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QUANTUM-LIKE CLASSICAL MECHANICS IN NON-COMMUTATIVE PHASE SPACE

Daniela DRAGOMAN

University of Bucharest, Physics Department, P.O. Box MG-11, 077125 Bucharest, Romania
E-mail: `danieladragoman@yahoo.com`

Quantum-like evolution laws for observables can be derived from classical Hamiltonian equations with the only additional assumption that the phase space is non-commutative. The derivation is possible for Hamiltonians that are polynomial functions in position and momentum variables, and supports the use of phase space distributions functions in both quantum and classical theories that rely on extended states in phase space.

Key Words: Phase space; Quantum-classical correspondence; Non-commutativity.

1. INTRODUCTION

Quantum mechanics is probably the only fundamental theory that is not based on a specific physical principle that could (or not) be tested experimentally. It offers only exceptionally successful recipes of quantizing the dynamics of physical systems described by known classical Hamiltonians. The recipe itself is simple: describe the state of the localized particle by a vector in the Hilbert space (a non-localized wavefunction in the Schrödinger formalism) and replace the classical dynamical variables of position **x** and momentum **p** and any function $A(\mathbf{x},\mathbf{p})$ by linear operators $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$ and $\hat{A}$, respectively, so that the Poisson bracket of two functions $[A,B]_P$ transforms into the commutation relation $-i\hbar^{-1}[\hat{A},\hat{B}] = -i\hbar^{-1}(\hat{A}\hat{B}-\hat{B}\hat{A})$. In particular, for one-dimensional systems $[\hat{x},\hat{p}] = i\hbar$. This substitution can be consistently applied as quantization method if the Hamiltonian of the classical commutative system is at most quadratic in x and p, but it holds (if x and p are expressed in Cartesian coordinates and symmetric averages of various possible orders are taken) for the quantization of classical wave fields and particles in arbitrary commutative phase spaces [1].

This mathematical procedure obscures the physical differences between quantum and classical mechanics embodied in the quantum measurement theory, which implies the "collapse" of the wavefunction and forbids the simultaneous precise determination of canonically conjugated variables. Although it is commonly accepted that classical mechanics can be recovered from quantum mechanics as the $\hbar \to 0$ limit, it is less clear how to relate the quantum wavefunction to parameters of an ensemble of classical particles and how to derive the Schrödinger equation starting from classical evolution laws (we consider in this paper only the non-relativistic case and non-entangled states). This long-standing problem is periodically revived and several attempts have been made to solve it. For example, it is possible to define a quantum wavefunction-like complex quantity, related to the position probability density of an ensemble of classical particles and the corresponding average velocity, which satisfies the Schrödinger equation in certain conditions. These conditions include random momentum fluctuations of the classical ensemble that scale inversely with the uncertainty in the particle position expressed by the position probability density [2], frictionless Brownian motion with a diffusion coefficient $\hbar/2m$ of classical point-like particle of mass m [3], or stochastic forces that produce a departure ΔE from the particle's classical energy, in energy conserving trajectories, which can persist for an average time $\Delta t \cong \hbar/(2\Delta E)$ [4]. These assumptions allegedly explain quantum phenomena, such as the zero-point energy of oscillators, the stability of atoms, the slit diffraction, and the tunneling effect starting from models of classical ensembles of particles that are highly improbable and not based on experimental observations. Such unconvincing attempts to derive quantum behavior from classical mechanics has prompted the identification of the essence of quantum mechanics with the principle of superposition of states [5], although experiments and theory show that superpositions of classical fields have the same phase space signature as quantum cat states [6]. Alternatively, the difference between quantum and classical physics has been shown to reside in the phase space volume occupied by a state [7]. This approach identified Planck's constant as the phase space quantum, i.e. the minimum phase space area of the quantum state projection on each plane spanned by conjugate variables, required for the existence of a quantum state, and allowed a phase space formulation of the axioms of quantum mechanics that avoids the separation between quantum systems and (classical) measuring devices (see also [8]).

