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STUDY OF THE PREDATOR-PREY MODELS PROPERTIES

Nina DRAGOESCU (CAZACU)

"Ovidius" University of Constanța, 900527, Constanța, Romania
E-mail: cazanina@yahoo.com

The paper analyzes the stability of the equilibrium points for the Lotka-Volterra mathematical model, by applying a simple method in order to find an analytical solution for the control problem, near the origin, proved to be a point of instability. Considerations refer to the theory of the linear analysis and, ultimately, the algebraic theory of groups.

Key words: Equilibrium; Stability; Optimality.

1. OPTIMAL CONTROL STABILITY

1.1. Lotka-Volterra model as a dynamic system

The mathematical model for the existence of the Volterra type predator-prey species [7-14] explains the variability of the existing native fish populations in the Adriatic Sea. The Volterra model is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N(a - bP), \ \frac{\mathrm{d}P}{\mathrm{d}t} = P(cN - d), \tag{1}$$

where $N(t)$ is the population of the prey type, $P(t)$ is the population of the predators that depend on time t, and a,b,c,d are positive constants, having the following meanings:

a) In the absence of any predation, the prey grows unboundedly, this aspect been revealed by the aN term;

b) The component $-bNP$ which is proportional to both populations, represents the effect of the predation to the prey's growth rate;

c) In the absence of any prey, the predator's death rate results in the exponential decay, as shown by $-\mathrm{d}P$;

d) The prey's contribution to the predators' growth rate is revealed by cNP and it is proportional to both populations.

The NP term represents the conversion of energy from one source to another: bNP is taken from the prey and cNP accrues to the predators. (a proportional term to both populations, approached to the *potential source* at the moment, but modified in one or another direction by the constant coefficients b and c). By using the notations

$$v(\tau) = \frac{bP(t)}{a}, \quad \tau = at, \quad \alpha = d/a, \tag{2}$$

the system (1) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = u(1-v), \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = \alpha v(u-1). \tag{3}$$

The system (3) have two singular points, i.e. $(0,0)$ and $(1,1)$. The linearization/dimensionless scheme is based on the singularities, and determines the state of the stability. Murray [4] has considered the first state of the system $(u,v)=(0,0)$. If x_1 and x_2 are *small perturbations* around the origin $(0,0)$, we obtain $u=x_1$ and $v=x_2$. Retaining only the linear terms, we obtain the state form

$$\begin{pmatrix} \dfrac{\mathrm{d}x_1}{\mathrm{d}\tau} \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}\tau} \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{4}$$

The solution of this system is

$$\begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} = D\mathrm{e}^{\lambda\cdot\tau}, \tag{5}$$

where D is a constant column vector, and the eigenvalues $\lambda_1=1$, $\lambda_2=-\alpha$ are given by the characteristic polynomial of the matrix A. Since at least one eigenvalue is positive, $x_1(\tau)$, $x_2(\tau)$ grow exponentially, and the point $u=v=0$ determines a linear system instability.

We now consider the system state $(u,v)=(1,1)$. By denoting $u=1+x_1$, $v=1+x_2$, the linear system (3) becomes

$$\begin{pmatrix} \dfrac{dx_1}{d\tau} \\ \dfrac{dx_2}{d\tau} \end{pmatrix} \approx \begin{pmatrix} 0 & -1 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6}$$

This time, the eigenvalues are $\lambda_1 = \mathrm{i}\sqrt{\alpha}$, $\lambda_2 = -\mathrm{i}\sqrt{\alpha}$, and the system solution is

$$\begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} = l\exp(\lambda_1\tau) + m\exp(\lambda_2\tau), \tag{7}$$

where l and m are the eigenvalues vectors. Therefore, the solutions near the point $u=v=1$ are periodic with the period $2\pi/\sqrt{\alpha} = T = 2\pi/\sqrt{a/d}$. The period is proportional to the square root of the ratio of linear growth a of the prey, and invert proportional with the mortality rate d of the predators. Even for *small disturbance* around the point, it can be seen that the period depends on the rate of the multiplication, namely *mortality*. A growth rate for the prey will induce the breeding period and a decreased mortality for predators.

