THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 12, Number 1/2011, pp. 9–15

A GENERALIZATION OF THE WEAK AMENABILITY OF SOME BANACH ALGEBRAS

Kazem HAGHNEJAD AZAR *, Abdolhamid. RIAZI **

[*] Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, Iran
Email: haghnejad@aut.ac.ir
[**] Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, Iran
Email: riazi@aut.ac.ir

Let A be a Banach algebra and A^{**} be the second dual of it. We show that by some new conditions, A is weakly amenable whenever A^{**} is weakly amenable. We will study this problem under generalization, that is, if $(n+2)^{\text{th}}$ dual of A, $A^{(n+2)}$, is $T-S-$ weakly amenable, then $A^{(n)}$ is $T-S-$ weakly amenable where T and S are continuous linear mappings from $A^{(n)}$ into $A^{(n)}$ and $n \geq 0$ is even number. We have some conclusions regarding the Arens regularity of Banach algebras.

Key words: Arens regularity; Weak amenability; Bilinear mapping; Topological center; Second dual.

1. INTRODICTION

Let A be a Banach algebra and A^*, A^{**}, respectively are the first and second duals of A. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa', respectively, the functionals in A^* defined by $\langle a'a, b\rangle = \langle a', ab\rangle$ and $\langle aa', b\rangle = \langle a', ba\rangle$, for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle - \langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$. Arens [1] has shown that given any Banach algebra A, there exist two algebra multiplications on the second dual of A which extend multiplication on A. In the following, we introduce both multiplications which are given in [13]. The first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''\, b'', a'\rangle = \langle a'', b''a'\rangle,$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of a'', $b'' \in A^{**}$ shall be defined by:

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'ob''\rangle,$$

for all $a, b \in A$ and $a' \in A^*$.

We say that A is Arens regular if both multiplications are equal. Let a'' and b'' be elements of A^{**}. By the Goldstin's theorem [6, p.425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_{\alpha}\lim_{\beta}\langle a', a_\alpha b_\beta\rangle = \langle a'' b'', a'\rangle$$

and

$$\lim_{\beta}\lim_{\alpha}\langle a', a_\alpha b_\beta\rangle = \langle a'' o\, b'', a'\rangle.$$

Thus A is Arens regular if and only if for every $a' \in A^*$, we have

$$\lim_{\alpha}\lim_{\beta}\langle a', a_\alpha b_\beta\rangle = \lim_{\beta}\lim_{\alpha}\langle a', a_\alpha b_\beta\rangle.$$

For more detail see [6, 13, 15].

Let X be a Banach A-bimodule. A *derivation* from A into X is a bounded linear mapping $D : A \longrightarrow X$ such that

$$D(xy) = xD(y) + D(x)y \quad \text{for all } x, y \in A.$$

The space of continuous derivations from A into X is denoted by $Z^1(A, X)$. Easy examples of derivations are the inner derivations, which are given for each $x \in X$ by

$$\delta_x(a) = ax - xa \quad \text{for all } a \in A.$$

The space of inner derivations from A into X is denoted by $N^1(A, X)$. The Banach algebra A is said to be *amenable*, when for every Banach A-bimodule X, the inner derivations are only derivations existing from A into X^*. It is clear that A is amenable if and only if $H^1(A, X^*) = Z^1(A, X^*)/N^1(A, X^*) = \{0\}$, for every Banach A-bimodule X. The concept of amenability for a Banach algebra A, introduced by Johnson in 1972, has proved to be of enormous importance in Banach algebra theory, see [10]. A Banach algebra A is said to be *weakly amenable*, if every derivation from A into A^* is inner. Equivalently, A is weakly amenable if and only if $H^1(A, A^*) = Z^1(A, A^*)/N^1(A, A^*) = \{0\}$. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [2] for commutative Banach algebras, and was extended to the noncommutative case by Johnson, see [11].