The identification of the element that leads to quantum mechanics from the classical theory of motion is still of interest since it is expected to enrich our understanding of quantum phenomena. The aim of this paper is to show that the evolution law of quantum observables can be derived from the classical Hamiltonian with the unique assumption that the position and momentum variables do not commute. This derivation method is inspired by the Feynman's proof of Maxwell's equations [9] and strengthens the results in [7]. Thus, classical non-commutative phase spaces and quantum states share the same key property of nonlocality and a phase space treatment of both cases becomes feasible.

2. QUANTUM MECHANICS FROM NON-COMMUTATIVE CLASSICAL MECHANICS

Let us consider for simplicity a one-dimensional classical particle with position x and momentum p, subject to a time-independent Hamiltonian H, i.e. to the classical equations of motion

$$\dot{p} = -\partial H / \partial x , \qquad \dot{x} = \partial H / \partial p , \tag{1}$$

which moves in a non-commutative phase space, in which

$$[x,x] = 0 , \quad [p,p] = 0 , \quad [x,p] = xp - px = \gamma . \tag{2}$$

Note that in (2) x and p are not considered operators but c-numbers; in this sense (2) are quantum-like commutation relations.

The derivation of the quantum-like evolution laws is much simplified if the Hamiltonian can be separated in a kinetic and a potential term, separation that is always possible in the non-relativistic classical mechanics for a particle of mass m subject to a force that derives from a potential $V(x)$. Then, for $H = p^2 / 2m + V(x)$, we have

$$\dot{x} = p / m , \qquad \dot{p} = -\partial V / \partial x , \tag{3}$$

and, after straightforward calculations, it follows from (2) that

$$[x,H] = [x, p^2 / 2m] = \gamma p / 2m + (p / 2m)[x,p] = \gamma p / m = \gamma \partial H / \partial p . \tag{4}$$

The quantum-like evolution law for position in a system with a time-independent Hamiltonian:

$$i\hbar \dot{x} = i\hbar \partial H / \partial p = [x,H] , \tag{5}$$

is then recovered if we put $\gamma = i\hbar$; we will comment later on this equality. For now, we keep the γ notation. Note that in (5) x and p are still not operators but the position and momentum variables in a non-commutative phase space. We refer to x and p as observables, since we assume that they can be observed and measured similar to the position and momentum variables in commutative classical phase spaces. This designation corresponds to the quantum one since x and p satisfy the quantum-like commutation relation (2).

The evolution law for the momentum observable similar to (5) can be found by employing the Jacobi identity

$$[H,[x,p]] + [x,[p,H]] + [p,[H,x]] = 0 , \tag{6}$$

in which the first term in the left-hand side vanishes since $[x,p]$ is a constant, and by noting that (1) and (2) imply

$$[\dot{x},p]+[x,\dot{p}]=0=[\partial H/\partial p,p]+[x,-\partial H/\partial x]\,. \tag{7}$$

By multiplying (7) with $-\gamma$ and adding it to (6) one obtains

$$[x,([p,H]+\gamma\partial H/\partial x)]+[p,([H,x]+\gamma\partial H/\partial p)]=0 \tag{8}$$

from which, using (5), it follows that

$$i\hbar\dot{p}=-\gamma\partial H/\partial x=[p,H]\,. \tag{9}$$

Again, this equation is identical to the corresponding quantum relation if $\gamma=i\hbar$. In fact, from (8) it only results that $[p,H]+\gamma\partial H/\partial x$ is constant or a function of *x*, but this function can be shown to vanish if $V(x)$ has certain forms. In particular, if the potential is a polynomial function of *x*, $V(x)=\alpha x^n$, then

$$\begin{aligned}[p,H]&=[p,\alpha x^n]=\alpha(px\cdot x^{n-1}-x^{n-1}\cdot xp)=\alpha(xp\cdot x^{n-1}-\gamma x^{n-1}-x^{n-1}\cdot xp)\\&=\alpha(x^2p\cdot x^{n-2}-2\gamma x^{n-1}-x^{n-1}\cdot xp)=\ldots=-n\alpha\gamma x^{n-1}=-\gamma\partial H/\partial x\,.\end{aligned} \tag{10}$$

Moreover, both equations (5) and (9) are valid for the motion of a particle with charge *e* in the presence of an electromagnetic field with vector and scalar potentials $\boldsymbol{A}$ and φ, respectively, at least when these potentials are polynomial functions of the spatial coordinate. For the one-dimensional non-relativistic case the Hamiltonian in this situation is $H(x,p)=[p-eA(x)]^2/2m+e\varphi(x)$, and the above method of finding the quantum evolution laws for *x* and *p* outlined above can be easily replicated. These two Hamiltonian forms cover the vast majority of situations in non-relativistic physics.

