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ON THE FEKETE-SZEGÖ THEOREM FOR THE GENERALIZED OWA-SRIVASTAVA OPERATOR

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Abstract. Let A be the class of analytic functions in the open unit disk $U=\{z:|z|<1\}$. The sharp bound is obtained for the coefficient functional $\left|a_3-\mu a_2^2\right|$, where $\mu\in\mathbb{C}$ or $\mathbb{R}$ and a_2, a_3 are respectively the second and the third coefficient for f belonging to a certain subclass $R_{\alpha,\beta(\lambda,\rho)}$ defined by a fractional operator. By specializing the parameters α,β,λ and ρ, many consequence results are obtained. Further, an improvement for the estimation of $\left|a_3-\mu a_2^2\right|$ is investigated by dividing the intervals of $\mu\in\mathbb{R}$. In addition, sharp estimates for the first few coefficients of the inverse functions of $R_{\alpha,\beta(\lambda,\rho)}$ are derived.

Key words: Fekete-Szegö inequality, Hankel determinant, Fractional derivative, Coefficient inequality.

1. INTRODUCTION

Let A be the class of functions analytic in the open unit disk $U=\{z:|z|<1\}$ and let S be the family of univalent functions f in A of the form

$$f(x)=z+\sum_{n=2}^{\infty}a_n z^n \quad (z\in U). \tag{1}$$

The class S is very important in the area of univalent function theory. Back in 1916, Bieberbach enriches the class S by conjecturing an elegant problem based on Koebe functions and become hot topics for several decades. The starting point of the problems was to look at the absolute value for the coefficient a_n for all $n\geq 2$ of functions $f\in S$. It was neatly proved by De'Branges in 1984-85. Along the years in 1933, Fekete and Szegö [6] solved the functional $\left|a_3-\mu a_2^2\right|$ for functions $f\in S$ and for $\mu\in\mathbb{R}$. It was a very great combination of the two coefficients which describe the area problems posted earlier by Gronwall in 1914-15. Though the problems are solved, many new results are obtained for different types of classes defined by different means. Here, we will study this classical problem for functions f in A defined by operator via convolutions (or Hadamard product).

For the functions f and g given by the power series

$$f(z)=z+\sum_{n=0}^{\infty}a_n z^n,\ g(z)=\sum_{n=0}^{\infty}b_n z^n \quad (z\in U), \tag{2}$$

their Hadamard product (or convolution), denoted by $f*g$, is defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_n b_n z^n \quad (z\in U).$$

Note that $f*g\in A$. By making use of the Hadamard product, Carlson-Shaffer [4] defined the linear operator $\mathcal{L}(a,b):A\to A$.

By

$$\mathcal{L}(a,b)f(z)=\psi(a,b;z)*f(z)(f\in A),$$

where $\psi(a,b;z)$ be given by $\psi(a,b;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}z^{n+1}(z\in U, b\neq 0,-1,-2,...)$ and $(x)_n$ is the Pohhammer symbol defined by

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1, & \text{for } n=0\\ x(x+1)...(x+n-1), & \text{for } n\in N.\end{cases}$$

Note that $\mathcal{L}(a,a)$ is the identity operator and $\mathcal{L}(a,c)=\mathcal{L}(a,b)\mathcal{L}(b,c),\ (b,c\neq -1,-2...)$.

It is well known that if $b>a>0$, then $\mathcal{L}$ maps A into itself. The concept of linear operator will be used in finding the sharp functions for the coefficient problems to be discussed in the subsequent sections.

In this present paper, we will find sharp upper bounds for the functional $\left|a_3-\mu a_2^2\right|$ for functions f in A as mentioned above. Few coefficients of the inverse functions will also be derived. Details regarding the problems will also be discussed in the next section.

1.1. Preliminary results

The following definitions will be used in our paper.

Definition 1 (Owa & Srivastava [13]; Owa [14], see also Srivastava & Owa [17, 18]). Let the function f be analytic in a simply-connected domain of the z-plane containing the origin. The fractional derivative of f of order α is defined by

$$D_z^{\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)}d\zeta,\ \ 0\leq\alpha\leq 1, \tag{3}$$

where the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\zeta>0$.

