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ANALITICAL METHODS TO ASSESS LINEAR MODELS FOR EXPERIMENTAL HYSTERETIC LOOPS

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In this paper is proposed a linearization method for hysteretic characteristics developed on the basis of the experimental loops, by considering the most general case, when there is no mathematical model associated with the hysteretic behavior. The method employs a classical differential linear model which depends on three parameters. The model is studied for all physically realizable combinations of the involved parameters. The linearization criteria are formulated in terms of the variation of frequency response function over an interval containing a predicted dominant value. The method allows choosing the best model configuration, in terms of force amplification factor and dissipated energy errors, for a given experimental hysteretic curve.

Key words: Hysteretic characteristics; Frequency response functions; Dissipated energy; Equivalent linearization.

1. INTRODUCTION

The behaviour of materials, structural elements or vibration isolators is described by hysteretic loops that are treated in a unified manner by a single nonlinear differential equation with no need to distinguish different phases of the applied loading pattern. In practice, the Bouc-Wen model [1, 2] is mostly used within the following inverse problem approach: given a set of experimental input–output data, how to adjust the Bouc-Wen model parameters so that the output of the model matches the experimental data. Various methods where developed to identify the model parameters from the experimental data of periodic vibration tests: analytical approaches, for example [3] and different methods based on genetic algorithms for extended versions of Bouc-Wen model, [4, 5, 7] and [8]. Although, the nonlinear models can predict with good approximation the dynamic behaviour of hysteretic structures, in many cases, the linear equivalent models are still of practical interest for complex systems having many degrees of freedom and complex types of excitations.

One of the most efficient techniques for approximating non-linear models within the operating domain is the equivalent linearization method, both in deterministic and stochastic approach. There are many studies about equivalent linearization of hysteretic characteristics [6, 8, 9, 10] that have proved the efficiency of this approach. In the linearization techniques literature, the linear model is obtained by taking as reference the non-linear equation which portrays the hysteretic loop.

This paper presents a linearization method that is developed only on the basis of experimental hysteretic loop, considering the general case, when there is no mathematical model associated to the hysteretic behavior. The linear equivalent model is described by ordinary differential equation of first order which can describe better the memory properties of hysteresis than the visco-elastic equivalent model. The linear model parameters are determined for a predicted dominant frequency component in the response spectrum of the mechanical structure equipped by the considered hysteretic device.

2. LINEARIZATION METHODS

Consider that the experimental hysteretic characteristic is a symmetric loop $-F_{\mathrm{m}} \le F(x) \le F_{\mathrm{m}}$, obtained for a periodic motion $-x_{\mathrm{m}}(t) \le x(t) \le x_{\mathrm{m}}(t)$, imposed between the mounting ends of the tested element and $E_{\mathrm{d}}^{\mathrm{f}}$ the corresponding dissipated energy. By introducing the dimensionless magnitudes

$$\xi(t) = x(t)/x_{\mathrm{u}}, \ \Phi(\xi) = F(x_{\mathrm{u}}\xi)/F_{\mathrm{u}}, \quad \xi_{\mathrm{m}} = \max\left|\xi(t)\right|, \ \Phi_{\mathrm{m}} = \max\left|\Phi(\xi)\right|, \tag{1}$$

where T is the period of the imposed cyclic motion and x_{u}, F_{u} are displacement and force reference units, a generic plot of the symmetric hysteresis loop $\Phi = \Phi(\xi)$ can be represented as shown in figure 1. Here the imposed cyclic displacement is $\xi = \xi_{\mathrm{m}} \sin(\omega_{\mathrm{exp}} t)$, with $\xi_{\mathrm{m}} = \dfrac{x_{\mathrm{m}}}{x_{\mathrm{u}}}$, $\Phi_{\mathrm{m}} = \dfrac{F_{\mathrm{m}}}{F_{\mathrm{u}}}$ is the maximum force and the dissipated energy per cycle by experimental loop is $E_{\mathrm{d}} = \oint \Phi(\xi)\mathrm{d}\xi$ – the area of surface enclosed by the hysteretic loop.

2.1. First linearization method

In this case the linear equivalent hysteretic force is given by: $\Phi_{\mathrm{le}}(\xi, \dot{\xi}) = z(\xi, \dot{\xi})$, where

$$\dot{z} = az + b\dot{\xi} + c\xi, \tag{2}$$

so, z can be written $z = a\int z\mathrm{d}t + b\xi + c\int \xi \mathrm{d}t \Rightarrow b \ge 0$ (physically realizable stiffness characteristic); $a < 0$ derived from the stability condition, b and c cannot be simultaneously equal to zero (because one obtains a exponential solution for $z(t)$) and $[a] = T^{-1}$, $[c] = T^{-1}$ and b is a dimensionless parameter.