We consider now a new approach for the dynamical system associated with the first steady state equations (4). Let us reconsider the equations by adding an external disturbing factor like a *push* acting on the prey, with optimal effect on the predators. In order to study the instability of the linear system near the origin, we introduce an external control denoted by ω. By including the *push* factor ω with its corresponding matrix B, the system (4) may be written as

$$\begin{aligned} \dot{x}(\tau) &\approx \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} x(\tau) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} = Ax(\tau) + B\omega(\tau), \\ y(\tau) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x(\tau) = Cx(\tau), \end{aligned} \tag{8}$$

The system (8) has the general form

$$f(x,\omega,t) = A(x) + B(t)\omega, \quad y = C(t)x, \tag{8'}$$

where $x(\tau)$ is the state vector, $\omega(\tau)$ is the command (the input vector), $y(\tau)$ is the output vector, t is the time, $f(x,\omega,t)$ is a self-adjoint operator (symmetrical) in the finite dimensional case. In our case A, B, C are symmetric 2D matrices.

The optimal control problem is limited to the determining of the control ω(t) which minimizes the quadratic cost function for any initial pair $(t_0, x_0) \in (T_1, T_2) \times X$. If $p:(T_1,T_2) \to L(X,X)$ is continuously differentiable, then $\overline{\omega}(t,x) = \frac{1}{2} < x, p(t)x >$ isalso continuously differentiable, according to the Kalman's theory. Also,if a Hamilton Iacobi solution $\overline{\omega}(t,x)$ is found, then the optimal control problem can be solved.

1.2. An analytical solution to the problem of optimal control in the origin

The dynamic system described by (8) in terms of the Kalman's theory [2], [3], [5], [6] is determining the optimal solution of the dynamic system with quadratic cost function, by solving an equation of Riccati type or, in particular, a Bernoulli-type equation with initial conditions

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A^* = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix},\ B^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A,B,C \in \mathrm{M}_2(R),\ C(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ X = Y = R^2,\ \tau_0 = 0,\quad \tau_1 = 1,\quad (\tau_0,\tau_1) \subset (T_1,T_2) \subset R,$$

$$\rho(\tau) = 0,\ \sigma(\tau) = 1,\quad \sigma^{-1}(\tau) = 1, x_0 = I_2,\quad p(\tau) = p^*(\tau),\quad S = B \cdot \sigma^{-1} \cdot B^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{9}$$

$$A + A^* = \begin{pmatrix} 2 & 0 \\ 0 & -2\alpha \end{pmatrix}.$$

Let us consider the particular case $\alpha = d/a = 1$, $b = 1$and $c = 3$. The condition $a = d$ represents that the rate of growth/multiplication in the absence of predator species is supposed to be equal to the rate decreasing predator species in the absence of prey, and the condition $c = 3$ means that the growth rate of species as a result of the action of prey, is supposed to be three times smaller than the predators death rate. The *push* $\omega(\tau, x)$ corresponding to the coefficients b and c, is regarded as outer order command for the Volterra system and depends on the time factor and the trajectory $x(\cdot)$ of the *small perturbation*. The optimal command will be calculated starting from the solution $p(\tau)$, of the following Bernoulli type equation [3], [5]

$$\dot{p}(\tau) + \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} p(\tau) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} p^2(\tau) = 0_2 \tag{10}$$

Consequently

$$p(\tau) = [C_1 - e^{(A+A^*)\cdot\tau}, C_2]^{-1},\ C_1 = S(A+A^*)^{-1}\ ,\ C_2 = -e^{-2\cdot A - A^*} + e^{-A-A^*} S(A+A^*)^{-1}\ . \tag{11}$$