Using Definition 1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [13] introduced the fractional differintegral operator $\Omega^{\alpha}:A\to A$ defined by

$$\begin{aligned}\Omega^{\alpha}f(z)&=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha},\ \alpha\neq 2,3,...z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}=\\&=\psi(2,2-\alpha;\ z)*f(z)=\mathcal{L}(2,2-\alpha)f(z).\end{aligned} \tag{4}$$

Note that $\Omega^0 f(z)=f(z)$ and $\Omega^1 f(z)=zf'(z)$.

Recently, Al-Refai and Darus [2] generalized the Owa-Srivastava operator as follows:

Definition 2 (see Al-Refai and Darus [3]). Let f be in A. Then the operator $\theta^{\alpha,\beta}:A\to A$ is defined by $(\theta^{\alpha,\beta}f)(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}\left(\Gamma(2-\beta)z^{\beta}D_z^{\beta}f(z)\right),(\alpha,\beta\neq 2,3,...)$ where D_z^{γ} is the fractional derivative of f of order γ.

From Definition 2, we note that

$$\theta^{\alpha,\beta}f(z)=\theta^{\beta,\alpha},\ \theta^{0,0}f(z)=f(z),$$

$$\theta^{\alpha,0}f(z)=\Omega^{\alpha}f(z),\ \theta^{0,1}f(z)=z\left(\Omega^{\alpha}f(z)\right)',$$

and

$$\begin{aligned}\theta^{\alpha,\beta} f(z) &= z+\sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\times\frac{\Gamma(n+1)\Gamma(2-\beta)}{\Gamma(n+1-\beta)}\right)a_n z^n = \\ &= \mathcal{L}(2,2-\beta)\Omega^{\alpha} f(z)=\Omega^{\beta}\left(z/(1-z)\right)*\Omega^{\alpha} f(z)= \\ &= \Omega^{\beta}\left(\Omega^{\alpha} f(z)\right)=\Omega^{\alpha}\left(\Omega^{\beta} f(z)\right).\end{aligned} \tag{5}$$

Making use of Definition 2, Al-Refai and Darus have studied interesting classes of analytic univalent functions for various geometric properties (see Al-Refai and Darus [1–3]). Many other properties are gained from these classes.

Definition 3 (Al-Refai and Darus [2]). The function $f \in A$ is said to be in the class $R_{\alpha,\beta}(\lambda,\rho)$ $\left(0\le\alpha<1,\ 0\le\beta<1,\ -\frac{\pi}{2}<\lambda<\frac{\pi}{2},\ 0\le\rho\le 1\right)$ if it satisfies the inequality

$$\operatorname{Re}\left(e^{i\lambda}\frac{\theta^{\alpha,\beta f(z)}}{z}\right)>\rho\cos\lambda \quad (z\in U). \tag{6}$$

Write

$$R_{\alpha,0}(0,\rho)=R_{\alpha}(\rho),\ \text{and}\ R_{\alpha,0}(\lambda,\rho)=R_{\alpha}(\lambda,\rho).$$

Let $\mathcal{P}$ be the family of functions $p\in A$ satisfying $p(0)=1$ and $\operatorname{Re}(p(z))>0,\ z\in U$.

It is known from (Al-Refai and Darus [2]) that

$$f\in R_{\alpha,\beta}(\lambda,\rho)\Leftrightarrow\left[(1-\rho)p(z)\rho\right]\cos\lambda+i\sin\lambda \tag{7}$$

where $-\frac{\pi}{2}<\lambda<\frac{\pi}{2}$, and $p\in\mathcal{P}$.

We note that,

$$R_{\alpha,\beta}(0,\rho)=\left\{f\in A:\operatorname{Re}\left\{\theta^{\alpha,\beta} f(z)/z\right\}>\rho\right\} \tag{8}$$

$$R_{\alpha,1}(\lambda,\rho)=\left\{f\in A:\operatorname{Re}\left\{e^{i\lambda}\left(\frac{(1-\alpha)\Omega^{\alpha+1} f(z)}{z}+\alpha\Omega^{\alpha} f(z)\right)\right\}>\rho\cos\lambda\right\}. \tag{9}$$

The classes $R_{\alpha}(\rho)$ and $R_{\alpha}(\lambda,\rho)$ have been studied in [10] and [11] respectively.