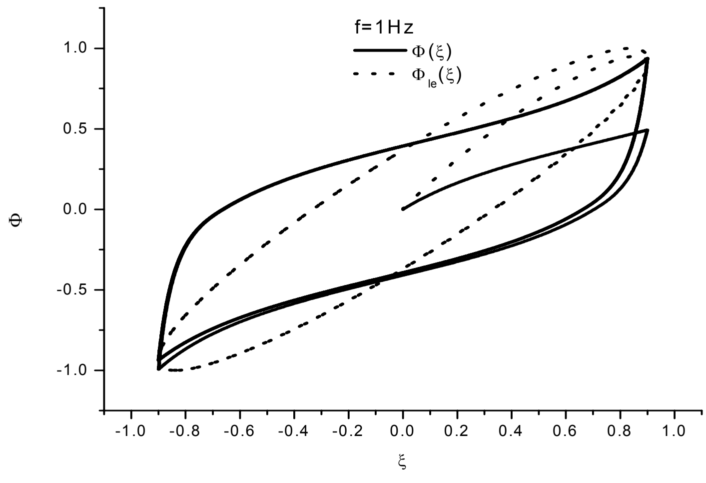

Fig. 1 Hysteretic loops: experimental (—–) and linear equivalent (- - -).

We consider $\xi(t) = \xi_{\mathrm{m}} \mathrm{e}^{i\omega t}$, $\dot{\xi}(t) = \mathrm{i}\omega\xi_{\mathrm{m}} \mathrm{e}^{i\omega t}$ and denote $H(\omega) = |H(\omega)| \exp[\mathrm{i}\theta(\omega)]$ the frequency response function corresponding to the input $\xi(t)$ and output $\Phi_{\mathrm{le}}(t) = \Phi_{\mathrm{le}}(\xi(t), \dot{\xi}(t))$. Therefore, the relation (2) implies: $|H(\omega)| = \sqrt{\dfrac{b^2\omega^2 + c^2}{\omega^2 + a^2}}$, $\theta(\omega) = \tan^{-1}\left(\dfrac{-(ab+c)\omega}{b\omega^2 - ac}\right)$ and $\sin\theta(\omega) = \dfrac{-(ab+c)\omega}{\sqrt{(\omega^2 + a^2)(b^2\omega^2 + c^2)}}$.

The energy dissipated per cycle by linear equivalent hysteretic force is: $E_{\mathrm{le}}(\omega)=\pi\left|H(\omega)\right|\xi_{\mathrm{m}}^{2}\sin\theta(\omega)$. As $E_{\mathrm{le}}>0$, one can derive that $ab+c<0$. Next are investigated the possible combinations of parameters a, b, c and the corresponding solutions of the linearization problem. In each case are studied the extreme points with respect to ω, of functions $\left|H(\omega)\right|$, $\sin\theta(\omega)$ and $E_{\mathrm{le}}(\omega)$.

I.1. $a<0$, $b>0$, $c\neq 0$ such that $ab+c<0$. With these assumptions, the frequency response function, $\sin\theta(\omega)$ and the dissipated energy become: $\left|H(\omega)\right|=\sqrt{\dfrac{b^2\omega^2+c^2}{\omega^2+a^2}}$, $\sin\theta(\omega)=\dfrac{-(ab+c)\omega}{\sqrt{(\omega^2+a^2)(b^2\omega^2+c^2)}}$ and $E_{\mathrm{le}}(\omega)=-\pi\xi_{\mathrm{m}}^{2}\dfrac{(ab+c)\omega}{\omega^2+a^2}$. As $c\neq 0$, one must consider two cases: $c>0$ and $c<0$.

I.1.1. $a<0$, $b>0$, $c>0$, $ab+c<0$. In this case one can prove that $\left|H(\omega)\right|$ is a monotonous increasing function on [0, ∞), therefore $\left|H(\omega)\right|$ has no extreme point.