For a Banach A-bimodule X, the quotient space $H^1(A, X)$ of all continuous derivations from A into X modulo the subspace of inner derivations is called the first cohomology group of A with coefficients in X.

Suppose that A is a Banach algebra and X is a Banach A-bimodule X. According to [5, p. 27 and 28], X^{**} is a Banach A^{**}-bimodule, where A^{**} is equipped with the first Arens product.

Let $A^{(n)}$ and $X^{(n)}$ be n^{th} duals of A and X, respectively. By [19, pp. 4132–4134], if $n \geq 0$ is an even number, then $X^{(n)}$ is a Banach $A^{(n)}$ bimodule. Then for $n \geq 0$, we define $X^{(n)}X^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for all $x^{(n)} \in X^{(n)}$, $x^{(n-1)} \in X^{(n-1)}$ and $a^{(n-2)} \in A^{(n-2)}$ we define

$$\langle x^{(n)}x^{(n-1)}, a^{(n-2)}\rangle = \langle x^{(n)}, x^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, then for $n \geq 1$, we define $X^{(n)}X^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $x^{(n)} \in X^{(n)}$, $x^{(n-1)} \in X^{(n-1)}$ and $a^{(n-1)} \in A^{(n-1)}$ we define

$$\langle x^{(n)}x^{(n-1)}, a^{(n-1)}\rangle = \langle x^{(n)}, x^{(n-1)}a^{(n-1)}\rangle.$$

If $n = 0$, we take $A^{(0)} = A$ and $X^{(0)} = X$.

Now let X be a Banach A-bimodule and $D : A \to X$ be a derivation. A problem which is of interest is under what conditions D'' is again a derivation. In [14, Theorem 5.9], this problem has been studied for the special case $X = A$, and they showed that D'' is a derivation if and only if $D''(A^{**})A^{**} \subseteq A^{*}$. We study this problem in the generality, that is, if $A^{(n+2)}$ is $T''-S''-$weakly amenable, then $A^{(n)}$ is $T-S-$weakly amenable where T and S are continuous linear mappings from $A^{(n)}$ into $A^{(n)}$ and $n \geq 0$.
The main results of this paper can be summarized as follows:

a) Assume that A is a Banach algebra and $A^{(n+2)}$ has $T-w^{*}w$ property. If $A^{(n+2)}$ is weakly $T''-S''-$amenable, then $A^{(n)}$ is weakly $T-S-$amenable.
b) Let X be a Banach A-bimodule and let $T, S : A^{(n)} \to A^{(n)}$ be continuous linear mappings. Let the mapping $a^{(n+2)} \to x^{(n+2)}T''(a^{(n+2)})$ be weak*-to-weak continuous for all $x^{(n+2)} \in X^{(n+2)}$. Then if $D : A^{(n)} \to X^{(n+1)}$ is a $T-S-$derivation, it follows that $D'' : A^{(n+2)} \to X^{(n+3)}$ is a $T''-S''-$derivation.
c) Let X be a Banach $A-$bimodule and the mapping $a'' \to x''a''$ be weak*-to- weak continuous for all $x'' \in X^{**}$. If $D : A \to X^{*}$ is a derivation, then $D''(A^{**})X^{**} \subseteq A^{*}$.
d) Let X be a Banach $A-$bimodule and $D : A \to X^{*}$ be a derivation. Suppose that $D'' : A^{**} \to X^{***}$ is surjective derivation. Then the mapping $a'' \to x''a''$ is weak*-to- weak continuous for all $x'' \in X^{**}$.
e) Suppose that X is a Banach $A-$bimodule and A is Arens regular. Assume that $D : A \to X^{*}$ is a derivation and surjective. Then $D'' : A^{**} \to X^{***}$ is a derivation if and only if the mapping $a'' \to x''a''$ is weak*-to-weak continuous for all $x'' \in X^{**}$.

In every parts of this paper, $n \geq 0$ is even number.