Definition 4 (Noonan and Thomas [12]). For the function f given by (1) and $q\in N:=\{1,2,3,...\}$, the q^{th} Hankel determinant of f is defined by

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}. \tag{10}$$

It is well known (Duren [5]) that for $f\in S$ and given by (1), the sharp inequality $|a_3-a_2^2|\le 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. For a given family *f* of functions in *S*, the more general problem of finding sharp estimates for $\left|a_3-\mu a_2^2\right|(\mu\in\mathbb{R}\text{ or }\mu\in\mathbb{C})$ is popularly known as the Fekete-Szegö problem for *f* again as mentioned earlier. In fact, for the general class *S*, we have

Theorem 1 (Fekete and Szegö [6]). *For each* $f \in S$,

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 4\mu - 3, & \text{if } \mu \ge 1 \\ 1 + 2e^{-2\mu/1-\mu}, & \text{if } \mu \le \mu \le 1. \\ 3 - 4\mu, & \text{if } \mu \le 0. \end{cases}$$

The inequality is sharp.

The Fekete-Szegö problem for the families $S, S^*, \mathcal{C}, \mathcal{K}$ has been completely solved by many authors including (Fekete and Szegö [6]) and (Keogh and Merkes [7]; Koepf [8]; Koepf [9]). In fact, many problems related to these problems are discussed by many authors for different types of classes and families.

2. THE FEKETE-SZEGO PROBLEM FOR $R_{\alpha,\beta}(\lambda,\rho)$

To establish our results, we need the following:

Lemma 1 (Duren [5]). *Let the function* $p \in \mathcal{P}$ *and be given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + ..., \quad (z \in U). \tag{11}$$

Then, the sharp estimate $|c_k| \le 2 \ (k \in N)$ *holds.*

Lemma 2 (Ravichandran at al. [16]). *Let the function* $p \in \mathcal{P}$ *be given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + ..., \ (z \in U).$$

Then, for any complex number μ,

$$\left|c_2 - \mu c_1^2\right| \le \max\{1, |2\mu - 1|\}. \tag{12}$$

and the result is sharp for the functions given by $p(z) = \dfrac{1+z^2}{1-z^2}, \ p(z) = \dfrac{1+z}{1-z}$.

Lemma 3 (Duren [5]). *Let* $p \in \mathcal{P}$, *where*

$$p(z) = 1 + c_1 z + c_2 z^2 + ..., \ (z \in U).$$

Then

$$\left|c_2 - \frac{1}{2}\mu c_1^2\right| \le 2 + \frac{1}{2}\left(|\mu - 1| - 1\right)|c_1|^2. \tag{13}$$

2.1. Investigating the problem where $\mu \in \mathbb{C}$

We prove the following:

Theorem 2. *Let the function* f *given by (1) be in the class* $R_{\alpha,\beta}(\lambda,\rho)$ $\left(0 \le \alpha < 1, \ 0 \le \beta < 1, \ -\dfrac{\pi}{2} < \lambda < \dfrac{\pi}{2}, \ 0 \le \rho \le 1\right)$. *Then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)\cos\lambda}{36} \max\left\{1, \left|\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu e^{-i\lambda}\cos\lambda}{2(3-\alpha)(3-\beta)} - 1\right|\right\}.$$

The result is sharp.