- The analytical study of $\sin\theta(\omega)$: $\sin\theta(\omega)\geq 0$, $\sin\theta(0)=0$, $\lim\limits_{\omega\to\infty}\sin\theta(\omega)=0$, $\dfrac{\mathrm{d}\left[\sin\theta(\omega)\right]}{\mathrm{d}\omega}=-\dfrac{(ab+c)(b^2\omega^4-a^2c^2)}{\left[(\omega^2+a^2)(b^2\omega^2+c^2)\right]^{\frac{3}{2}}}=0 \;\Rightarrow\; \omega_{\mathrm{ext1}}=\sqrt{-\dfrac{ac}{b}}$ and $\left[\sin\theta(\omega)\right]_{\max}=\dfrac{ab+c}{ab-c}$, $\left|H(\omega_{\mathrm{ext1}})\right|=\sqrt{-\dfrac{bc}{a}}$, $E_{\mathrm{le}}(\omega_{\mathrm{ext1}})=\pi\xi_{\mathrm{m}}^{2}\dfrac{ab+c}{ab-c}\sqrt{-\dfrac{bc}{a}}$.
- The analytical study of $E_{\mathrm{le}}(\omega)$: $E_{\mathrm{le}}(0)=0$, $\lim\limits_{\omega\to\infty}E_{\mathrm{le}}(\omega)=0$, $\dfrac{\mathrm{d}\left[E_{\mathrm{le}}(\omega)\right]}{\mathrm{d}\omega}=\pi\xi_{\mathrm{m}}^{2}\dfrac{(ab+c)(\omega^2-a^2)}{(\omega^2+a^2)^2}=0 \;\Rightarrow\; \omega_{\mathrm{ext2}}=|a|=-a$ and $\left[E_{\mathrm{le}}(\omega)\right]_{\max}=E_{\mathrm{le}}(\omega_{\mathrm{ext2}})=\pi\xi_{\mathrm{m}}^{2}\dfrac{ab+c}{2a}$, $\left|H(\omega_{\mathrm{ext2}})\right|=-\dfrac{1}{a}\sqrt{\dfrac{a^2b^2+c^2}{2}}$, $\sin\theta(\omega_{\mathrm{ext2}})=-\dfrac{ab+c}{\sqrt{2(a^2b^2+c^2)}}$. We make the following notations: $\mu=\dfrac{E_{\mathrm{d}}^{\mathrm{f}}}{\pi F_{\mathrm{m}}x_{\mathrm{m}}},\eta=\dfrac{\Phi_{\mathrm{m}}}{\xi_{\mathrm{m}}}=\dfrac{F_{\mathrm{m}}}{x_{\mathrm{m}}}\dfrac{x_{\mathrm{u}}}{F_{\mathrm{u}}}$. By using the definition of dissipated energy and the relations given in (1), one derive $\mu=\dfrac{E_{\mathrm{d}}^{\mathrm{f}}}{\pi F_{\mathrm{m}}x_{\mathrm{m}}}=\dfrac{E_{\mathrm{d}}}{\pi\Phi_{\mathrm{m}}\xi_{\mathrm{m}}}$. It is easy to remark that, in the most cases, the experimental hysteresis loops have the property $\mu<1$, so, in this paper, we assume that this condition holds. The parameters of linear equivalent model are determined for two cases: **a-** when the dominant frequency is chosen ω_{ext1} and **b** – for ω_{ext2}.

I.1.1.a. The parameters are obtained by solving the system

$$\sqrt{-\frac{ac}{b}}=\omega_{\mathrm{ext1}},\left|H(\omega_{\mathrm{ext1}})\right|=\sqrt{-\frac{bc}{a}}=\frac{\Phi_{\mathrm{m}}}{\xi_{\mathrm{m}}},E_{\mathrm{le}}(\omega_{\mathrm{ext1}})=\pi\xi_{\mathrm{m}}^{2}\frac{ab+c}{ab-c}\sqrt{-\frac{bc}{a}}=E_{\mathrm{d}}\,. \tag{3}$$

The solution of (3) is given by: $a=-\omega_{\mathrm{ext1}}\sqrt{\dfrac{1+\mu}{1-\mu}},b=\dfrac{\Phi_{\mathrm{m}}}{\xi_{\mathrm{m}}}\sqrt{\dfrac{1+\mu}{1-\mu}},c=\omega_{\mathrm{ext1}}\dfrac{\Phi_{\mathrm{m}}}{\xi_{\mathrm{m}}}$.

I.1.1.b. In this case the parameters are obtained using the relations $a=-\omega_{\mathrm{ext2}},\left|H(\omega_{\mathrm{ext2}})\right|=-\dfrac{1}{a}\sqrt{\dfrac{a^2b^2+c^2}{2}}=\dfrac{\Phi_{\mathrm{m}}}{\xi_{\mathrm{m}}},E_{\mathrm{le}}(\omega_{\mathrm{ext2}})=\pi\xi_{\mathrm{m}}^{2}\dfrac{ab+c}{2a}=E_{\mathrm{d}}$.

This solution is feasible if $\mu<\dfrac{\sqrt{2}}{2}$, which is not in general true.