The constant values for the numerical considered case are given by

$$C_2 == ((-0.00123938,\ -0.0676676\},\ \{-0.00915782,\ -0.5)),$$

$$C_1 = ((0.5,\ 0),\ \{0,\ 0)).$$

The optimal command $\omega^0(\tau, x)$ is given by

$$\omega^0(\tau,x) = -\sigma^{-1}(\tau) B^* p(\tau) x(\tau) = -B^* p(\tau) x(\tau), \tag{12}$$

where $x(\tau)$ is the solution to the linear differential equation

$$\dot{x}(\tau) = [A - Sp(\tau)]x(\tau),\qquad x(\tau) = x_0 \exp\left(\int_0^\tau [A - S(C_1 - \exp\left((A+A^*)\nu\right)]\mathrm{d}\nu\right). \tag{13}$$

The evolution of $x(\tau)$ on different intervals around the origin, is illustrated in Figs. 1 and 2

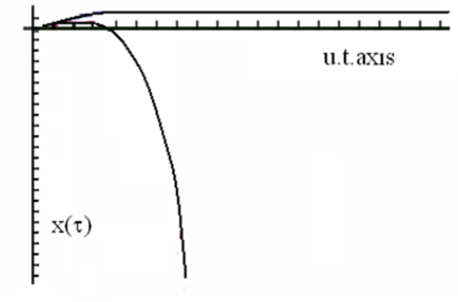

Fig. 1 – The evolution of $x(\tau)$ on [0,1] (u.t.).

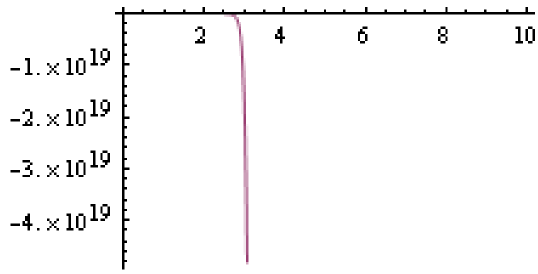

Fig. 2 – The evolution of $x(\tau)$ on [0,10] (u.t.).

Divided on components, the evolution of $x(\tau)$ values on the interval $[-2, 3]$ shows the existence of a maximum in the vicinity of the origin, followed by a significant decrease from positive to negative values, thus, marking an irregular oscillation behavior around this point (Fig. 3).

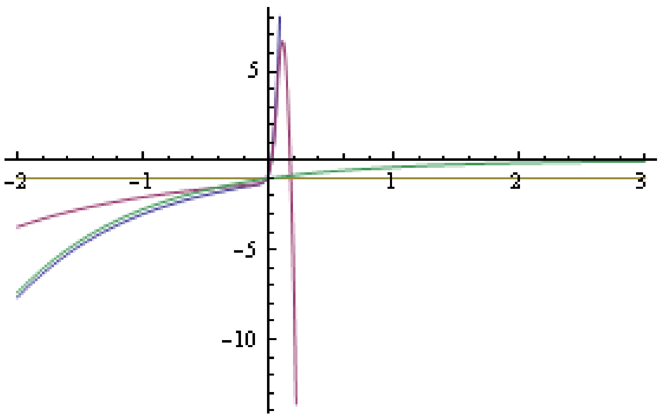

Fig. 3 – Evolution of $x(\tau)$ around (0, 0). The maximum of $x(\tau)$ is between 0.1 and 0.2 (u.t.)

Representations by points (Fig. 4), compared to a linear increase of the argument time on the range (0, 1), demonstrates the existence of a point of *minimum* for the command ω^0 (on the range (0.01, 0.2)).

