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THE SOLUTION OF A PARAMTERIZED SIXTH ORDER BOUNDARY VALUE PROBLEM BY THE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

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A new analytical method, namely the Optimal Homotopy Asymptotic Method (OHAM) is applied to a parameterized sixth order two-point boundary value problem. This method develops a series solution whose convergence is controlled optimally and the convergence region can be adjusted according to the demand of the problem. A comparison with the solution obtained by Homotopy Analysis Method (HAM) is made and the results show high accuracy and reliability. The results reveal that the proposed method is effective, explicit and easy to use.

Key words: Parameterized sixth order boundary value problems, Boundary value problems, Optimal homotopy asymptotic method.

1. INTRODUCTION

Most of the physical phenomena are generally modeled by differential equations along with appropriate conditions. Since exact solutions to these differential equations are very rare so researchers always look for the best approximate solutions. Numerical methods and series solution methods are the tools to find the approximate solutions.

The recent literature for the solution of differential equations contains: The Adomian decomposition method (ADM) [1–3], the differential transform method (DTM) [4], the variational iteration method (VIM) [5], the successive iteration [6], the splines [7,8], the homotopy perturbation method (HPM) [9,10], the homotopy analysis method (HAM) [11] etc. The classical perturbation methods are restricted to small or large parameters and hence their use is confined to a limited class of problems. The HPM as well as HAM, which are the elegant combination of homotopy from topology and perturbation techniques, overcome the restrictions of small or large parameters in the problems.

Marinca and Herisanu [12–15] introduced the Optimal Homotopy Asymptotic Method (OHAM), which uses the more generalized auxiliary function $H(p)$. They reported different forms of auxiliary function that can be expressed in a compact form as $H(p)=f(r)g(p,C_i)$. Here $g(p,C_i)$ is the power series in p, and the unknown constants C_i's, which control the convergence of the approximating series solution, are optimally determined. For $f(r)g(p,C_i)=-p$, OHAM becomes HPM and for $f(r)=\varphi(r)$, where φ is a smoothing function in HAM and $g(p,C_i)=p\hbar$, it becomes the HAM. Thus HPM and HAM are the special cases of OHAM. One feels great freedom as the convergence of the OHAM solution does not depend on the initial guess; it depends only on the auxiliary function.

Javed Ali et al. [16], used this method for the solutions of multi-point boundary value problems. The same author used this method for the solution of special twelfth order boundary value problems [17].

In this paper, we solve a parameterized sixth-order boundary value problem by OHAM and the results are compared with those of exact solution and the solution obtained by HAM. The structure of this paper is organized as follows. Section 2 is devoted to the analysis of the proposed method. Solution of the selected problem is presented in Section 3. In Section 4, we conclude by discussing results of the numerical simulation by using Mathematica.

2. ANALYIS OF THE METHOD

Consider the following differential equation:

$$L(u(r)) + \phi(r) + N(u(r)) = 0, \quad r \in \Omega \tag{2.1}$$

along with the boundary conditions: $B\left(u, \frac{\partial u}{\partial n}\right) = 0$ on $\partial\Omega$, where: L is a linear operator, r denotes independent variable, $u(r)$ is an unknown function , $\phi(r)$ is a known function, N is a nonlinear operator and B is a boundary operator.

According to OHAM we construct a homotopy $H(v,p): \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$\begin{aligned} &(1-p)[L(v(r,p)) + \phi(r)] = h(p)[L(v(r,p)) + \phi(r) + N(v(r,p))], \\ &B\left(v(r,p), \frac{\partial v(r,p)}{\partial n}\right) = 0, \end{aligned} \tag{2.2}$$

where $r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, $h(p)$ is a nonzero auxiliary function for $p \neq 0$, $h(0) = 0$ and $v(r,p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v(r,0) = u_0(r)$ and $v(r,1) = u(r)$ respectively.

Thus, as p varies from 0 to 1, the function $v(r,p)$ approaches from the starting value $u_0(r)$ to the solution function $u(r)$, where $u_0(r)$ is obtained from Eq (2.2) for $p = 0$ and we have

$$L(u_0(r)) + \phi(r) = 0, \qquad B\left(u_0, \frac{\partial u_0}{\partial n}\right) = 0 . \tag{2.3}$$

Next, we choose auxiliary function $h(p)$ in the form

$$h(p) = pC_1 + p^2C_2 + p^3C_3 + ... \tag{2.4}$$

where $C_1, C_{2,}...$ are constants to be determined.