I.1.2. $a<0$, $b>0$, $c<0$ (the inequality $ab+c<0$ is automatically satisfied). For this case it is easy to check that $\left|H(\omega)\right|$ is a monotonous increasing on [0, ∞) if $a>\frac{c}{b}$ and monotonous decreasing on [0, ∞) if $a<\frac{c}{b}$, therefore $\left|H(\omega)\right|$ has no extreme point.

- The analytical study of $\sin\theta(\omega)$: $\sin\theta(\omega)\geq 0$, $\sin\theta(0)=0$, $\lim_{\omega\to\infty}\sin\theta(\omega)=0$, $\frac{\mathrm{d}\left[\sin\theta(\omega)\right]}{\mathrm{d}\omega}=-\frac{(ab+c)\left(b^2\omega^4-a^2c^2\right)}{\left[\left(\omega^2+a^2\right)\left(b^2\omega^2+c^2\right)\right]^{\frac{3}{2}}}=0 \Rightarrow \omega_{\mathrm{ext1}}=\sqrt{\frac{ac}{b}}$ and $\left[\sin\theta(\omega)\right]_{\max}=\sin\theta\left(\omega_{\mathrm{ext1}}\right)=1$ $\Rightarrow\theta\left(\omega_{\mathrm{ext1}}\right)=\frac{\pi}{2} \Rightarrow z\left(\xi,\dot{\xi}\right)$ is a pure viscous-elastic characteristic for any imposed cyclic motion $\xi(t)=\xi_{\mathrm{m}}\sin\left(\omega_{\mathrm{ext1}}t\right)$, $\left|H\left(\omega_{\mathrm{ext1}}\right)\right|=\frac{c}{a}$, $E_{\mathrm{le}}\left(\omega_{\mathrm{ext1}}\right)=\pi\xi_{\mathrm{m}}^2\frac{c}{a}$.
- The analytical study of $E_{\mathrm{le}}(\omega)$ is analogous to the **I.1.1** case ($\omega_{\mathrm{ext2}}=|a|=-a$).

I.1.2.a. The parameters must fulfil a set of conditions that imply $\mu=1$ which is contradictory to the assumption $\mu<1$. So, this case is not possible.

I.1.2.b. The system for finding the parameters of linear system is identical with **I.1.1.b**. There are two possible solutions:

– the firs one is identical with the solution of **I.1.1.b** and is feasible if $\mu\in\left[\frac{\sqrt{2}}{2},1\right]$, which is rather restrictive;

– the second is given by $a=-\omega_{\mathrm{ext2}}, b=\eta\left(\mu-\sqrt{1-\mu^2}\right), c=\omega_{\mathrm{ext2}}\eta\left(-\mu-\sqrt{1-\mu^2}\right)$ - feasible if $\mu\in\left[\frac{\sqrt{2}}{2},1\right]$.

I.2. $a<0$, $b>0$, $c=0$. For these hypotheses about the linear model parameters, the frequency response function, $\sin\theta(\omega)$ and the dissipated energy become: $\left|H(\omega)\right|=\frac{b\omega}{\sqrt{\omega^2+a^2}}$, $\sin\theta(\omega)=\frac{-a}{\sqrt{\left(\omega^2+a^2\right)}}$ and $E_{\mathrm{le}}(\omega)=-\pi\xi_{\mathrm{m}}^2\frac{ab\omega}{\omega^2+a^2}$. In this case one can prove that $\left|H(\omega)\right|$ is a monotonous increasing on [0, ∞) and $\sin\theta(\omega)$ is a monotonous decreasing function on [0, ∞).

- The analytical study of $E_{\mathrm{le}}(\omega)$: $E_{\mathrm{le}}(0)=0$, $\lim_{\omega\to\infty}E_{\mathrm{le}}(\omega)=0$, $\frac{\mathrm{d}\left[E_{\mathrm{le}}(\omega)\right]}{\mathrm{d}\omega}=\pi\xi_{\mathrm{m}}^2ab\frac{\omega^2-a^2}{\left(\omega^2+a^2\right)^2}=0 \Rightarrow \omega_{\mathrm{ext2}}=|a|=-a$ and $\left|H\left(\omega_{\mathrm{ext2}}\right)\right|=\frac{b}{\sqrt{2}}$, $\sin\theta\left(\omega_{\mathrm{ext2}}\right)=\frac{1}{\sqrt{2}}$ $\Rightarrow\theta\left(\omega_{\mathrm{ext2}}\right)=\frac{\pi}{4}$, $\left[E_{\mathrm{le}}(\omega)\right]_{\max}=E_{\mathrm{le}}\left(\omega_{\mathrm{ext2}}\right)=\frac{\pi\xi_{\mathrm{m}}^2b}{2}$. Therefore, $a=-\omega_{\mathrm{ext2}}, b=\frac{\Phi_{\mathrm{m}}}{\xi_{\mathrm{m}}}\sqrt{2}$ with the condition $\mu=\frac{\sqrt{2}}{2}$ that cannot be satisfied in the general case.

