DOMINATION NUMBERS OF THE COMPLETE GRID GRAPHS $P_{13} \times P_n$

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This paper concerns the domination numbers $\gamma(P_k \times P_n)$ for the complete $P_k \times P_n$ grid graphs for $k = 13$ and $n \geq 1$. These numbers were previously established for $1 \leq k \leq 12$ and $n \geq 1$ [3], [4]. The dominating set decision problem is NP-Complete [2]. There is no formula for the domination number of a graph. In this paper, we introduce the concept of transforming the domination from a vertex in a dominating set $D$ of a graph $G = (V, E)$ to a vertex in $V - D$, where $G$ is a simple connected graph. We give an algorithm using this transformation to obtain a dominating set of a graph $G$.

Key words: Dominating set; Domination numbers; Transformation of a dominating set; Cartesian product of two paths.

1. INTRODUCTION

A graph $G = (V, E)$ is a mathematical structure which consists of two sets $V$ and $E$, where $V$ is a finite and nonempty, and every element of $E$ is an unordered pair $\{u, v\}$ of distinct elements of $V$; we simply write $uv$ instead of $\{u, v\}$.

The elements of $V$ are called vertices, while the elements of $E$ are called edges. The order of $G$ is the cardinality $|V|$ of its vertex-set, the size of $G$ is the cardinality $|E|$ of its edges-set [1].

Two vertices $u$ and $v$ of a graph $G$ are said to be adjacent if $uv \in E$. For a vertex $v$ of $G$, the neighborhood of $v$ is the set of all vertices of $G$ which are adjacent to $v$; the neighborhood of $v$ is denoted by $N(v)$. The closed neighborhood of $v$ is $\overline{N}(v)$, $\overline{N}(v) = N(v) \cup \{v\}$.

If $D$ is a set of vertices of $G$, then the neighborhood of $D$ is $N(D) = \bigcup_{v \in D} N(v)$, and $\overline{N}(D) = \bigcup_{v \in D} \overline{N}(v)$.

The degree of a vertex $v$ is $d(v) = |N(v)|$.

Let $G = (V, E)$ be a graph; a set $D \subseteq V$ is called a dominating set of $G$ if every vertex in $V - D$ is adjacent to at least one vertex of $D$; i.e. if $\overline{N}(D) = V$.

A dominating set $D$ of $G$ is said to be a minimum dominating set of $G$ if $|D| \leq |D|$ for any dominating set $D$ of $G$.

A minimal dominating set in a graph $G$ is a dominating set that contains no dominating set as a proper subset. The cardinality of a minimum dominating set of $G$ is known as the domination number of $G$, and is denoted by $\gamma(G)$.
2. DEFINITIONS

Let $D$ be a dominating set of a graph $G=(V,E)$.

1. We define the function $C_D$, which we call the weight function, as follows: $C_D : v \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, $C_D(v) = |N(v)|$, where $N(v) = \{ w \in D : v \in E \text{ or } w = v \}$, i.e. the weight of $v$ is the number of vertices in $D$ which dominate $v$.

2. We say that $v \in D$ has a moving domination if there exists a vertex $w \in N(v) - D$ such that $wu \in E$ for every vertex $u \in \{ x \in N(v) : C_D(x) = 1 \}$.

3. We say that a vertex $v \in D$ is a redundant vertex of $D$ if $C_D(w) \geq 2$ for every vertex $w \in N(v)$.

4. If $v \in D$ has a moving domination, we say that $v$ is inefficient if transforming the domination from $v$ to any vertex in $N(v)$ would not produce any redundant vertex.

3. COMPLETE GRID GRAPH $P_k \times P_n$

For two vertices $v_0$ and $v_n$ of a graph $G$, a $v_0 - v_n$ walk is an alternating sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_n, v_n$ such that consecutive vertices and edges are incident.

A path is a walk in which no vertex is repeated. A path with $n$ vertices is denoted by $P_n$, it has $n-1$ edges; the length of $P_n$ is $n-1$; the Cartesian product $P_k \times P_n$ of two paths is the complete grid graph with vertex set $V = \{ (i,j) : 1 \leq i \leq k, 1 \leq j \leq n \}$, where $(u_1, u_2)(v_1, v_2)$ is an edge of $P_k \times P_n$ if $|u_1 - v_1| + |u_2 - v_2| = 1$ [4].

If $D$ is a dominating set of $P_k \times P_n$ which has no redundant vertex, then a vertex $v \in D$ has a moving domination if and only if one of the following two cases occurs:

Case 1: for every vertex $w \in N(v)$, we have $C_D(w) \geq 2$. In this case the domination of $v$ can be transformed to any vertex in $N(v) - D$.

Case 2: there exists exactly one vertex $u \in N(v)$ such that $C_D(u) = 1$. In this case, the domination of $v$ can be transformed only to $u$.

