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SOME RESULTS ON THE GEOMETRY OF THE ZEROS OF POLYNOMIALS

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A necessary and sufficient condition for a polynomial with complex coefficients to have all its zeros on the unit circle, generalizing a classical result established by Schur in 1917, is presented. Furthermore, incomplete polynomials are revisited and as a consequence a generalization of a theorem of Cohn on location of zeros of polynomials is also given

Key words: Geometry of polynomials; Location of zeros; Convex hull; Reflection coefficients; Unit circle problem.

1. INTRODUCTION

The study of polynomials from a non-algebraic standpoint is to a great extent connected with the geometric theory of functions of a complex variable. This connection becomes clear when we examine the type of problems dealt with and the techniques used to solve them. Hereof, some results in the "Geometry of the Zeros of a Polynomial in a Complex Variable" recently published are revisited. They are used to establish new ones and to generalize classics results on location of the zeros of polynomials. A basic concept continuously taken into account when deriving these results is the characterization of polynomials by their reflection coefficients, obtained by using Schur-Cohn type recursions ([11, 12]). It can be found in [7] and the references therein, where some results on location of the zeros are presented. Polynomials characterized by reflection coefficients appear frequently in many areas of applied science and technology ([11, 10, 5]).

Examples can be found in electric engineering, seismology and in control theory applications ([9, 3, 13]). A first goal in this paper is to obtain conditions for a polynomial to have all its zeros on the unit circle. As a consequence, a generalization of a classical result presented by Schur in 1917 is also given. A second goal is to revisit incomplete polynomials, introduced and efficiently used to obtain a generalization of the well-known Gauss-Lucas theorem in [8], in order to generalize another classical result established by Cohn in 1922.

2. NOTATION AND BASIC CONCEPTS

Some notation and concepts that will be used hereafter are recalled. Let $A_n(z)$ be a monic polynomial with complex coefficients of degree n, namely,

$$A_n(z) = z^n + \sum_{k=0}^{n-1} a_{nk} z^k .$$

The reciprocal polynomial $A_n^*(z)$ of $A_n(z)$ is defined by

$$A_n^*(z) = z^n \overline{A_n(1/\bar{z})} = \sum_{k=0}^{n} \bar{a}_{n,n-k} z^k = \bar{a}_{n0} \prod_{k=1}^{n} (z - z_k^*),$$

where $\bar{a}$ denotes the complex conjugate of a. Its zeros, $z_n^* = 1/\overline{z_k}$, are the inverse of the zeros of $A_n(z)$ in the unit circle. An immediate consequence of the well-known fact that $|A_n(z)| = |A_n^*(z)|$ for all z with $|z| = 1$ (see [12]), is that if $A_n(z)$ has no zeros of modulus 1; then $A_n^*(z)$ has no zeros on the unit circle $|z| = 1$. In this case, if $A_n(z)$ has $m, (m \le n)$ zeros inside the unit circle, then $A_n^*(z)$ has $n - m$ zeros in $|z| < 1$ The reflection coefficients α_k's, also known in the literature as Schur-Szegö parameters [5] or partial correlation (PARCOR) coefficients [10] can be obtained from $A_n(z)$ by using backward Levinson's recursion ([11, 15]).

$$zA_{k-1}(z) = \frac{1}{1 - |a_k|^2}\left[A_k(z) - \alpha_k A_k^*(z)\right], \tag{1}$$

where $\alpha_k = a_{k0}$. From (1) and some and some straightforward algebra forward Levinson's recursion

$$A_k(z) = zA_{k-1}(z) + \alpha_k A_{k-1}^*(z), \tag{2}$$

is obtained.

Now, for ease of reference, we give the definition of the characterization of $A_n(z)$ by its reflection coefficients as given in [7].

Definition 1. The characterization of a monic complex polynomial $A_n(z)$ using reflection coefficients is given by

$$A_n(z) = \left[A_j(z); \alpha_{j+1}, \alpha_{j+2}, ..., \alpha_n\right],$$

where $A_j(z)$, called base polynomial, is either $A_j(z) = A_0(z) = 1$ for $j = 0$, or $A_j(z)$ is a non-self-inversive unitary polynomial (i.e., $|a_{j0}| = 1$) for $1 \le j \le n$. The $\alpha_k \in \mathrm{C}$, $k = j+1,\ j+2,\ \dots,\ n$, are reflection coefficients.

2. A GENERALIZATION OF A THEOREM OF SCHUR

Before dealing with the main result of this section, some basic results will be stated. We begin with a Lemma on Blaschke's partial products.

Lemma 1. *Let* $z_{n-1,k}$ *and* ξ *be complex numbers*. If $|z_{n-1,k}| < 1$ *and* $|\xi| < 1$, *then* $|w_k(\xi) < 1|$, *where* $w_k(\xi) = \dfrac{\xi - z_{n-1,k}}{1 - \xi\overline{z}_{n-1,k}}$.