To get an approximate solution, we expand $v(r,p)$ in Taylor's series about p in the following manner,

$$v(r,p,C_i) = u_0(r) + \sum_{k=1}^{\infty} u_k(r, C_1, C_2, ..., C_k)\, p^k \tag{2.5}$$

Substituting Eq. (2.5) into Eq. (2.2) and equating the coefficient of like powers of p, we obtain the following linear equations.

Zeroth order problem is given by Eq. (2.3) and the first order problem is given by Eq. (2.6):

$$L(u_1(r)) + \phi(r) = C_1 N_0(u_0(r)), \; B\left(u_1, \frac{\partial u_1}{\partial n}\right) = 0 \tag{2.6}$$

The general governing equations for $u_k(r)$ are given by:

$$\begin{aligned} &L(u_k(r)) - L(u_{k-1}(r)) = C_k\, N_0(u_0(r)) + \\ &\sum_{i=1}^{k-1} C_i\left[L(u_{k-i}(r)) + N_{k-i}(u_0(r), u_1(r), ..., u_{k-1}(r))\right], \\ &B\left(u_k, \frac{\partial u_k}{\partial n}\right) = 0 \qquad\qquad k = 2,3,..., \end{aligned} \tag{2.7}$$

where $N_m(u_0(r), u_1(r), ..., u_m(r))$ is the coefficient of p^m in the expansion of $N(v(r,p))$ about the embedding parameter p,

$$N(v(r,p,C_i)) = N_0(u_0(r)) + \sum_{m=1}^{\infty} N_m(u_0, u_1, u_2, ..., u_m)\, p^m\ . \tag{2.8}$$

The convergence of the series (2.5) depends upon the auxiliary constants $C_1, C_2, ...$. If it is convergent at $p = 1$, one has

$$v(r, C_i) = u_0(r) + \sum_{k=1}^{\infty} u_k(r, C_1, C_2, ..., C_k), \tag{2.9}$$

and the result of the m^{th}-order approximations are given by

$$\tilde{u}(r, C_1, C_2, ..., C_m) = u_0(r) + \sum_{i=1}^{m} u_i(r, C_1, C_2, ..., C_i) \tag{2.10}$$

By substituting Eq. (2.10) into Eq. (2.1), the resulting residual is

$$R(r, C_1, C_2, ..., C_m) = L(\tilde{u}(r, C_1, C_2, ..., C_m)) + \phi(r) + N(\tilde{u}(r, C_1, C_2, ..., C_m)) \tag{2.11}$$

$\tilde{u}$ will be the exact solution if $R = 0$. But it doesn't happen in nonlinear problems.

To find the optimal values of C_i, we first construct the functional

$$\mathrm{v}(C_1, C_2, ..., C_m) = \int_{r_a}^{r_b} R^2(r, C_1, C_2, ..., C_m)\,\mathrm{d}r\ . \tag{2.12}$$

And then minimizing it, we have

$$\frac{\partial \nu}{\partial C_1} = \frac{\partial \nu}{\partial C_2} = ... = \frac{\partial \nu}{\partial C_m} = 0, \tag{2.13}$$

where r_a and r_b are in the domain of the problem. With these constants known, the approximate solution (of order m) is well-determined.

3. PARAMETARIZED SIXTH ORDER BOUNDARY VALUE PROBLEM

Consider the following problem

$$u^{(6)}(x) = (1+c)u^{(4)}(x) - cu''(x) + cx\ , \quad 0 < x < 1, \tag{3.1}$$

with boundary conditions:

$$\begin{gathered} u(0) = 1\ ,\ u'(0) = 1\ ,\ u''(0) = 0, \\ u(1) = \frac{7}{6} + \sinh(1)\ ,\ u'(1) = \frac{1}{2} + \cosh(1)\ ,\ u''(1) = 1 + \sinh(1). \end{gathered} \tag{3.2}$$

The exact solution of this problem is

$$u(x) = 1 + \frac{1}{6}x^3 + \sinh(x). \tag{3.3}$$

We see that the exact solution of this problem does not depend on the parameter c but the problem itself does. This can be viewed by rewriting Eq.(3.1) as

$$\{u^{(6)}(x) - u^{(4)}(x)\} - c\{u^{(4)}(x) - u''(x) + x\} = 0,$$

which shows that, no matter what the value of c is, a solution of fourth-order problem is also a solution of the sixth-order problem.