2. WEAK AMENABILITY OF BANACH ALGEBRAS

Definition 2.1. Let X be a Banach $A-$bimodule and T, S be continuous linear mappings from A into itself. We say that a linear mapping $D : A \to X$ is $T-S-$derivation, if

$$D(xy) = T(x)D(y) + D(x)S(y) \text{ for all } x, y \in A.$$

Now let $x \in A$. Then we say that the linear mapping $\delta_x : A \to A$ is inner $T-S-$derivation, if for every $a \in A$, we have $\delta_x(a) = T(a)x - xS(a)$. The Banach algebra A is said to be a $T-S-$amenable, when for every Banach $A-$bimodule X, every $T-S-$derivations from A into X^{*} is inner $T-S-$derivations.
The definition of weakly $T-S-$amenable is similar.

Definition 2.2. Assume that A is a Banach algebra and $T : A \to A$ is a continuous linear mapping such that the mapping $b'' \to a''T''(b'') : A^{**} \to A^{**}$ is weak*-to- weak continuous where $a'' \in A^{**}$. Then we say that $a'' \in A^{**}$ has $T-w^{*}w$ property. We say that $B \subseteq A^{*}$ has $T-w^{*}w$ property, if every $b \in B$ has $T-w^{*}w$ property.

For a Banach algebra A, take the linear mapping $I : x \to x$ from A into A and assume that A^{**} has $I-w^{*}w$ property. Then, it is clear that A is Arens regular.

There are some non-reflexive Banach algebras whose second duals have $T-w^{*}w$ property where T is a continuous linear mappings from A into itself. If A is Arens regular, then, in general, A^{**} dose not have $T-w^{*}w$ property. In the following we give some examples of Banach algebras whose second duals have $T-w^{*}w$ property or not.

i) Let A *be a non-reflexive Banach space and suppose that* $\langle f, x\rangle = 1$ *for some* $f \in A^*$ *and* $x \in A$. *We define the product on* A *as* $ab = \langle f, a\rangle b$ *for all* $a, b \in A$. *It is clear that* A *is a Banach algebra with this product, then* A^{**} *has* $I - w^*w$ *property.*

ii) Every reflexive Banach algebra has $T - w^*w$ *property.*

iii) Consider the algebra $c_0 = (c_0, .)$ *is the collection of all sequences of scalars that convergence to* 0, *with the some vector space operations and norm as* ℓ_∞. *Then* $c_0^{**} = \ell_\infty$ *has* $I - w^*w$ *property.*

iv) $L^1(G)^{**}$ *and* $M(G)^{**}$ *have not* $I - w^*w$ *property whenever G is locally compact group, but when G is finite,* $L^1(G)^{**}$ *and* $M(G)^{**}$ *have* $I - w^*w$ *property.*

Theorem 2.1. *Assume that* A *is a Banach algebra and* $A^{(n+2)}$ *has* $T - w^*w$ *property. If* $D : A^{(n)} \to A^{(n+1)}$ *is a* $T - S -$*derivation, then* $D'' : A^{(n+2)} \to A^{(n+3)}$ *is a* $T'' - S'' -$*derivation.*

Proof. Let $a^{(n+2)}, b^{(n+2)} \in A^{(n+2)}$ and let $(a_\alpha^{(n)})_\alpha, (b_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n+2)}$ and $b_\beta^{(n)} \xrightarrow{w^*} b^{(n+2)}$. Due to $A^{(n+2)}$ has $T - w^*w$ property, we have $c^{(n+2)}T(a_\alpha^{(n)}) \xrightarrow{w} c^{(n+2)}T''(a^{(n+2)})$ for all $c^{(n+2)} \in A^{(n+2)}$. Using the weak*-to- weak* continuity of D'', we obtain

$$\begin{aligned}
&\lim_\alpha \lim_\beta \langle T(a_\alpha^{(n)}) D(b_\beta^{(n)}), c^{(n+2)}\rangle = \lim_\alpha \lim_\beta \langle D(b_\beta^{(n)}), c^{(n+2)} T(a_\alpha^{(n)})\rangle = \\
&= \lim_\alpha \langle D''(b^{(n+2)}), c^{(n+2)} T(a_\alpha^{(n)})\rangle = \langle D''(b^{(n+2)}), c^{(n+2)} T''(a^{(n+2)})\rangle = \\
&= \langle T''(a^{(n+2)}) D''(b^{(n+2)}), c^{(n+2)}\rangle.
\end{aligned}$$