Proof. We have to prove that $|w_k(\xi)| = \left|\dfrac{\xi - z_{n-1,k}}{1 - \xi\overline{z}_{n-1,k}}\right| < 1$, which is equivalent to proving that $|\xi - z_{n-1,k}| < |1 - \xi\overline{z}_{n-1,k}|$, or $|\xi - z_{n-1,k}|^2 < |1 - \xi\overline{z}_{n-1,k}|^2$, for $|z_{n-1,k}| < 1$ and $|\xi| < 1$. Let us denote by:

$$A = |\xi - z_{n-1,k}|^2 = |\xi|^2 - \xi\overline{z}_{n-1,k} - \overline{\xi}z_{n-1,k} + |z_{n-1,k}|^2$$

and by

$$B = |1 - \xi\overline{z}_{n-1,k}|^2 = 1 - \xi\overline{z}_{n-1,k} - \overline{\xi}z_{n-1,k} + |\xi|^2 |z_{n-1,k}|^2,$$

respectively. Then $B - A = (1 - |z_{n-1,k}|^2)(1 - |\xi|^2) > 0$ and the lemma is proved. □

A key and well-known result in the theory of polynomials and reflection coefficients ([1, 2, 14, 17]), that we will use further on, is

Theorem 1. Let $A_n(z) = \sum_{k=0}^{n} a_{nk} z^k$ *be a monic complex polynomial with reflection coefficients* $\alpha_1, \alpha_2, \ldots, \alpha_n$. *Then* $A_n(z)$ *has all its zeros inside the unit circle if and only if* $|\alpha_k| < 1$ for $k = 1, 2, \ldots, n.$

Another important result, published in [6], that will be needed, is

Theorem 2. *The polynomials whose characterization with the reflection coefficients is of the form* $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ *with* $|\alpha_k| < 1,\ 1 \le k \le n-1$, *and* $|\alpha_n| = 1$ *have all their zeros on the unit circle and they are simple.*

Now, we state and prove our main result. It gives a necessary and sufficient condition for a polynomial to have all its zeros on the unit circle.

Theorem 3. *Let* $A_n(z)$ *be a monic polynomial with complex coefficients. It has all its zeros on the unit circle if and only if there exists a polynomial* $A_{n-j}(z) \equiv \left[\alpha_1, \alpha_2, ..., \alpha_{n-j}\right]$, $|\alpha_k| < 1$, $k = 1, 2, ..., n-j-1$, *such that* $A_n(z) \equiv \left[\alpha_1, \alpha_2, ..., \alpha_{n-j}, 0, ..., 0, \alpha_n\right], |\alpha_n| = 1$ *for some* $j \ge 0$.

Proof. $\Rightarrow$) If $A_n(z)$ has all its zeroes on $|z| = 1$, then $A_n(z)$ is self inversive. That is, $A_n(z) = \alpha_n(z) A_n^*(z)$, $|\alpha_n| = 1$. Then, taking $j = 0$ and $A_{n-j}(z) = \frac{1}{2} A_n(z)$, we have:

$$z^j A_{n-j}(z) + \alpha_n A_{n-j}^*(z) = \frac{1}{2} A_n(z) + \frac{1}{2}\alpha_n A_n^*(z) = \frac{1}{2} A_n(z) + \frac{1}{2} A_n(z) = A_n(z)$$

and the necessary condition is proved.

$\Leftarrow$) By applying Theorem 1 and Theorem 2 to the polynomial $A_{n-j}(z)$, we have that all its zeros lie inside or on the unit circle. Hence,

$$A_{n-j}(z) = \prod_{k=1}^{n-j} (z - z_k) \text{ and } A_{n-j}^*(z) = \prod_{k=1}^{n-j} (1 - z\bar{z}_k) .$$

Now we claim that $A_n(z)$ only has unitary zeroes. In fact, let ξ be a zero of $A_n(z)$. Then $A_n(\xi) = \xi^j A_{n-j}(\xi) + \alpha_n A_{n-j}^*(\xi) = 0$ and $|\xi^j A_{n-j}(\xi)| = |\alpha_n A_{n-j}^*(\xi)|$. That is,

$$|\xi|^j \prod_{k=1}^{n-j} |\xi - z_k| = \prod_{k-1}^{n-j} \left|1 - \xi\bar{z}_k\right| . \tag{3}$$