4. AN ALGORITHM FOR FINDING A DOMINATING SET OF A GRAPH $P_k \times P_n$ USING A TRANSFORMATION OF DOMINATION OF VERTEXES

1. Let $P_k \times P_n = (V,E)$ be a graph of order greater than 1; $|V| = m$.

2. Let $D = V$ be a dominating set of $P_k \times P_n$. Then, for any vertex $v \in D$ we have $C_D(v) = d(v) + 1 \geq 2$.

3. Pick a vertex $v_1$ of $D$, and delete from $D$ all vertices $w$, $w \in N(v_1)$. Then, for $1 < n < \frac{m}{2}$, pick a vertex $v_e \in D - \bigcup_{v_j \in v} N(v_j)$ and delete from $D$ all vertices $w$, $w \in N(v_e) - \bigcup_{v_j \in v} N(v_j)$.

4. If $D$ contains a redundant vertex, then delete it. Repeat this process until $D$ has no redundant vertex.

5. Transform domination from vertices of $D$ which have moving domination to vertices in $V - D$ to obtain redundant vertices and go to step 4. If no redundant vertex can be obtained by a transformation of domination of vertices of $D$, then stop, and the obtained dominating set $D$ satisfies: for every $v \in D$, $\exists w \in N(v)$ such that $C_D(w) = 1$. 
Example:
1 – Let \((k,n)\) be the vertex in the \(k^{th}\) row and in the \(n^{th}\) column of the graph \(G = P_{13} \times P_{12} ; \vert V \vert = 156\).
2 – Let \(D = V\), dominating set of \(G\).
3 – Pick a vertex \(v_i = (1,2) \in D\), and delete from \(D\) all vertices \(w, w \in N(v_i)\), then, for \(1 < n < \frac{156}{2}\),
   pick a vertex \(v_n, v_n \in D - \bigcup_{i=1}^{n-1} N(v_i)\), and delete from \(D\) all vertices \(w, w \in N(v_n) - \bigcup_{i=1}^{n-1} N(v_i)\). We obtain the dominating set \(D\) (black circles) in Fig. 1.

![Fig. 1]

4 – Since for every vertex \(v \in D, \exists w \in \overline{N}(v)\) such that \(C_D(w) = 1\), \(D\) has no redundant vertices.
5 – Transform the domination from the vertex \((7,12)\) to the vertex \((8,12)\) and delete, from \(D\), the resulting redundant vertex \((9,12)\). Note that the vertices \((1,2),(1,7),(1,12),(9,4),(10,5)\) are the only vertices of \(D\) with moving domination, but inefficient.
Therefore, the set \(D\) indicated in Fig. 2 (black circles) is a dominating set of \(G = P_{13} \times P_{12} \). Note that \(D\) is a minimum dominating set (see [3]), \(\gamma(P_{13} \times P_{12}) = 38\).

![Fig. 2]

4 \(n = 12; \gamma(P_{13} \times P_{12}) = 3 \times 10 + 8 = 3(n-2) + 8 = 3n + 2\).

And so, we gradually get domination numbers of \(P_{13} \times P_{n} , 1 \leq n \leq 16\).
Table 1

Domination numbers $\gamma(P_{13} \times P_n)$ for $1 \leq n \leq 16$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\gamma(P_{13} \times P_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{n-1}{3}(4+3+2)+4$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{n-1}{3}(4+3+2)+4$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{n-1}{3}(4+3+2)+4$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{n-1}{3}(4+3+2)+4$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{n-1}{3}(4+3+2)+4$</td>
</tr>
</tbody>
</table>

Note: The expressions for $\gamma(P_{13} \times P_n)$ are derived from the given equations and simplified for clarity.
Table 2

Known number of elements in the 1st column

<table>
<thead>
<tr>
<th>Grid Graph ( P_1 \times P_n )</th>
<th>( P_{13} \times P_2 )</th>
<th>( P_{13} \times P_3 )</th>
<th>( P_{13} \times P_4 )</th>
<th>( P_{13} \times P_5 )</th>
<th>( P_{13} \times P_6 )</th>
<th>( P_{13} \times P_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements of dominating set in 1st column</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
Hence:

\[
\gamma(P_n \times P_n) = \begin{cases} 
5 & \text{for } n = 1 \\
7 & \text{for } n = 2 \\
10 & \text{for } n = 3 \\
3n + 2 & \text{for } n = 3t - 1, 3t, t \geq 2 \\
3n + 1 & \text{for } n = 3t + 1, t \geq 1 
\end{cases}
\]

where \( t \) is a positive integer.

We plan to deal with \( P_n \times P_n \) grid graphs, for \( n \geq 1 \) and \( k \geq 14 \), in a forthcoming paper.

REFERENCES

4. TONY YU CHANG, W. EDWIN CLARK., *The domination numbers of \( 5 \times n \) and \( 6 \times n \) grid graphs*, Journal of graph theory, 17, 1, pp. 81–107, 1993.

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