I.3. $a<0$, $b=0$, $c<0$. In this case one can write:

$$\left|H(\omega)\right|=-\frac{c}{\sqrt{\omega^2+a^2}},\quad \sin\theta(\omega)=\frac{\omega}{\sqrt{\left(\omega^2+a^2\right)}}\quad\text{and}\quad E_{\mathrm{le}}(\omega)=-\pi\xi_{\mathrm{m}}^2\frac{c\omega}{\omega^2+a^2}.$$

It is easy to prove that $\left|H(\omega)\right|$ is a monotonous decreasing on [0, ∞) and $\sin\theta(\omega)$ is a monotonous increasing function on [0, ∞).

- The analytical study of $E_{\text{le}}(\omega)$: $E_{\text{le}}(0)=0$, $\lim\limits_{\omega\to\infty} E_{\text{le}}(\omega)=0$, $\dfrac{\mathrm{d}\left[E_{\text{le}}(\omega)\right]}{\mathrm{d}\omega} = \pi\xi_{\text{m}}^2 c\dfrac{\omega^2-a^2}{\left(\omega^2+a^2\right)^2}=0$

$\Rightarrow$ $\omega_{\text{ext2}}=|a|=-a$ and $\left|H(\omega_{\text{ext2}})\right|=\dfrac{c}{a\sqrt{2}}$, $\sin\theta(\omega_{\text{ext2}})=\dfrac{1}{\sqrt{2}}$ $\Rightarrow\theta(\omega_{\text{ext2}})=\dfrac{\pi}{4}$ as in the, previous case, $\left[E_{\text{le}}(\omega)\right]_{\max}=E_{\text{le}}(\omega_{\text{ext2}})=\dfrac{\pi\xi_{\text{m}}^2 c}{2a}$. The solution is $a=-\omega_{\text{ext2}}, c=-\omega_{\text{ext2}}\dfrac{\Phi_{\text{m}}}{\xi_{\text{m}}}\sqrt{2}$ with the assumption $\mu=\dfrac{\sqrt{2}}{2}$, which is possible only in particular cases.

2.2. Second linearization method

In this case the linear equivalent hysteretic force is given by:

$$\Phi_{\text{le}}\left(\xi,\dot{\xi}\right)=\alpha\xi+(1-\alpha)z,\alpha\in[0,1) \tag{4}$$

and z is obtained using the relation (2), where $a<0$. As b is a redundant parameter of this model, one can take $b=0$ so the equation (2) becomes: $\dot{z}=az+c\xi$. Consequently, the frequency response function, $\sin\theta(\omega)$ and the dissipated energy have the forms:

$$\left|H(\omega)\right|=\sqrt{\frac{\alpha^2\omega^2+\left[a\alpha-c(1-\alpha)\right]^2}{\omega^2+a^2}},\quad \sin\theta(\omega)=\frac{-c(1-\alpha)\omega}{\left(\omega^2+a^2\right)\left|H(\omega)\right|}\quad\text{and}\quad E_{\text{le}}(\omega)=-\pi\xi_{\text{m}}^2\frac{c(1-\alpha)\omega}{\omega^2+a^2}.$$

As the dissipated energy is positive, $E_{\text{le}}>0$, then the condition $c<0$ must be fulfilled. In this case $\left|H(\omega)\right|$ has the sign of term $\left(2a\alpha-c(1-\alpha)\right)$, which implies that $\left|H(\omega)\right|$ has no extreme point.

