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NEW SUFFICIENT CONDITIONS FOR B-PREINVEXITY AND SOME EXTENSIONS

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We give some sufficient conditions for B-preinvexity for locally Lipschitz functions defined on a invex set of a Banach space. Further, a general class of Lipschitz functions of B-preinvexity type is introduced for which some properties and results are given.

Key words: B-vex functions; B-preinvex functions; (B,ρ,d)-preinvex functions, Locally Lipschitz functions.

1. INTRODUCTION

The convexity and generalized convexity are very important in optimization. See for example [1–11, 15–17] and their references. Thus, the convexity was generalized to quasiconvexity, pseudoconvexity [11], invexity [5,7], F-convexity [14], (F,ρ)-convexity [15], B-vexity [1], preinvexity [8,17], B-preinvexity [16], and so on.

In the following we consider the case of locally Lipschitz functions of B-preinvexity type [10,4,12,13]. Thus, we give some general sufficient conditions for B-preinvexity and properties and results for a new class introduced in this paper, the class of locally Lipschitz (B,ρ,d)-preinvex functions. We extend many results of B-vexity type stated in literature, for example [1,2,9,10,16] and their references.

2. PRELIMINARIES

Let f be a locally Lipschitz real-valued function defined on a Banach space X. According to [4], the Clarke generalized directional derivative of f at a point $y \in X$ with respect to a direction d is

$$f^0(y;d) = \limsup_{\substack{x \to y \\ \lambda \downarrow 0}} \frac{f(x+\lambda d) - f(x)}{\lambda}. \tag{1}$$

Thus, for any $y \in X$, the mapping $f^0(y;\cdot): X \to \mathrm{R}$ is finite, positively homogeneous and subadditive [4]. Also, the set

$$\partial f(y) = \{\xi \in X^* : \langle \xi, d \rangle \le f^0(y;d), \forall d \in X\} \tag{2}$$

a subset of the topological dual X^* of X, is the Clarke generalized gradient of the mapping f at y. This set $\partial f(y)$ is nonempty, convex and weak compact, and further

$$f^0(y;d) = \max_{\xi \in \partial f(y)} \langle \xi, d \rangle; \qquad \forall d \in X. \tag{3}$$

According to [4], a locally Lipschitz function f is regular at y if there exists the directional derivative $f'(y;\cdot)$ and $f^0(y;\cdot) = f'(y;\cdot)$, where

$$f'(y;d)=\lim_{\lambda\downarrow 0}\frac{f(y+\lambda d)-f(y)}{\lambda} \tag{4}$$

with $d\in X$.

3. SUFFICIENT CONDITIONS FOR B-PREINVEXITY

In this section, following the ideas of [10], we consider more general sufficient conditions for B-preinvexity locally Lipschitz function $f:D\to R$. Thus, some results are extended [1,10].

Theorem 1. *Let* h *be a mapping from* $D\times D$ *into the set* $(0,\infty)\subset R$. *Also, consider the mappings* $\rho_1:D\times D\to R$ and $\rho_2:D\times D\times D\to R$. *Suppose that for every* $x,y\in D$ and $\lambda\in[0,1]$:

(i_1) $h(y,z)\left[f(y)-f(z)\right]\geq f^0(z;\eta(y,z))+\rho_1(y,z)$;

(i_2) $f^0(z;(1-\lambda)\eta(y,z)+\lambda\eta(x,z))\geq\rho_2(x,y,z)$;

(i_3) $\lambda\rho_1(y,z)+(1-\lambda)\rho_1(x,z)+\rho_2(x,y,z)\geq 0$,

where $z=z(x,y,\lambda)=y+\lambda\eta(x,y)$. *Then* f *is B-preinvex at* y *with respect to* η *and some* b.

Proof. Since D is a η-invex set, we have $y+\lambda\eta(x,y)\in D$ for every $x\in D$ and $\lambda\in[0,1]$. Using (i_1) we have

$$\begin{aligned}&h(y,y+\lambda\eta(x,y))\left[f(y)-f(y+\lambda\eta(x,y))\right]\geq\\&\geq f^0(y+\lambda\eta(x,y);\eta(y,y+\lambda\eta(x,y)))+\rho_1(y,y+\lambda\eta(x,y)),\end{aligned} \tag{5}$$

$$\begin{aligned}&h(x,y+\lambda\eta(x,y))\left[f(x)-f(y+\lambda\eta(x,y))\right]\geq\\&\geq f^0(y+\lambda\eta(x,y);\eta(x,y+\lambda\eta(x,y)))+\rho_1(x,y+\lambda\eta(x,y)).\end{aligned} \tag{6}$$