For the problem under discussion, we follow the procedure of OHAM in section 2 and we select the following auxiliary function.

$$h(p) = pC_1 + p^2C_2 + p^3C_3$$

Accordingly we obtain the following linear problems:

Zeroth Order Problem:

$$\begin{gathered} u_0^{(6)}(x) = cx, \\ u_0(0) = 1,\ u_0^{(1)}(0) = 1,\ u_0^{(2)}(0) = 0, \\ u_0(1) = \frac{7}{6} + \sinh(1),\ u_0^{(1)}(1) = \frac{1}{2} + \cosh(1),\ u_0^{(2)}(1) = 1 + \sinh(1). \end{gathered} \tag{3.4}$$

First Order Problem:

$$\begin{gathered} u_1^{(6)}(x) = (1+C_1)u_0^{(6)}(x) - C_1(1+c)u_0^{(4)}(x) + C_1cu_0^{(2)}(x) - c(1+C_1), \\ u_1(0) = 0,\ u_1^{(1)}(0) = 0,\ u_1^{(2)}(0) = 0, \\ u_1(1) = 0,\ u_1^{(1)}(1) = 0,\ u_1^{(2)}(1) = 0. \end{gathered} \tag{3.5}$$

Second order problem:

$$\begin{gathered} u_2^{(6)}(x) = (1+C_1)u_1^{(6)}(x) + C_2u_0^{(6)}(x) - C_1(1+c)u_1^{(4)}(x) - C_2(1+c)u_0^{(4)}(x) \\ +C_1cu_1^{(2)}(x) + C_2c(u_0^{(2)}(x) - x), \\ u_2(0) = 0,\ u_2^{(1)}(0) = 0,\ u_2^{(2)}(0) = 0, \\ u_2(1) = 0,\ u_2^{(1)}(1) = 0,\ u_2^{(2)}(1) = 0. \end{gathered} \tag{3.6}$$

Third order problem:

$$\begin{gathered} u_3^{(6)}(x) = (1+C_1)u_2^{(6)}(x) + C_2u_1^{(6)}(x) + C_3u_0^{(6)}(x) - C_1(1+c)u_2^{(4)}(x) - C_2(1+c)u_1^{(4)}(x) \\ -C_3(1+c)u_0^{(4)}(x) + C_1cu_2^{(2)}(x) + C_2cu_1^{(2)}(x) + C_3c(u_0^{(2)}(x) - x), \\ u_3(0) = 0,\ u_3^{(1)}(0) = 0,\ u_3^{(2)}(0) = 0, \\ u_3(1) = 0,\ u_3^{(1)}(1) = 0,\ u_3^{(2)}(1) = 0. \end{gathered} \tag{3.7}$$

Solutions to the zeroth and the first order problems are given by Eqs. (3.8) and (3.9) respectively.

$$\begin{gathered} u_0(x) = 1 + x + 0.333957x^3 - 0.000595238c\,x^3 - 0.00165465x^4 + \\ +0.0015873c\,x^4 + 0.00956585x^5 - 0.00119048c\,x^5 + 0.000198413c\,x^7. \end{gathered} \tag{3.8}$$

$$\begin{aligned}u_1(x,C_1) &= 0.00062812C_1x^3 - 0.000594872C_1cx^3 - 4.69906\times10^{-6}C_1c^2x^3 - 0.0016566C_1x^4 + \\ &+ 0.00158697C_1cx^4 + 2.14732\times10^{-6}C_1c^2x^4 + 0.00120109C_1x^5 - 0.00119192C_1c\,x^5 + \\ &+ 0.0000300053C_1c^2x^5 + 0.0000551552C_1x^6 + 2.2451\times10^{-6}C_1c\,x^6 - 0.0000529101C_1c^2x^6 - \\ &- 0.000227758C_1x^7 + 0.000198154C_1cx^7 + 0.0000276361C_1c^2x^7 - 9.84913\times10^{-7}C_1c^2x^8 + \\ &+ 9.44822\times10^{-7}C_1c^2x^8 + 4.0758\times10^{-7}C_1cx^9 - 3.14941\times10^{-6}C_1c^2x^9 + 2.50521\times10^{-8}C_1c^2x^{11}.\end{aligned} \quad (3.9)$$