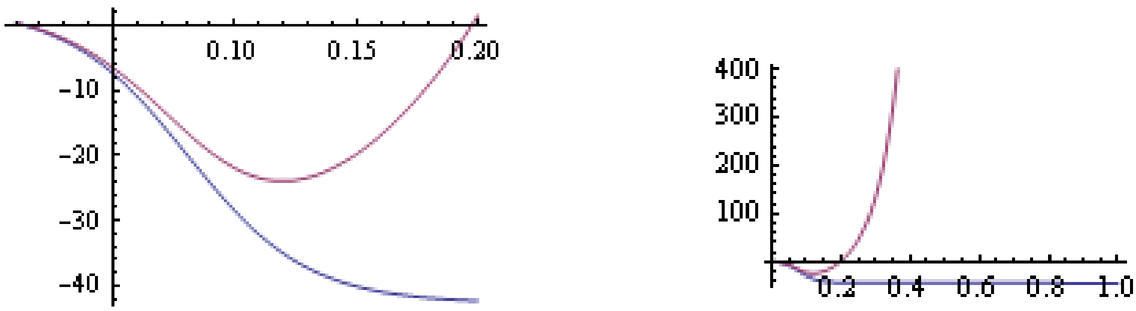

Fig. 4 – Minimum for ω^0 : between 0.10 and 0.15 (u.t.).

The points of minimum for ω^0, and those of the maximum for $x(\tau)$ are located approximately (as it can observe in Fig.4) between the same limits of ranges. The external negative action, i.e. the decrease in the number of predators, will lead to an increase multiplication rate for the prey population. Consequently, as the two segments of the population coexist, it should see that the first (prey population) grows faster than the second that drops (predatory population). The charts are made using the appropriate source code found in the file: "*Dynamic models of Lotka-Volterra type_Optimale solutions.nb*".

2. ASYMPTOTIC STABILITY OF THE LOTKA-VOLTERRA MODEL

Returning to the Lotka-Volterra system we consider the linearized dynamical system associated with new steady state approach. We recall that the solution near the point $u = v = 1$ is periodic, with the period $2\pi/\sqrt{\alpha} = T = 2\pi/\sqrt{a/d}$. On this observation, the following considerations are considered, starting from Lotka-Volterra equations, and taking into account that N and P ($N = x_1$ and $P = x_2$) are viewed as components in R^2 of state vector x.

By denoting

$$\varphi(x,t) = \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} \approx \begin{pmatrix} 0 & -1 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

it results

$$\dot{x} = \varphi(x,t), \quad \varphi(x,t+T) = \varphi(x,t), \quad x \in R^2, \tag{14}$$

with the right member periodically in time. We assimilate the small perturbations (x_1, x_2) with a pendulum motion whose parameters are changing periodically (cradle)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega^2(t) \cdot x_1, \quad x \in R^2 \tag{15}$$

with $\omega(t+T) = \omega(t) = \sqrt{-\alpha}$ and $A_1 = -A$, $A_1 = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}$.

We thus obtain the linearized system (6), written in a new form, i.e. as an equation with periodical coefficients. We assume that all solutions of (14) is extended unbounded, as the system is still linear. Periodicity of the right member of the equation is expressed by special properties for (6) [1]

LEMMA 1. *Phase space transformation given by the evolution $g_{t_1}^{t_2} : R^n \to R^n$ from time t_1 to time t_2 does not change, if t_1 and t_1 are simultaneously increased with T period of the right member.*

The translated $\psi(x,t) = \varphi(x,t+T)$ of the solution $\varphi(x,t)$ with time T is also a solution. But the translation of the T-axis time in the extended phase space of directions leaves unchanged the field equation (14). Therefore, an integral curve is tangent to T, translated all over the field direction and thus remains integral curve. It results: $g_{t_1+T}^{t_2+T} = g_{t_1}^{t_2}$ q.e.d.

We assume that $A = g_0^T : R^n \to R^n$ is the evolution of the dynamic system at the time T.