Proof. Let $R_{\alpha,\beta}(\lambda,\rho)$ $\left(0 \le \alpha < 1, \ 0 \le \beta < 1, \ -\dfrac{\pi}{2} < \lambda < \dfrac{\pi}{2}, \ 0 \le \rho \le 1\right)$. Then, by (7),

$$e^{i,\lambda} \frac{\theta^{\alpha,\beta} f(z)}{z} = \left[(1-\rho)p(z) + \rho\right]\cos\lambda + i\sin\lambda, \tag{14}$$

where $p \in \mathcal{P}$ and is given by (11). Using (5), we rewrite (14) as

$$e^{i\lambda}\left\{1+\sum_{n=2}^{\infty}\frac{\left(\Gamma(n+1)\right)^{2}\Gamma(2-\alpha)\Gamma(2-\beta)}{\Gamma(n+1-\alpha)\Gamma(n+1-\beta)}a_n z^{n-1}\right\}=\left\{(1-\rho)\left(1+\sum_{n=1}^{\infty}c_n+\rho\right)\right\}\cos\lambda+i\sin\lambda\,. \tag{15}$$

Comparing the coefficient, we get

$$a_2=\frac{(2-\alpha)(2-\beta)(1-\rho)\cos\lambda}{4e^{i\lambda}}c_1 \tag{16}$$

$$a_3=\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)\cos\lambda}{36e^{i\lambda}}c_2\,. \tag{17}$$

Therefore, using (16) and (17), and after simplification we get

$$\left|a_3-\mu a_2^2\right|=\frac{(2-\alpha)(2-\beta)(3-\alpha)(1-\rho)\cos\lambda}{36}\left|c_2-\nu c_1^2\right|,$$

where

$$\nu=\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu e^{-i\lambda}\cos\lambda}{4(3-\alpha)(3-\beta)}\,.$$

Therefore, using Lemma 2, we obtain

$$\left|a_3-\mu a_2^2\right|\le$$

$$\le\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)\cos\lambda}{36}\max\left\{1,\left|\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu e^{-i\lambda}\cos\lambda}{2(3-\alpha)(3-\beta)}-1\right|\right\}, \tag{18}$$

where the equality occurs for the function f given by

$$f(z)=\mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{ze^{-i\lambda}\left[(1-\rho)\frac{1+z}{1-z}\right]\cos\lambda+i\sin\lambda\right\}$$

and also for

$$f(z)=\mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{ze^{-i\lambda}\left[(1-\rho)\frac{1+z^2}{1-z^2}+\rho\right]\cos\lambda+i\sin\lambda\right\}.$$

The proof of Theorem 2 is complete. Many interesting corollaries can be found by choosing suitably values of β and ρ.

2.2. Investigating for the case $\mu\in\mathbb{R}$

For $\lambda=0$ and $\mu\in\mathbb{R}$, we have the following

Theorem 3. *Let the function $f\in A$ be in the class $R_{\alpha,\beta}(\lambda,\rho)$. Then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(\dfrac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{(3-\alpha)(3-\beta)}-2\right) & (\mu\ge\sigma),\\[2ex] \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18} & (0\le\mu\le\sigma),\\[2ex] \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(2-\dfrac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{(3-\alpha)(3-\beta)}\right) & (\mu\le 0)\end{cases} \tag{19}$$

where for convenience,

$$\sigma = \frac{4(3-\alpha)(3-\beta)}{9(2-\alpha)(2-\beta)(1-\rho)}$$

Each of the estimates in (19) is sharp.

Proof. Using (16) and (17), we write

$$\left|a_3 - \mu a_2^2\right| = \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left|\nu c_1^2 - c_2\right|, \tag{20}$$

where

$$\nu = \frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}.$$

Using triangle inequality, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(\left|c_1^2 - c_2\right| + |\nu - 1||c_1|^2\right). \tag{21}$$

If $\mu \ge \sigma$, then $\nu \ge 1$. Thus, applying Lemma 2 and Lemma 3, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{(3-\alpha)(3-\beta)} - 2\right), \tag{22}$$

which is the first part of assertion (19). Equality in (21) or equivalently (22) holds true if and only if $\left|c_2\right| = 2$. Thus the function f is given by

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{z\left[(1-\rho)\frac{1+z}{1-z} + \rho\right]\right\}$$

or one of its rotations for $\mu > \sigma$.