Solutions to the second and third order problems can be obtained easily.

For $p=1$, the third order approximate solution by OHAM is

$$\tilde{u}(x) = u_0(x) + u_1(x,C_1) + u_2(x,C_1,C_2) + u_3(x,C_1,C_2,C_3). \quad (3.10)$$

Using Eq.(3.10), residual of the solution is

$$R = \tilde{u}^{(6)}(x) - (1+c)\tilde{u}^{(4)}(x) + c\tilde{u}''(x) - cx\,. \quad (3.11)$$

Now using Eq.(2.12) and Eq.(2.13) in section 2, and taking $r_a=0$ and $r_b=1$, we obtain the following values of the C_i's for $c = 1$.

$$C_1 = -0.935780212,\ C_2 = -0.001555044,\ C_3 = 0.000043847.$$

By considering these values our approximate solution becomes:

$$\begin{aligned}\tilde{u}(x) &= 1 + x + 0.333333x^3 + 1.29887\times10^{-6}x^4 + 0.00832295x^5 + 0.0000416114x^6 + \\ &+ 0.000096176x^7 + 0.000162708x^8 - 0.000162731x^9 + 0.0000992639x^{10} - \\ &- 0.0000267093x^{11} - 2.10774\times10^{-6}x^{12} + 2.11295\times10^{-6}x^{13} + 1.05849\times10^{-8}x^{14} - \\ &- 2.3803\times10^{-8}x^{15} - 1.33648\times10^{-11}x^{16} + 7.66735\times10^{-11}x^{17} - 6.73639\times10^{-14}x^{19}.\end{aligned} \quad (3.12)$$

For c = 1 000, the following values of the C_i's are obtained:

$$C_1 = -0.135830834,\ C_2 = -0.025540511,\ C_3 = 0.002007572.$$

In this case our solution is

$$\begin{aligned}\tilde{u}(x) &= 1 + x + 0.274438x^3 + 0.833256x^4 - 4.9068x^5 + 16.3271x^6 - 34.4276x^7 + \\ &+ 482972x^8 - 44.7369x^9 + 25.2817x^{10} - 6.55572x^{11} - 0.59442x^{12} + 0.552725x^{13} + \\ &+ 0.00328518x^{14} - 0.00652957x^{15} - 4.5582\times10^{-6}x^{16} + 0.0000221568x^{17} - \\ &- 2.06016\times10^{-8}x^{19}.\end{aligned} \quad (3.13)$$

In Tables 1 and 2, numerical results of the OHAM third-order solution are compared with the exact solution and the error is compared with the error of the fifth-order HAM solution. It is clear from this table that OHAM is more effective and explicit than HAM.

Table 1

Numerical Results for c = 10

x	Exact solution	Solution by OHAM	Error*(HAM[4])	Error*(OHAM)
0.1	1.10033	1.10033	6.9E-11	-2.6E-11
0.2	1.20267	1.20267	2.6E-10	-1.9E-10
0.3	1.30902	1.30902	1.1E-09	-3.4E-10
0.4	1.42142	1.42142	1.7E-09	-2.9E-10
0.5	1.54193	1.54193	1.9E-09	-7.0E-11
0.6	1.67265	1.67265	1.5E-09	1.4E-10
0.7	1.81575	1.81575	7.6E-10	1.8E-10
0.8	1.97344	1.97344	1.6E-10	8.8E-11
0.9	2.14802	2.14802	3.5E-11	2.9E-11