Equations (5) and (9), which are identical to the evolution laws of quantum observables for time-independent Hamiltonians, although referring to classical variables in a non-commutative phase space, can be generalized to any function of *x* and *p*. Using (5) and (9), it is a straightforward task to show that

$$\gamma d(x^k p^l)/dt=[x^k p^l,H]\,, \tag{11}$$

for *k*, *l* integers, and that a similar equation is satisfied by any function *F* of *x* and *p*, which does not explicitly depend on time:

$$\dot{F}(x,p)=[F,H]/\gamma\,. \tag{12}$$

After this result is established, an explicit time dependence can be easily accounted for. Note that in equations (11) and (12) the order of *x* and *p* is important, since they do not commute; for instance $\mathrm{d}(x^2p)/\mathrm{d}t=\mathrm{d}(xxp)/\mathrm{d}t=\dot{x}xp+x\dot{x}p+xx\dot{p}$. For $k=l=1$ and a polynomial energy potential, (11) becomes $\gamma\mathrm{d}(xp)/\mathrm{d}t=[xp,H]=\gamma(p^2/m-n\alpha x^n)$.

Since the classical evolution law for any function of *x* and *p* is given by the Poisson bracket, (12) implies that in a classical non-commutative phase space the Poisson bracket

$$[F,H]_P=(\partial F/\partial x)(\partial H/\partial p)-(\partial H/\partial x)(\partial F/\partial p) \tag{13}$$

has to be replaced by $[F,H]/\gamma$. This result is identical to that in quantum mechanics for $\gamma=i\hbar$, except that no use has been made of the operatorial mathematical formalism. It shows that quantum non-commutativity can be replaced by phase space non-commutativity, the non-locality property of physical states being shared by the two cases. In fact, this conclusion is not new. It is well known that quantum mechanical formalism is equivalent to a phase space treatment in terms of the Wigner distribution function [10], defined on *c*-number variables (see [7, 11–15] for its properties and relations to the standard formulations of quantum mechanics). Such a phase-space treatment of quantum mechanics not only allows a re-writing of the known quantum mechanical axioms but provides the physical principle that distinguishes quantum mechanics from classical mechanics: quantum states have phase space projection areas of at least $\hbar/2$ on any plane spanned by non-commuting variables, while classical states are points in phase space. The employment of the Wigner distribution function to describe quantum states is perfectly suited for the description of quantum phenomena, including the measurement problem [7,8], as long as the Wigner distribution function is interpreted as a quasi-probability distribution in phase space. Negative value regions of this distribution, which correspond to dark rays in optics, do not only lead to positive probability values if averaged on the phase space area associated to the quantum state, but are necessary for the consistency of the theory since they are intimately related to wave-like phenomena such as diffraction and interference [16].

3. DISCUSSIONS

The results obtained in this paper show that the point-like classical particles in commutative phase spaces can be replaced in quantum-like theories by extended particles that evolve according to classical laws but are localized in finite phase space areas, of $\gamma/2$. This value follows from the identity $[x,p]/2 = x \wedge p$, where $x \wedge p = (xp - px)/2$ is the outer product of *x* and *p*, which equals the oriented area of the parallelogram in the non-commutative phase space with sides *x* and *p* [17]. Such a result supports and strengthens the phase space treatment of quantum mechanics in [7].