Next, if $\mu \le 0$, we use triangle inequality for (20) to obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(|c_2| - \nu|c_1|^2\right). \tag{23}$$

The estimates $|c_2| \le 2$ and $|c_1| \le 2$, after simplification yield the second part of the assertion (19), in which equality holds if and only if f is a rotation of

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{z\left[(1-\rho)\frac{1+z}{1-z} + \rho\right]\right\}.$$

If $\mu = 0$, then equality holds true if and only if $\left|c_2\right| = 2$. Equivalently, we have

$$p_1(z) = \frac{1+2tz+z^2}{1-z^2} \quad (0 \le t \le 1;\ z \in U).$$

Thus, the equality function f is given by

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{z\left[(1-\rho)\frac{1+2tz+z^2}{1-z^2} + \rho\right]\right\} \quad (0 \le t \le 1;\ z \in U)$$

or one of its rotations.

If $\mu = \sigma$, then $\nu = 1$. Therefore, equality holds true if and only if $\left|c_1^2 - c_2\right| = 2$. This happens if and only if

$$p_1(z)=\frac{1-z^2}{1+2tz+z^2}\quad(0\le t\le 1;\ z\in U).$$

Thus the equality function f is given by

$$f(z)=\mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{z\left[(1-\rho)\frac{1-z^2}{1+2tz+z^2}+\rho\right]\right\}\ (0\le t\le 1;\ z\in U)$$

or one of rotation.

Finally, we see that

$$\left|a_3-\mu a_2^2\right|\le\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left|c_2-c_1^2+(1-\nu)c_1^2\right|.$$

and $\max|1-\nu|\le 1\ (0\le\mu\le\sigma)$. Therefore, using Lemma 3, we get

$$\left|a_3-\mu a_2^2\right|\le\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left\{2-|c_1|^2+|c_1|^2\right\}=$$
$$=\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18}\quad(0<\mu\le\mu).$$

If $\sigma_1<\mu<\sigma_2$, then equality holds true if and only if $|c_1|=0$ and $|c_2|=0$. Equivalently, we have

$$p_1(z)=\frac{1+tz^2}{1-tz^2}\ (0\le t\le 1; z\in U).$$

Thus the equality function f is given by

$$f(z)=\mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{z\left[(1-\rho)\frac{1+tz^2}{1-tz^2}+\rho\right]\right\}\ (0\le t\le 1;\ z\in U)$$

or one of this rotations. The proof of Theorem 3 is evidently complete.

Furthermore, by setting $\beta=0,\ \beta=1,\ \rho=0$ in Theorem 3 respectively, we shall obtain interesting corollaries which can be seen easily from the theorem.

3. IMPROVEMENT OF THE ESTIMATION

The second part of assertion (19) can be improved as follows:

Theorem 4. *Let* $f\in\mathcal{R}_{\alpha,\beta}(\rho)(0\le\sigma<1,\ 0\le\beta<1,\ 0\le\rho\le 1)$. *If* $(\sigma/2\le\mu\le\sigma)$, *then*

$$\left|a_3-\mu a_2^2\right|+\left\{\frac{4(3-\alpha)(3-\beta)}{9(1-\rho)(2-\alpha)(2-\beta)}-\mu\right\}|a_2|^2\le\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18}\tag{24}$$

and if $(0\le\mu\le\sigma/2)$, *then*

$$\left|a_3-\mu a_2^2\right|+\mu|a_2|^2\le\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18}\tag{25}$$

where, for convenience,

$$\sigma=\frac{4(3-\alpha)(3-\beta)}{9(2-\alpha)(2-\beta)(1-\rho)}$$

Proof. For the values of μ prescribed in (24), we have

$$\left|\mu a_2^2-a_3\right|+\left\{\frac{4(3-\alpha)(3-\beta)}{9(1-\rho)(2-\alpha)(2-\beta)}-\mu\right\}\left|a_2\right|^2=$$

$$=\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left\{\left|c_2-\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}c_1^2\right|+\left(1-\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}\right)\left|c_1\right|^2\right\}\leq$$

$$\leq\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left\{2+\left(\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}-1\right)\left|c_1\right|^2+\left(1-\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}\right)\left|c_1\right|^2\right\}=$$

$$=\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18}\quad\left(\frac{\sigma}{2}\leq\mu\leq\sigma\right),$$