LEMMA 2. *Transformations g_0^{nT} form a group $g_0^{nT} = A^n$, $g_0^{nT+s} = g_0^s \cdot g_0^{nT}$.*

Taking into account the Lemma 1, it results: $g_{nT}^{nT+s} = g_0^s$, and consequently $g_0^{nT+s} = g_{nT}^{nT+s} \cdot g_0^{nT} = g_0^s \cdot g_0^{nT}$. If $s = T$ we obtain $g_0^{(n+1)T} = A \cdot g_0^{nT} \Rightarrow g_0^{nT} = A^n$ (by induction).

Some remarks are given next.

The linearized dynamic system of Lotka-Volterra type for particular cases of the variables is compatible with the transformations occurring in the real plan:

1) Rotation, with associated homogeneous system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1, \text{ with the system matrix } A_1 = -A, \quad A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_1 = -1 = -\alpha, \tag{16}$$

is similar with the linearization of the initial system, in the vicinity of (1,1).

2) Hyperbolic rotation, with the associated homogeneous system

$$\dot{x}_1 = x_1, \quad \dot{x}_2 = -x_2 \text{, with the system matrix } A_1 = A, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_1 = 1 = \alpha(a = d), \tag{17}$$

is similar with the linearization of the initial system, in the vicinity of (0,0).

It is also known that in R^2, the rotation is a stable application and the hyperbolic rotation is an unstable application.

Various properties of solutions of (14) correspond to similar properties of the application A for (6).

THEOREM. 1) *x_0 is a fixed point to A* $(Ax_0 = x_0) \Leftrightarrow x(t)$, with $x(0) = x_0$, *has the property* $x(t+T) = x(t)$

2) *x_0 (the fixed point) is stable $\Leftrightarrow x(t)$ is Lyapunov stable (asymptotically stable)*

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \text{a.i.}: \ \|x - x_0\| < \delta_\varepsilon \Rightarrow \ \|A^n x - A^n x_0\| < \varepsilon, \forall n < \infty \quad \Leftrightarrow \ A^n x_{n\to\infty} \to A^n x_0 .$$

3) *If the system (14) is linear then* $\det(\varphi(x,t) = \varphi(t)x$ *is a linear function related to x and the application A is linear.*

4) *If* $\mathrm{tr}(\varphi(t))$ *is null, then the application A preserves the volume* $\det A = 1$.

The proof is given by V.I. Arnold [1] for paragraphs 1), 2), 3) and for 4) it results from Liouville's theorem.

Stability conditions: Applying the theorem in the case of considered numerical example, we propose the following:

COROLLARY: *Application A is linear and preserves the area (*$\det A = 1$*). The linearized system solution at the point of equilibrium (singular) is stable if and only if application A is stable.*

3. CONCLUSIONS

Following the previous considerations, we will state the main conclusions:

1) The solution in the vicinity of $(0,0)$ of the linearized Lotka-Volterra system is unstable. The application of (8) leads to a hyperbolic rotation, and the intervention of the *disturbing factor* or the *external impulse/push* on the system is necessary and possible. The matrix A is symmetric [2], [3] and the optimal control command has the analytic expression expressed by (12).

2) The solution in the vicinity of (1,1) of the linearized Lotka-Volterra system is stable. The application of (6) ia equivalent to a rotation with period $T = 2\pi/\sqrt{\alpha}$. As a result, the intervention of the *disturbing factor* or the *external impulse* has no reason to be applied and the condition $\lambda_1 \cdot \overline{\lambda}_2 = \lambda_2 \cdot \overline{\lambda}_1 = |\lambda_1|^2 = |\alpha| < 2 = 1$ is verified.

3) According to (14), the periodical behavior of the linearized system (6) can be compared to the pendulum motion, if denoting by $\alpha = -\omega^2(t)$ and $A = -A_1$, with a rotation of angle $\sqrt{\alpha}$ $(x_{1,2} = \exp(\pm i\sqrt{\alpha})$. In this context, according to Lemma 2, the linearized dynamic system evolution in the vicinity of (1, 1) forms an algebraic group structure. The properties of solutions are similar to the properties of the application A (matrix of the system).

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