Moreover, it is also clear that for every $c^{(n+2)} \in A^{(n+2)}$, we have

$$\lim_\alpha \lim_\beta \langle D(a_\alpha^{(n)}) S(b_\beta^{(n)}), c^{(n+2)}\rangle = \langle D''(a^{(n+2)}) S''(b^{(n+2)}), c^{(n+2)}\rangle.$$

Notice that in the latest equalities, we didn't need $S - w^*w$ property for $A^{(n+2)}$. In the following, we take the limit with respect to the weak$^*-$ topologies. Thus

$$\begin{aligned}
&D''(a^{(n+2)} b^{(n+2)}) = \lim_\alpha \lim_\beta D(a_\alpha^{(n)} b_\beta^{(n)}) = \lim_\alpha \lim_\beta T(a_\alpha^{(n)}) D(b_\beta^{(n)}) + \\
&\lim_\alpha \lim_\beta D(a_\alpha^{(n)}) S(b_\beta^{(n)}) = T''(a^{(n+2)}) D''(b^{(n+2)}) + D''(a^{(n+2)}) S''(b^{(n+2)}).
\end{aligned}$$

Theorem 2.2. *Assume that* A *is a Banach algebra and* $A^{(n+2)}$ *has* $T - w^*w$ *property. If* $A^{(n+2)}$ *is weakly* $T'' - S'' -$*amenable, then* $A^{(n)}$ *is weakly* $T - S -$*amenable.*

Proof. Let $D : A^{(n)} \to A^{(n+1)}$ be a $T - S -$derivation. Then by using Theorem 2.3, $D'' : A^{(n+2)} \to A^{(n+3)}$ is a $T'' - S'' -$derivation. Since $A^{(n+2)}$ is weakly $T'' - S'' -$amenable, $D'' : A^{(n+2)} \to A^{(n+3)}$ is an inner $T'' - S'' -$derivation. It follows that for every $a^{(n+2)} \in A^{(n+2)}$, we have

$$D''(a^{(n+2)}) = T''(a^{(n+2)}) a^{(n+3)} - a^{(n+3)} S''(b^{(n+2)}),$$

for some $a^{(n+3)} \in A^{(n+3)}$. Take $a^{(n+1)} = a^{(n+3)}|_{A^{(n)}}$ Then for every $a^{(n)} \in A^{(n)}$, we have

$$D(a^{(n)}) = T(a^{(n)})a^{(n+1)} - a^{(n+1)}S(b^{(n)}),$$

It follows that D is inner $T-S-$derivation, and this completes the proof.

Corollary 2.1. *Let A be a Banach algebra. If A^{**} has $I-w^*w$ property and A^{**} is weakly amenable, then A is weakly amenable.*

Corollary 2.2. *Let A be a Banach algebra. If $A^{***}A^{**} \subseteq A^*$ and A^{**} is weakly amenable, then A is weakly amenable.*

Proof. It is enough to show that A^{**} has $I-w^*w$ property. Suppose that $a'', b'' \in A^{**}$ and $b''_\alpha \xrightarrow{w^*} b''$. Let $c''' \in A^{***}$. Sine $c'''a'' \in A^*$, we have

$$\langle c''', a''b''_\alpha\rangle = \langle c'''a'', b''_\alpha\rangle = \langle b''_\alpha, c'''a''\rangle \to \langle b'', c'''a''\rangle = \langle c''', a''b''\rangle.$$

We conclude that $a''b''_\alpha \xrightarrow{w^*} a''b''$. So A^{**} has $I-w^*w$ property. By using Corollary 2.1, A is weakly amenable.