Multiplying these inequalities by $(1-\lambda)$ and λ, respectively, and then adding the obtained inequalities, we get

$$\begin{aligned}&\lambda h(x,y+\lambda\eta(x,y))\cdot f(x)+(1-\lambda)h(y,y+\lambda\eta(x,y))\cdot f(y)-\\&-\left[\lambda h(x,y+\lambda\eta(x,y))+(1-\lambda)h(y,y+\lambda\eta(x,y))\right]\cdot f(y+\lambda\eta(x,y))\geq\\&\geq\lambda f^0(y+\lambda\eta(x,y);\eta(x,y+\lambda\eta(x,y)))+(1-\lambda)f^0(y+\lambda\eta(x,y);\eta(y,y+\lambda\eta(x,y)))+\\&+\lambda\rho_1(x,y+\lambda\eta(x,y))+(1-\lambda)\rho_1(y,y+\lambda\eta(x,y)).\end{aligned} \tag{7}$$

Using the convexity of the mapping $f^0(y+\lambda\eta(x,y);\cdot)$ in the second argument, we obtain

$$\begin{aligned}&f^0(y+\lambda\eta(x,y);\lambda\eta(x,y+\lambda\eta(x,y))+(1-\lambda)\eta(y,y+\lambda\eta(x,y)))\leq\\&\leq\lambda f^0(y+\lambda\eta(x,y);\eta(x,y+\lambda\eta(x,y)))+(1-\lambda)f^0(y+\lambda\eta(x,y);\eta(y,y+\lambda\eta(x,y))).\end{aligned} \tag{8}$$

Now, by (i_2) we have

$$\begin{aligned}&\lambda f^0(y+\lambda\eta(x,y);\eta(x,y+\lambda\eta(x,y)))+(1-\lambda)f^0(y+\lambda\eta(x,y);\eta(y,y+\lambda\eta(x,y)))\geq\\&\geq\rho_2(x,y,\eta(y,y+\lambda\eta(x,y))).\end{aligned} \tag{9}$$

Hence,

$$\begin{aligned}&\lambda h\left(x,y+\lambda\eta(x,y)\right)\cdot f(x)+(1-\lambda)h\left(y,y+\lambda\eta(x,y)\right)\cdot f(y)-\\&-\left[\lambda h\left(x,y+\lambda\eta(x,y)\right)+(1-\lambda)h\left(y,y+\lambda\eta(x,y)\right)\right]\cdot f\left(y+\lambda\eta(x,y)\right)\geq\\&\geq\lambda\rho_1\left(x,y+\lambda\eta(x,y)\right)+(1-\lambda)\rho_1\left(y,y+\lambda\eta(x,y)\right)+\\&+\lambda\rho_1\left(x,y+\lambda\eta(x,y)\right)+\rho_2\left(x,y,\eta\left(y,y+\lambda\eta(x,y)\right)\right).\end{aligned}\tag{10}$$

From (i_3) and this inequality, we get as in [10] that

$$f\left(y+\lambda\eta(x,y)\right)\leq b(x,y,\lambda)\cdot f(x)+\left(1-b(x,y,\lambda)\right)\cdot f(y),\tag{11}$$

with

$$b(x,y,\lambda)=\frac{\lambda h\left(x,y+\lambda\eta(x,y)\right)}{(1-\lambda)h\left(y,y+\lambda\eta(x,y)\right)+\lambda h\left(x,y+\lambda\eta(x,y)\right)}\tag{12}$$

i.e., f is B-preinvex at y relative to η and b.

Since $f^0(z;d)=\max_{\xi\in\partial f(z)}\langle\xi,d\rangle$, $\forall d\in X$, we see that in this theorem conditions (i_1) and (i_2) are equivalent to

$$h(y,z)\left[f(y)-f(z)\right]\geq\langle\xi,\eta(y,z)\rangle+\rho_1(y,z),\ \forall y\in D,\ \xi\in\partial f(z),\tag{13}$$

and there exists $\overline{\xi}\in\partial f(z)$ with

$$\left\langle\overline{\xi},\lambda\eta(x,z)+(1-\lambda)\eta(y,z)\right\rangle\geq\rho_2(x,y,z),\ \forall x,y\in D,\tag{14}$$

respectively.

Corollary 1. *If in Theorem 1 we assume that* $\rho_1=0$ *and* $\rho_2=0$ *we obtain Theorem 4.1 of [10].*

Also, using the above theorem we obtain a new criteria for B-preinvexity.

