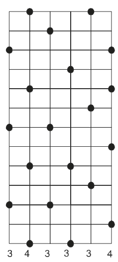
n=4

$$\gamma(P_{13} \times P_4) = (\tfrac{n-1}{3})(4+3+2)+4 \\ = 3n+1 \\ = 13$$



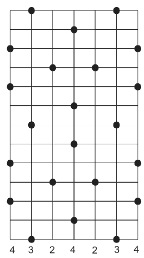
n=5

$$\gamma(P_{13} \times P_5) = 3(n-2)+8 \\ = 3n+2 \\ = 17$$



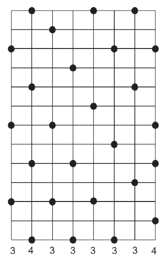
n=6

$$\gamma(P_{13} \times P_6) = 3(n-2)+8 \\ = 3n+2 \\ = 20$$



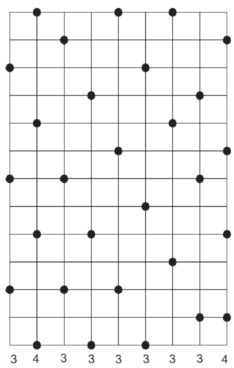
n=7

$$\gamma(P_{13} \times P_7) = (\tfrac{n-1}{3})(4+3+2)+4 \\ = 3n+1 \\ = 22$$



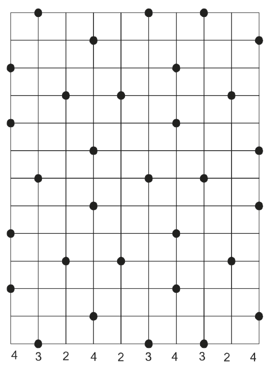
n=8

$$\gamma(P_{13} \times P_8) = 3(n-2)+8 \\ = 3n+2 \\ = 26$$



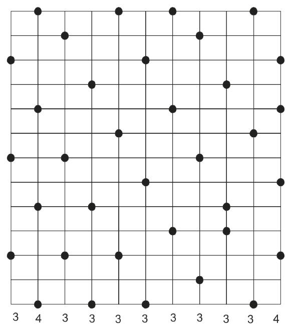
n=9

$$\gamma(P_{13} \times P_9) = 3(n-2)+8 \\ = 3n+2 \\ = 29$$



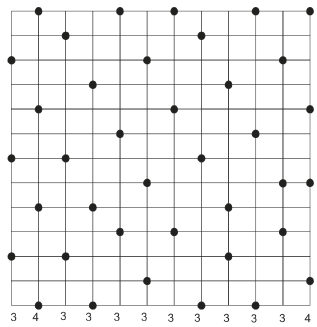
n=10

$$\gamma(P_{13} \times P_{10}) = (\tfrac{n-1}{3})(4+3+2)+4 \\ = 3n+1 \\ = 31$$



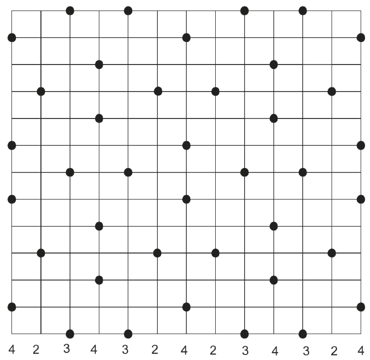
n=11

$$\gamma(P_{13} \times P_{11}) = 3(n-2)+8 = 3n+2 = 35$$



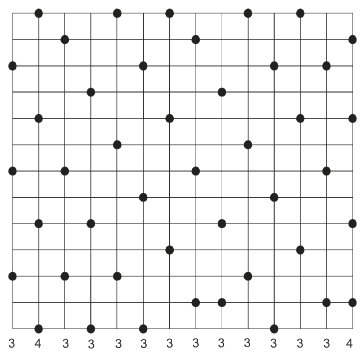
n=12

$$\gamma(P_{13} \times P_{12}) = 3(n-2)+8 = 3n+2 = 38$$



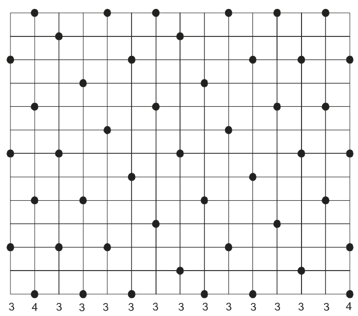
n=13

$$\gamma(P_{13} \times P_{13}) = (\tfrac{n-1}{3})(4+3+2)+4 = 3n+1 = 40$$



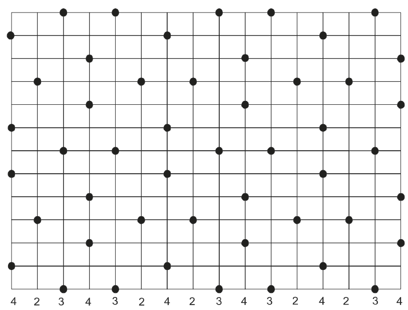
n=14

$$\gamma(P_{13} \times P_{14}) = 3(n-2)+8 = 3n+2 = 44$$



n=15

$$\gamma(P_{13} \times P_{15}) = 3(n-2)+8 = 3n+2 = 47$$



n=16

$$\gamma(P_{13} \times P_{16}) = (\tfrac{n-1}{3})(4+3+2)+4 = 3n+1 = 49$$

Table 2

Known number of elements in the 1st column

Grid Graph $(P_k \times P_n)$	$P_{13} \times P_2$	$P_{13} \times P_3$	$P_{13} \times P_4$	$P_{13} \times P_5$	$P_{13} \times P_6$	$P_{13} \times P_7$	....
Number of elements of dominating set in 1-st column	3	3	4	3	3	4	....

Hence:

$$\gamma\left(P_{13}\times P_n\right)=\begin{cases}5 & \text{for } n=1\\ 7 & \text{for } n=2\\ 10 & \text{for } n=3\\ 3n+2 & \text{for } n=3t-1, 3t, t\geq 2\\ 3n+1 & \text{for } n=3t+1, t\geq 1\end{cases},$$

where t is a positive integer.

We plan to deal with $P_k \times P_n$ grid graphs, for $n \geq 1$ and $k \geq 14$, in a forthcoming paper.

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