which establishes (24). Similarly, for the value of μ prescribed in (25), we have

$$\left|\mu a_2^2-a_3\right|+\mu\left|a_2\right|^2=$$

$$=\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left\{\left|c_2-\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}c_1^2\right|+\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}\left|c_1\right|^2\right\}\leq$$

$$\leq\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left\{2-\left(\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}-1\right)\left|c_1\right|^2+\frac{9(2-\alpha)(2-\beta)(1-\rho)\mu}{4(3-\alpha)(3-\beta)}\left|c_1\right|^2\right\}=$$

$$=\frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18}\quad\left(0\leq\mu\leq\frac{\sigma}{2}\right),$$

which proves (25).

Again, many corollaries shall be obtained by choosing $\beta=0,\ \beta=1,\ \rho=0$ in Theorem 4.

4. COEFFICIENT ESTIMATES FOR THE INVERSE FUNCTIONS OF $f\in\mathcal{R}_{\alpha,\beta}(\rho)$

We first state the following theorem.

Theorem 5. *Let the function f given by (1), be in the class $f\in\mathcal{R}_{\alpha,\beta}(\rho)$. Also let the function f^{-1}, defined by $f^{-1}(f(z))=z=f(f^{-1}(z))$, be the inverse of f if*

$$f^{-1}(w)=w+\sum_{n=2}^{\infty}d_nw^n\quad\left(|w|<r_0;\ r_0>\frac{1}{4}\right). \tag{26}$$

Then

$$\left|d_2\right|\leq\frac{(2-\alpha)(2-\beta)(1-\rho)}{2} \tag{2.7}$$

and

$$\left|d_3\right|\leq\begin{cases}\dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(\dfrac{18(2-\alpha)(2-\beta)(1-\rho)}{(3-\alpha)(3-\beta)}-2\right) & \sigma\leq 2\\[2ex] \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18} & \sigma\geq 2\,.\end{cases} \tag{2.8}$$

Each of the estimates in (27) and (28) is sharp.

Proof. Relation (26) gives

$$d_2 = -a_2, \text{ and } d_3 = 2a_2^2 - a_3$$

Thus, making use of (16) and (17), we get

$$|d_2| = \left| -\frac{(2-\alpha)(2-\beta)(1-\rho)}{4} c_1 \right|$$

Using the estimate $|c_1| \le 2$, we obtain

$$|d_2| \le \frac{(2-\alpha)(2-\beta)(1-\rho)}{2}.$$

Equality occurs for the inverse of

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{ z\left[(1-\rho)\frac{1+z}{1-z} + \rho \right] \right\}$$

By using Theorem 3, we have

$$|d_3| \le \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{36}\left(\frac{18(2-\alpha)(2-\beta)(1-\rho)}{(3-\alpha)(3-\beta)} - 2 \right) \quad \text{for } \sigma \le 2,$$

and

$$|d_3| \le \frac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)(1-\rho)}{18} \quad \text{for } \sigma \ge 2, \text{ where } \sigma = \frac{4(3-\alpha)(3-\beta)}{9(2-\alpha)(2-\beta)(1-\rho)}.$$

Since each estimate of Theorem 3 is sharp, then the estimate of $|d_3|$ is also sharp.

For $\sigma > 2$, the equality occurs for

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{ z\left[(1-\rho)\frac{1+tz^2}{1-tz^2} + \rho \right] \right\} \ (0 < t < 1;\ z \in U).$$

For $\sigma = 2$, the equality occurs for

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{ z\left[(1-\rho)\frac{1+2tz+z^2}{1-z^2} + \rho \right] \right\} \ (0 \le t \le 1;\ z \in U).$$

For $\sigma < 2$ the equality occurs for

$$f(z) = \mathcal{L}(2-\alpha,2)\mathcal{L}(2-\beta,2)\left\{ z\left[(1-\rho)\frac{1+z}{1-z} + \rho \right] \right\} \ (z \in U).$$

The proof of Theorem 5 is evidently complete.

For $\beta = 0,\ \beta = 1,\ \rho = 0$ respectively in Theorem 5, we shall obtain interesting corollaries.

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