Example $c_0 = (c_0, .)$ is weakly amenable.

Proof. Since $\ell_\infty = c_0^{**}$ is weakly amenable and ℓ_∞ has $I-w^*w$ property by Corollary 2.1, the proof complete.

Theorem 2.3. *Suppose that A is a Banach algebra and B is a closed subalgebra of $A^{(n+2)}$ that is consisting of $A^{(n)}$ where $n \geq 0$. If B has $T-w^*w$ property and is weakly $T''-S''-$amenable, then $A^{(n)}$ is weakly $T-S-$amenable.*

Proof. Suppose that $D: A^{(n)} \to A^{(n+1)}$ is a $T-S-$derivation and $P: A^{(n+3)} \to B^*$ is the restriction map, defined by $P(a^{(n+3)}) = a^{(n+3)}|_B$ for every $a^{(n+3)} \in A^{(n+3)}$. Since B has $T-w^*w$ property, $\tilde{D} = PoD''|_B \to B^*$ is a $T''-S''-$derivation. Since B is weakly $T''-S''-$amenable, there is $b' \in B^*$ such that $\tilde{D} = \delta_{b'}$. We take $a^{(n+1)} = b'|_{A^{(n)}}$, then $D = \tilde{D}$ in $A^{(n)}$. Consequently, we have $D = \delta_{a^{(n+1)}}$.

Theorem 2.4. *Let X be a Banach A-bimodule and let $T, S: A^{(n)} \to A^{(n)}$ be continuous linear mappings. Let the mapping $a^{(n+2)} \to x^{(n+2)}T''(a^{(n+2)})$ be weak*-to-weak continuous for all $x^{(n+2)} \in X^{(n+2)}$. If $D: A^{(n)} \to X^{(n+1)}$ is a $T-S-$derivation, then $D'': A^{(n+2)} \to X^{(n+3)}$ is a $T''-S''-$derivation.*

Proof. Let $a^{(n+2)}, b^{(n+2)} \in A^{(n+2)}$ and let $(a_\alpha^{(n)})_\alpha, (b_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n+2)}$ and $b_\beta^{(n)} \xrightarrow{w^*} b^{(n+2)}$. Then for all $x^{(n+2)} \in X^{(n+2)}$, we have $x^{(n+2)}T(a_\alpha^{(n)}) \xrightarrow{w} x^{(n+2)}T''(a^{(n+2)})$. Consequently, we have

$$\lim_\alpha \lim_\beta \langle T(a_\alpha^{(n)})D(b_\beta^{(n)}), x^{(n+2)}\rangle = \lim_\alpha \lim_\beta \langle D(b_\beta^{(n)}), x^{(n+2)}T(a_\alpha^{(n)})\rangle =$$
$$= \lim_\alpha \langle D''(b^{(n+2)}), x^{(n+2)}T(a_\alpha^{(n)})\rangle = \langle D''(b^{(n+2)}), x^{(n+2)}T''(a^{(n+2)})\rangle =$$
$$= \langle T''(a^{(n+2)})D''(b^{(n+2)}), x^{(n+2)}\rangle.$$

For every $x^{(n+2)} \in X^{(n+2)}$, we have also the following equalities

$$\lim_{\alpha}\lim_{\beta}\langle D(a_{\alpha}{}^{(n)})S(b_{\beta}{}^{(n)}),x^{(n+2)}\rangle = \lim_{\alpha}\lim_{\beta}\langle D(a_{\alpha}{}^{(n)}),S(b_{\beta}{}^{(n)})x^{(n+2)}\rangle =$$
$$= \lim_{\alpha}\langle D''(a_{\alpha}{}^{(n)}),S(b^{(n+2)})x^{(n+2)}\rangle = \langle D''(a^{(n+2)}),S(b^{(n+2)})x^{(n+2)}\rangle =$$
$$= \langle D''(a^{(n+2)})S(b^{(n+2)}),x^{(n+2)}\rangle.$$