Error* = Exact solution – Approximate solution

Table 2

Numerical Results for $c = 1000$

x	Exact solution	Solution by OHAM	Error*(HAM[4])	Error*(OHAM)
0.1	1.10033	1.10032	9.1E-06	1.1E-05
0.2	1.20267	1.20267	1.6E-04	3.3E-06
0.3	1.30902	1.30903	4.4E-04	-1.4E-05
0.4	1.42142	1.42141	6.8E-04	5.2E-06
0.5	1.54193	1.54189	7.3E-04	4.2E-05
0.6	1.67265	1.67260	5.8E-04	5.7E-05
0.7	1.81575	1.81570	3.2E-04	4.9E-05
0.8	1.97344	1.97339	9.8E-05	4.5E-05
0.9	2.14802	2.14799	4.7E-06	2.4E-05

Error* = Exact solution – Approximate solution.

4. CONCLUSIONS

In this paper we have used OHAM to find the approximate analytic solution to parameterized sixth order two-point boundary value problems. It is observed that the method is explicit, effective and reliable. It works well for higher order problems and represents the fastest convergence as well as a remarkable low error. The OHAM also provides us with a very simple way to control and adjust the convergence of the series solution using the auxiliary constants $C_i's$ which are optimally determined. Furthermore, the results of the method show excellent agreement with the exact solution. This method has a great potential to attract researchers, scientists and engineer of every field.

REFERENCES

1. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition method*, Vol. 60 of *Fundamental Theories of Physics*, Kluwer Academic Publishers, Boston, Mass, USA, 1994.
2. G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations*, Computer Math, Appl., **21**, *5*, pp. 101-127, 1991.
3. A. M. Wazwaz, *Approximate solutions to boundary value problems of higher-order by the modified decomposition method*, Computer Math, Appl., **40**, pp. 679-691, 2000.
4. Siraj-Ul-Islam, Sirajul-Haq and Javed Ali, *Numerical solution of special 12th-order boundary value problems using differential transform method*, Comm.Nonl.Sc.Nu.Sim., **14**, *4*, pp. 1132-1138, 2009.
5. J. H He, *Variational approach to the sixth-order boundary value problems*, Appl. Math. Comput., **143**, *2-3*, pp. 537-538, 2003.
6. Q. Yao, *Successive iteration and positive solution for nonlinear second-order three-point boundary value problem*, Comput. Math. Appl. **50**, pp. 433-444, 2005.
7. S. S. Siddique, E. H. Twizell, *Spline solutions of linear twelfth-order boundary value problems,* Computer Math.Appl., **78**, pp. 371-390, 1997.
8. S.S. Siddique, G. Akram, *Solutions of 12th order boundary value problems using non-polynomial spline technique*, Appl. Math. Comput., **199**, *1-2*, pp. 559-571, 2008.
9. Ji-Huan He, *Homotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering, **170,** *3-4*, pp. 257-262, 1999.
10. Ji-Huan He, *Homotopy perturbation method for solving boundary value problems*. Phys. Lett. A., **350**, pp. 87-88, 2006.
11. Shijun Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*. Chapman & Hall/CRC, Florida, 2004.
12. V. Marinca, N. Herisanu, I. Nemes, *Optimal homotopy asymptotic method with application to thin film flow*, Cent. Eur. J. Phys., **6**, *3,* pp. 648-653, 2008.
13. N. Herisanu, V. Marinca, T. Dordea, G. Madescu, *A new analytical approach to nonlinear vibration of an electric machine*, Proc. Romanian. Acad., Seies A, **9**, *3*, 2008.
14. V. Marinca, N. Herisanu, C. Bota, B.Marinca, *An optimal homotopy asymptotic method applied to the steady flow of fourth-grade fluid past a porous plate*, Appl. Math. Lett., **22**, *2*, pp. 245-251, 2009.
15. V. Marinca, N. Herisanu, *An optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer*, Int. Comm. Heat and Mass Transfer, **35**, pp. 710-715, 2008.
16. R. A. Shah, S. Islam, Gul Zaman, T. Hussain, and Gul Zaman, *Solution of stagnation point flow with heat transfer analysis by optimal homotopy asymptotic method*, Proc. Rom. Acad., Series A, **11**, *4*, pp. 312-321, 2010.

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