It should be mentioned that the idea of non-commutative phase spaces in classical physics is not new, but encountered in wave mechanics and electromagnetic field theory. This paper only generalizes this idea for the classical mechanics of material particles. A similar change in phase space topology, from a point to an area equal to $\lambda\!\!\!^{-}/2$, with $\lambda\!\!\!^{-} = \lambda/2\pi$ and λ the optical wavelength, characterizes the transition from ray optics to wave optics, for example. Moreover, an operator $\hat{p} = -i\lambda\!\!\!^{-}\partial/\partial x$, analogous to the quantum mechanical operator $\hat{p} = -i\hbar\partial/\partial x$, can be introduced in wave optics [18]. This operator is canonically conjugate to the transverse position in the Hamiltonian sense if the longitudinal spatial coordinate plays the role of time in quantum mechanics. In addition, the quantum uncertainty relation is expressed in optics through the beam quality factor *Q* [19].

The results obtained here are sustained by the demonstration that quantum wavefunction discontinuities propagate along classical trajectories in quantum mechanics, in a similar manner as electromagnetic field discontinuities propagate along rays in geometrical optics, the classical Hamilton-Jacobi equation corresponding to the eikonal equation in optics [20]. It should be noted that the recovery of quantum-like evolution law for observables from classical Hamiltonians in a non-commutative phase space demonstrated in this paper has a different meaning than the Ehrenfest theorem [21]. One difference is that the position and momentum observables in this paper are not average values for an ensemble of particles but refer to a single particle. Another one refers to the different mathematical treatment: Ehrenfest showed that the equations of motion of the average values of quantum observables are identical with the classical expressions based on quantum evolution laws and the non-commutativity of position and momentum operators, whereas here the quantum-like evolution laws were recovered starting from the classical Hamiltonian and phase space non-commutativity, with *x* and *p* as *c*-numbers.

The full recovery of quantum-like evolution laws in the present approach relies on the identification $\gamma = i\hbar$. The γ parameter should be in agreement with measurements. In fact, experimental tests of quantum uncertainty do not measure directly the commutation relations but the Heisenberg uncertainty relations, which can be written in the general form as $\Delta A \Delta B \geq (1/2)\,|\langle[\hat{A},\hat{B}]\rangle|$, where the standard deviations are defined as $\Delta X = (\langle \hat{X}^2\rangle - \langle \hat{X}\rangle^2)^{1/2}$, *X* = *A*, *B*, with $\langle ... \rangle$ the mean value of the observable in the given quantum state. In particular, $\Delta x \Delta p \geq |i\hbar|/2 = \hbar/2$ and from all diffraction experiments for quantum particles such as neutrons [22], atoms [23] and fullerenes [24] it only follows that $|\gamma| = \hbar$. So, $\gamma = i\hbar$ is not at odds with measurement results. It should be emphasized that the mentioned experimental data can be considered for the determination of γ since the quantum particles do not obey the classical laws of point particles in commutative phase spaces.

4. CONCLUSIONS

In this paper a demonstration of the classical-quantum correspondence rule is provided in detail for the position and momentum observables, endowing the classical variables with a single new property: that of phase space non-commutativity. This correspondence rule has been demonstrated for Hamiltonians that are polynomial functions of x and p of any order, which is important since the quantization method in standard quantum mechanics is actually proven only for Hamiltonians of the classical commutative system at most quadratic in x and p; this result is known as the Groenwald-van Hove theorem [25]. Although used for quantizing classical wave fields and particles in arbitrary commutative phase spaces [1], the replacement rule $[F,H]_P \to [F,H]/i\hbar$ produces relevant results, but is not actually demonstrated.

The quantum-like laws are obtained here without the need to introduce Hilbert spaces, quantum states or wavefunctions with debatable meaning. This is possible since classical non-commutative phase space and quantum mechanics share the non-locality property. The transition from classical to quantum mechanics is similar to the transition from ray optics to wave optics, at least from the point of view of deriving evolution equations for observables, and therefore does not need a special mathematical apparatus. More precisely, the Hilbert space and the operators and states defined on it in the common formulation of quantum mechanics can be replaced as mathematical tools by classical variables in a non-commutative phase space that describes extended classical particles. This result supports the use of c-number-defined Wigner distribution functions for describing the quantum behavior of extended states in phase space. Since this distribution function can be defined for both quantum states and extended classical states (in optics, up to now, and for classical particles, in this paper), it becomes possible to develop a common phase space mathematical formalism for both quantum and classical theories.

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