In the following, we take the limit with respect to the weak$^*-$ topologies. Using the weak*-to-weak* continuity of D'', we obtain

$$D''(a^{(n+2)}b^{(n+2)}) = \lim_{\alpha}\lim_{\beta} D(a_{\alpha}{}^{(n)}b_{\beta}{}^{(n)}) = \lim_{\alpha}\lim_{\beta} T(a_{\alpha}{}^{(n)})D(b_{\beta}{}^{(n)}) +$$
$$+\lim_{\alpha}\lim_{\beta} D(a_{\alpha}{}^{(n)})S(b_{\beta}{}^{(n)}) = T''(a^{(n+2)})D''(b^{(n+2)}) + D''(a^{(n+2)})S''(b^{(n+2)}).$$

Thus $D'': A^{(n+2)} \to X^{(n+3)}$ is a $T''-S''-$derivation. □

Corollary 2.3. *Let X be a Banach A-bimodule and the mapping $a'' \to x''a''$ be weak*-to-weak* continuous for all the $x'' \in X^{**}$. If $H^1(A^{**},X^{***}) = 0$,, then $H^1(A\ ,X^*) = 0$.*

Corollary 2.4. *Let X be a Banach $A-$bimodule and the mapping $a'' \to x''a''$ be weak*-to- weak continuous for all $x'' \in X^{**}$. If $D: A \to X^*$ is a derivation, then $D''(A^{**})X^{**} \subseteq A^*$.*

Proof. By using Theorem 2.9 and [14, Corollary 4.3], the proof complete. □

Theorem 2.5. *Let X be a Banach A-bimodule and $D: A \to X^*$ be a surjective derivation. Suppose that $D'': A^{**} \to X^{***}$ is also surjective derivation. Then the mapping $a'' \to x''a''$ is weak*-to- weak continuous for all $x'' \in X^{**}$.*

Proof. Let $a'' \in A^{**}$ such that $a''_{\alpha} \xrightarrow{w^*} a''$. We show that $x''a''_{\alpha} \xrightarrow{w} x''a''$ for all $x'' \in X^{**}$. Suppose that $x''' \in X^{***}$. Since $D''(A^{**}) = X^{***}$ by using [14, Corollary 4.3], we conclude that $X^{***}X^{**} = D''(A^{**})X^{**} \subseteq A^*$. Thus $x'''x'' \in A^*$, and so we have the following equalities

$$\langle x''',x''a''_{\alpha}\rangle = \langle x'''x'',a''_{\alpha}\rangle = \langle a''_{\alpha},x'''x''\rangle \to \langle a'',x'''x''\rangle = \langle x''',x''a''\rangle.$$

Corollary 2.5. *Suppose that X is a Banach A-bimodule and A is Arens regular. Assume that $D: A \to X^*$ is a derivation and surjective. Then $D'': A^{**} \to X^{***}$ is a derivation if and only if the mapping $a'' \to x''a''$ is weak*-to-weak continuous for all $x'' \in X^{**}$.*

Proof. By using Corollary 2.4, Theorem 2.5 and [14, Corollary 4.3], the proof complete. □

In the proceeding corollary, if we omit the Arens regularity of A, then we have also the following conclusion.

Assume that $D: A \to X^*$ is a surjective derivation. Then, $D''(A^{**})X^{**} \subseteq A^*$ if and only if the mapping $a'' \to x''a''$ is weak*-to-weak continuous for all $x'' \in X^{**}$.