II.a. The analytical study of $\sin\theta(\omega)$: $\sin\theta(0)=0$, $\lim\limits_{\omega\to\infty}\sin\theta(\omega)=0$ and $\dfrac{\mathrm{d}\left[\sin\theta(\omega)\right]}{\mathrm{d}\omega}=0$ implies $\omega_{\text{ext1}}=\sqrt{\dfrac{-a\left|a\alpha-c(1-\alpha)\right|}{\alpha}}$. The parameters a, c and α have to be determined from the system: $\left|H(\omega_{\text{ext1}})\right|=\sqrt{\dfrac{-\alpha\left|a\alpha-c(1-\alpha)\right|}{a}}=\dfrac{\Phi_{\text{m}}}{\xi_{\text{m}}}$, $E_{\text{le}}(\omega_{\text{ext1}})=\dfrac{-\pi\xi_{\text{m}}^2 c(1-\alpha)\sqrt{-a\alpha\left|a\alpha-c(1-\alpha)\right|}}{a\left(a\alpha-\left|a\alpha-c(1-\alpha)\right|\right)}=E_{\text{d}}$. One can observe that if $a\alpha-c(1-\alpha)\ge 0$ then $E_{\text{d}}=\pi\Phi_{\text{m}}\xi_{\text{m}}$ which is not feasible for an arbitrary experimental loop. Therefore, we assume $a\alpha-c(1-\alpha)<0$. Under this hypothesis, the previous relations become: $\dfrac{-a\left(c(1-\alpha)-a\alpha\right)}{\alpha}=\omega_{\text{ext1}}^2$, $\dfrac{-\alpha\left(a\alpha-c(1-\alpha)\right)}{a}=\eta^2$, $\dfrac{c(1-\alpha)}{2a\alpha-c(1-\alpha)}=\mu$, with the solution $\alpha=\eta\sqrt{\dfrac{(1+\mu)\omega_{\text{ext1}}\eta}{1-\mu}}$, $a=-\omega_{\text{ext1}}\sqrt{\dfrac{(1+\mu)\omega_{\text{ext1}}\eta}{1-\mu}}$, $c=\dfrac{-2\mu\omega_{\text{ext1}}^2\eta^2}{(1-\mu)\left(1-\eta\sqrt{\dfrac{(1+\mu)\omega_{\text{ext1}}\eta}{1-\mu}}\right)}$, which is applicable if

$\eta\sqrt{\dfrac{(1+\mu)\omega_{\text{ext1}}\eta}{1-\mu}}<1\Leftrightarrow\dfrac{x_{\text{u}}}{F_{\text{u}}}<\dfrac{x_{\text{m}}}{F_{\text{m}}}\sqrt[3]{\dfrac{1-\mu}{(1+\mu)\omega_{\text{ext1}}}}$ (which can be obtained by an appropriate choosing of the gains x_{u} and F_{u}).

II.b. The analytical study of $E_{\mathrm{le}}(\omega)$: $E_{\mathrm{le}}(0)=0$, $\lim_{\omega\to\infty} E_{\mathrm{le}}(\omega)=0$, $\dfrac{\mathrm{d}\left[E_{\mathrm{le}}(\omega)\right]}{\mathrm{d}\omega}=$ $=\pi\xi_{\mathrm{m}}^2 c(1-\alpha)\dfrac{\omega^2-a^2}{\left(\omega^2+a^2\right)^2}=0$ $\Rightarrow$ $\omega_{\mathrm{ext2}}=|a|=-a$ and $\left[E_{\mathrm{le}}(\omega)\right]_{\max}=E_{\mathrm{le}}(\omega_{\mathrm{ext2}})=\pi\xi_{\mathrm{m}}^2\dfrac{c(1-\alpha)}{2}$, $\left|H(\omega_{\mathrm{ext2}})\right|=-\dfrac{1}{a}\sqrt{\dfrac{a^2\alpha^2+\left[a\alpha-c(1-\alpha)\right]^2}{2}}$. In this case, the following relations are derived: $a=-\omega_{\mathrm{ext2}}$, $-\dfrac{1}{a}\sqrt{\dfrac{a^2\alpha^2+\left[a\alpha-c(1-\alpha)\right]^2}{2}}=\eta$, $\dfrac{\pi\xi_{\mathrm{m}}^2 c(1-\alpha)}{2a}=E_{\mathrm{d}}$. Then, α is a root of the equation: $\omega_{\mathrm{ext2}}\alpha^2-\dfrac{2\omega_{\mathrm{ext2}}^2 E_{\mathrm{d}}}{\pi\xi_{\mathrm{m}}^2}\alpha+\dfrac{2\omega_{\mathrm{ext2}}^2 E_{\mathrm{d}}{}^2}{\pi^2\xi_{\mathrm{m}}^4}-\omega_{\mathrm{ext2}}^2\eta^2=0$. As, $\mu=\dfrac{E_{\mathrm{d}}^{\mathrm{f}}}{\pi F_{\mathrm{m}}x_{\mathrm{m}}}=\dfrac{E_{\mathrm{d}}}{\pi\Phi_{\mathrm{m}}\xi_{\mathrm{m}}}\in(0,1)$, then the previous equation has the solutions: $\alpha_1=\eta\left(\mu+\sqrt{1-\mu^2}\right),\alpha_2=\eta\left(\mu-\sqrt{1-\mu^2}\right)$. One can prove that α_1 is a feasible solution if $\eta\sqrt{2}<1 \Leftrightarrow \dfrac{x_{\mathrm{u}}}{F_{\mathrm{u}}}<\dfrac{\sqrt{2}}{2}\dfrac{x_{\mathrm{m}}}{F_{\mathrm{m}}}$, that is always possible by an appropriate scaling. In the same way, α_2 is a possible solution if: $\dfrac{x_{\mathrm{u}}}{F_{\mathrm{u}}}<\dfrac{x_{\mathrm{m}}}{F_{\mathrm{m}}}$ (feasible by an appropriate scaling) and $\mu\ge\dfrac{\sqrt{2}}{2}$, which is more restrictive than the condition assumed in this paper ($\mu\in(0,1)$). After α is determined then c is given by $c=\dfrac{2a}{1-\alpha}\mu\eta$.