Since $|z - z_k| = |1 - z\bar{z}_k|$ when $|z_k| = 1$, then in (3) these factors simplify and we only have to consider factors with $|z_k| < 1$. Assume that $|\xi| > 1$. Then, by Lemma 1, we have

$$\prod_{k=1}^{s}|\xi - z_k| > \prod_{k=1}^{s}\left|1-\xi \bar{z}_k\right|, \quad s \le n-j$$

and

$$|\xi|\, a^j \prod_{k=1}^{s}|\xi - z_k| > |\xi|^j \prod_{k=1}^{s}\left|1-\xi \bar{z}_k\right| > \prod_{k=1}^{s}\left|1-\xi \bar{z}_k\right|,$$

which contradicts (3). On the other hand, if $|\xi| < 1$, again on account of Lemma 1, we have

$$\prod_{k=1}^{s}|\xi - z_k| < \prod_{k=1}^{s}\left|1-\xi \bar{z}_k\right|, s \le n-j\ ,$$

and

$$|\xi|^j \prod_{k=1}^{s}|\xi - z_k| < |\xi|^j \prod_{k=1}^{s}\left|1-\xi \bar{z}_k\right| < \prod_{k=1}^{s}\left|1-\xi \bar{z}_k\right|,$$

in contradiction with (3). Therefore, (3) holds only when $|\xi| = 1$. This leads us to the conclusion that all the zeros of $A_n(z)$ lie on the unit circle. Thus the sufficient condition is established and the theorem is proved.

Finally, we point out that the preceding theorem is a generalization of a result established by Schur in 1917 [16]. It was stated as

Corollary 1. *Let $A_n(z)$ be a monic polynomial with complex coefficients whose zeros lie within the unit circle. Then $A_n(z) + A_n^*(z)$ has all its zeros on the unit circle. More generally, the roots of the equation $A_n^*(z) + \lambda A_n(z) = 0$ lie outside the unit circle for $|\lambda| < 1$, inside for $|\lambda| > 1$ and on its boundary for $|\lambda| = 1$.*

3. A GENERALIZATION OF A THEOREM OF COHN

In [8] incomplete polynomials were defined and used to generalize the well-known Gauss-Lucas theorem. In this section incomplete polynomials will be considered again, to generalize a result of Cohn on location of the zeros of polynomials and its derivatives. We begin with

Definition 2. Let $A_n(z)$ be a monic polynomial with complex coefficients and zeros $z_1, z_2, \ldots, z_n$. A polynomial $A_{n-1}(z)$ is called a convex linear combination of incomplete polynomials associated with $A_n(z)$ if $A_{n-1}(z)$, also denoted by $A_n^\gamma(z)$, is given by

$$A_n^\gamma(z) = \sum_{k=1}^{n} \gamma_k g_k(z),$$

where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ is a vector which components are non negative real numbers such that $\sum_{k=1}^{n}\gamma_k = 1$,

and $g_k(z) = \prod_{\substack{j=1 \\ j\ne k}}^{n}(z - z_j)$ are the incomplete polynomials.

Now, we state a result presented in [8] dealing with the study of the convex hull of the zeros of incomplete polynomials. It will be used later on.

Theorem 4 (Gauss-Lucas generalized). *Let* $z_1, z_2, \ldots, z_n$ *be not necessarily distinct, complex numbers. Then, the polynomial* $A_n^{\gamma}(z) = \sum_{k=1}^{n} \gamma_k g_k(z)$ *has all its zeros in or on the convex hull* $H(z_1, z_2, \ldots, z_n)$ *of the zeros of* $A_n(z) = \prod_{k=1}^{n} (z - z_k)$.

The main result in this section is

Theorem 5. *The polynomial* $A_n(z)$ *has all its zeros on* $|z_k| = 1$ *if and only if there exists a convex linear combination of incomplete polynomials, associated with* $A_n(z)$, say $A_{n-1}(z)$, *such that it generates* $A_n(z)$ *by forward recursion* (2) *with an unitary reflection coefficient and has all its zeros inside or on the unit circle*.

Proof. First, we prove the necessary condition. In fact, if $A_{n-1}(z)$ has all its zeros in $|z_k| \le 1$ and we apply forward recursion (2) with α_n unitary, that is, $\alpha_n = a_{n0}$. Then, on account of Theorem 2, $A_n(z)$ has all its zeros on $|z_k| = 1$.

In order to prove the sufficient condition, we assume that $A_n(z)$ has all its zeros on $|z_k| = 1$: then, by Theorem 4, every convex linear combination of incomplete polynomials, namely,

$$A_{n-1}(z) = A_n^{\gamma}(z) = \sum_{k=1}^{n} \gamma_k g_k(z), \quad \sum_{k=1}^{n} \gamma_k = 1, \quad \gamma_k \ge 0,$$

has all its zeros in $H(z_1, z_2, \ldots, z_n) \subset U = \{z \in C : |z| \le 1\}$ and the theorem is proved.□

Corollary 1 (Cohn's Theorem). $A_n(z)$ *has all its zeros on the unit circle if and only if* $A_n(z)$ *is self-inversive and* $A'_n(z)$ *has all its zeros inside or on the unit circle*.

Proof. Since $A_n(z)$ is self-inversive, $|a_{n0}| = |\alpha_n| = 1$. By Theorem 5, if we choose $\gamma_k = \frac{1}{n}$, $1 \le k \le n$, we have:

$$A_n^{\gamma}(z) = \sum_{k=1}^{n} \gamma_k g_k(z) = \frac{1}{n} A_n^{'}(z), \quad \sum_{k=1}^{n} \gamma_k = 1,$$

and the result is proved.

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