Corollary 2.6. *Let A be a Banach algebra. Then we have the following results:*

i) Assume that A is Arens regular and $D: A \to A^*$ is a surjective derivation. Then $D'': A^{**} \to A^{***}$ is a derivation if and only if A has $I-w^*w$ property.

ii) *Assume that* $D : A \to A^*$ *is a surjective derivation. Then,* A *has* $I - w^*w$ *property if and only if* $D''(A^{**})A^{**} \subseteq A^*$. *So if* $D : A \to A^*$ *is a surjective derivation and* $D''(A^{**})A^{**} \subseteq A^*$, *then* A *is Arens regular.*

Problem. Let S be a semigroup. Does $C(S)^{**}$, $L^1(S)^{**}$ and $M(S)^{**}$ have $I - w^*w$ property ?

ACKNOWLEDGMENTS

We would like to thank the referee for her/his careful reading of our paper and many valuable suggestions.

REFERENCES

1. ARENSE, R.E., *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc, **2,** pp.839–848, 1951.
2. BADE W.G., CURTIS, P.C., DALES, H.G., *Amenability and weak amenability for Beurling and Lipschitz algebra*, Proc. London. Math. Soc**, 50**, pp.359–377, 1987.
3. BAKER, J., LAU, A.T., PYM, J.S., *Module homomorphism and topological centers associated with weakly sequentially compact Banach algebras,* Journal of Functional Analysis, **158**, pp.186–208, 1998.
4. BONSALL, F.F., DUNCAN, J., *Complete normed algebras*, Springer-Verlag, Berlin, 1973.
5. DALES, H.G., RODIGUES-PALACIOS, A., VELASCO, M.V., *The second transpose of a derivation*, J. London. Math. Soc., **64,** 2, pp. 707–721, 2001.
6. DALES, H.G., *Banach algebra and automatic continuity*, Oxford, 2000.
7. DUNFORD, N., SCHWARTZ, J.T., *Linear operators I,* Wiley, New York, 1958.
8. ESHAGHI GORDJI, M., FILALI, M., *Weak amenability of the second dual of a Banach algebra*, Studia Math, **182**, *3*, pp. 205–213, 2007.
9. HEWITT, E., ROSS, K.A., *Abstract harmonic analysis* I, Springer, Berlin, 1963.
10. JOHONSON, B.E., *Cohomology in Banach algebras,* Mem. Amer. Math. Soc, **127**, 1972.
11. JOHONSON, B.E., *Derivations from* $L^1(G)$ *into* $L^1(G)$ *and* $L^\infty(G)$, Lecture Note in Math, **1359**, pp. 191–198, 1988.
12. JOHONSON, B.E., *Weak amenability of group algebra*, Bull. London. Math. Soc., **23**, pp. 281–284, 1991.
13. LAU, A.T., ULGER, A., *Topological center of certain dual algebras*, Trans. Amer. Math. Soc., **348**, pp. 1191–1212, 1996.
14. MOHAMADZADIH, S., VISHKI, H.R.E., *Arens regularity of module actions and the second adjoint of a derivation*, Bulletin of the Australian Mathematical Society, **77**, pp. 465-476, 2008.
15. PYM, J.S., *The convolution of functionals on spaces of bounded functions*, Proc. London Math Soc., **15**, pp. 84–104, 1965.
16. ULGER, A., *Some stability properties of Arens regular bilinear operators*, Proc. Amer. Math. Soc., **34**, pp. 443–454, 1991.
17. ULGER, A., *Arens regularity of weakly sequentialy compact Banach algebras*, Proc. Amer. Math. Soc., **127**, *11*, pp. 3221–3227, 1999.
18. WONG, P.K., *The second conjugate algebras of Banach algebras*, J. Math. Sci., **17**, *1*, pp. 15–18, 1994.
19. ZHANG, Y., *Weak amenability of module extentions of Banach algebras*, Trans. Amer. Math. Soc., **354**, *10*, pp. 4131–4151, 2002.
20. ZHANG, Y., *Weak amenability of a class of Banach algebra*, Cand. Math. Bull, **44,** *4*, pp. 504–508, 2001.

Received November 22, 2010