3. APPLICATION OF THE LINEARIZATION METHODS

In this paper, the proposed linearization methods are illustrated for a hysteretic loop corresponding to a seismic protection device – Elastomeric Bearings (BIS), manufactured by the Italian Company FIPP INDUSTRIALE. The force-displacement curve of BIS device [12] was obtained by shear/ compression tests conducted on a column of two identical rubber bearings under a constant static vertical load (equal to the maximum vertical load for this isolator) and a cyclic horizontal load applied at the middle, such that to produce the maximum admissible shear deformation. For this device only the experimental data obtained for maximum working range were available. The hysteretic loop has the following parameters: $E_{\mathrm{d}}^{\mathrm{f}}=190\,\mathrm{kJ}$, $F_{\mathrm{m}}=760\,\mathrm{kN}$, $x_{\mathrm{m}}=0.175\,\mathrm{m}$, $\mu\cong 0.454$. The described linearization methods were applied assuming the dominant frequency $\omega_{\mathrm{ext1}}=\omega_{\mathrm{ext2}}=2\pi$. The scaling parameters chosen for each case and the obtained values of linear model parameters are given in Table 1.

Table 1

Case	x_{u} [m]	F_{u} [kN]	a [s^{-1}]	b	c [s^{-1}]	α
I.1.1.a	0.2	800	-10.262	1.773	6.821	-
I.1.1.b.	0.2	800	-6.283	1.46	2.973	-
II.a.	0.05	800	-13.402	0	-11.522	0.579
II.b.	0.1	800	-6.283	0	-11.503	0.73

The efficiency of linearization methods is assessed in terms of the relative errors of dissipated energy and amplification factor computed on a frequency range that includes the chosen dominant frequency: $\mathrm{er}E(\omega)=\dfrac{\left|E_{\mathrm{le}}(\omega)-E_{\mathrm{d}}\right|}{E_{\mathrm{d}}}$, $\mathrm{er}H(\omega)=\dfrac{\left|\left|H(\omega)\right|-\eta\right|}{\eta}$. Figures 2–5 are show the variation of relative errors versus the frequency for the cases specified in Table 1.

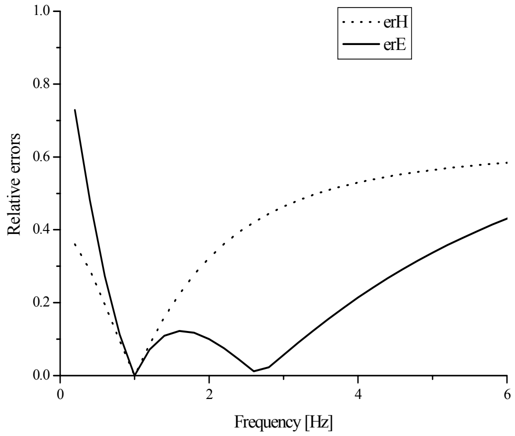

Fig. 2 – Relative errors in case **I.1.1.a,**

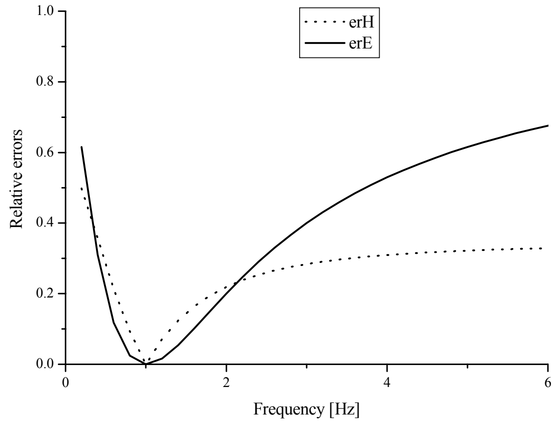

Fig. 3 – Relative errors in case **I.1.1.b.**

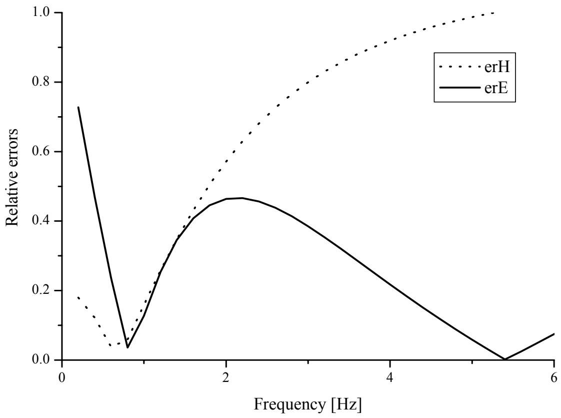

Fig. 4 – Relative errors in case **II.a.**

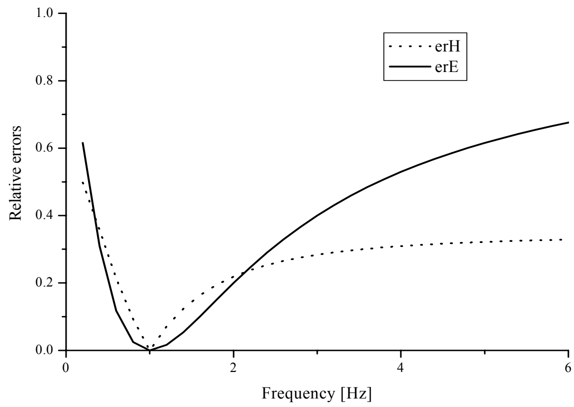

Fig. 5 – Relative errors in case **II.b.**

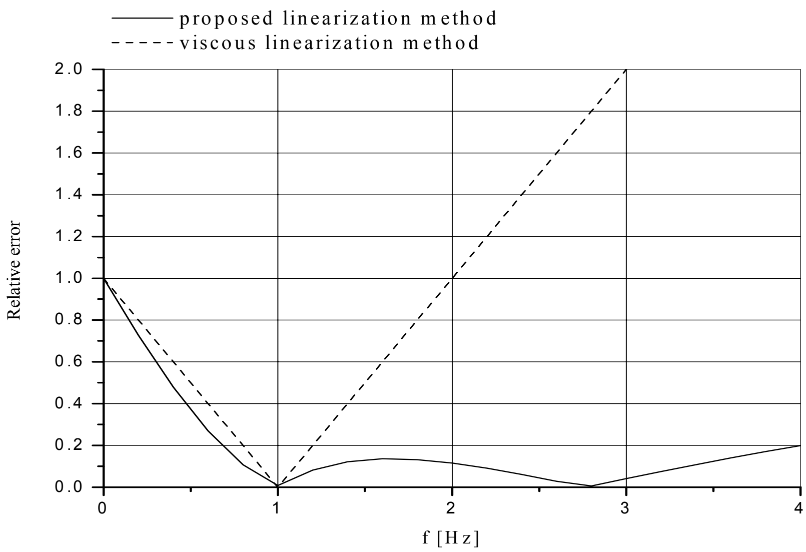

Fig. 6 – Relative errors of dissipated energy.

From these plots one can conclude that the most efficient linear equivalent model, for the considered hysteretic loop, is that corresponding to the case **I.1.1.a**. In order to outline the efficiency of the proposed linearization algorithm, the relative error of dissipated energy is compared with that obtained for a linear viscous equivalent model. Usually, the linear viscous damping $\lambda\dot{\xi}$, equivalent to the energy dissipated by a hysteretic device, is determined from: $\lambda = \frac{E_{\mathrm{d}}}{\pi\omega_{\mathrm{ext}}\xi_{\mathrm{m}}^2}$. In this case $E_{\mathrm{le}}(\omega) = \frac{\omega}{\omega_{\mathrm{ext}}} E_d$. Figure 6 shows comparatively the relative errors of dissipated energy given by the proposed method and by the equivalent viscous damping.

As one can see, the dissipated energy per cycle, which is constant for the experimental loop, is much better approximated by the obtained linear equivalent differential model over relatively large frequency range.

4. CONCLUSIONS

In this paper were developed analytical methods to assess equivalent differential linear models for experimental hysteretic loops. By studying the efficiency of all presented cases in terms of relative errors of the amplification factor and dissipated energy, one can find the most suitable linear model for a given hysteretic curve.

The essential property of hysteretic characteristics is that their dissipative properties are practically independent of the loading cycle frequency. The major drawback of linear representation of hysteretic loops is the dependence of dissipated energy on the input frequency. In this respect, the proposed linear equivalent differential model is superior to the visco-elastic equivalent model.

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