Theorem 2. *Let* h, f, ρ_1 *and* ρ_2 *defined as in Theorem 1. Suppose that for every* $x,y\in D$ *and* $\lambda\in[0,1]$,

(j_1) $h(x,y)\left[f(x)-f(y)\right]\geq f^0\left(y;\eta(x,y)\right)+\rho_1(x,y)$;

(j_2) $f^0\left(x+\lambda\eta(y,x);\lambda\eta\left(y,x+\lambda\eta(y,x)\right)+(1-\lambda)\eta\left(x,x+\lambda\eta(y,x)\right)\right)\geq\rho_2\left(y,x,x+\lambda\eta(y,x)\right)$;

(j_3) $\lambda\rho_1\left(x,x+\lambda\eta(y,x)\right)+(1-\lambda)\rho_1\left(y,x+\lambda\eta(y,x)\right)+\rho_2\left(y,x,x+\lambda\eta(y,x)\right)\geq 0$.

Then f *is a B-preinvex function on* D *with respect to* η *and some* b.

Remark 1. Hypotheses (j_1) and (j_2) are equivalent to

$$h(x,y)\left[f(x)-f(y)\right]\geq\langle\xi,\eta(x,y)\rangle+\rho_1(x,y),\ \forall\xi\in\partial f(y),\ \forall x,y\in D\tag{15}$$

and

$$\left\langle\overline{\xi},\lambda\eta\left(y,x+\lambda\eta(y,x)\right)+(1-\lambda)\eta\left(x,x+\lambda\eta(y,x)\right)\right\rangle\geq\rho_2\left(x,y,x+\lambda\eta(y,x)\right),\quad\forall x,y\in D\tag{16}$$

for some $\overline{\xi}\in\partial f\left(x+\lambda\eta(y,x)\right)$.

Remark 2. If $\rho_1=\rho_2$, we get Theorem 4.2 from [10]. Further if $\eta(x,y)=x-y$, according to [10], (j_2) is satisfied and then we have a result of [9] for B-vex functions.

Theorem 3. *Suppose that*

(k_1) $f\left(y+\eta(x,y)\right)\leq f(x)+\rho_{01}(x,y)$, $\forall x,y\in D$;

(k_2) $\langle \xi_1, \eta(x,y)\rangle h(x,y) - \langle \xi_2, \eta(x,y)\rangle h(y,x) \ge \rho_{02}(x,y)$, $\forall x, y \in D$, $\lambda \in (0,1)$, $\xi_1 \in \partial f(y + \lambda\eta(x,y))$, $\xi_2 \in \partial f(y)$;

(k_3) $f^0(x + \lambda\eta(y,x); \lambda\eta(y,x) + (1-\lambda)\eta(x, x + \lambda\eta(y,x))) \ge \rho_2(y, x, \eta(x, x + \lambda\eta(y,x)))$, $\forall x, y \in D$ and $\lambda \in [0,1]$;

(k_4) $\lambda\rho_{01}(x,y) + (1-\lambda)\rho_{02}(y, x + \lambda\eta(y,x)) + \rho_2(y, x, \eta(x, x + \lambda\eta(y,x))) \ge 0$, $\forall x, y \in D$ and $\lambda \in [0,1]$.

Then f is a B-preinvex function on D with respect to η and some b.

Proof. According to the Lebourg's theorem, for $x, y \in D$ there exists $\theta \in (0,1)$ such that

$$f(y + \eta(x,y)) - f(y) \in \langle \partial f(y + \theta\eta(x,y)), \eta(x,y)\rangle. \tag{16}$$

Hence there exists $\bar{\xi} \in \partial f(y + \theta\eta(x,y))$ such that

$$f(y + \eta(x,y)) - f(y) = \langle \bar{\xi}, \eta(x,y)\rangle \tag{18}$$

Now, by (k_1) we have

$$f(x) - f(y + \eta(x,y)) \ge \langle \bar{\xi}, \eta(x,y)\rangle + \rho_{01}(x,y). \tag{19}$$

Using this inequality, (k_2) and the assumption $h > 0$, we obtain

$$\begin{aligned} h(x,y)[f(x) - f(y)] &\ge \langle \bar{\xi}, \eta(x,y)\rangle h(x,y) + \rho_{01}(x,y)h(x,y) \ge \\ &\ge [\langle \xi_2, \eta(x,y)\rangle + \rho_{02}(x,y)] h(y,x) + \rho_{01}(x,y)h(z,y) = \\ &= \langle \xi_2, \eta(x,y)\rangle h(y,x) + \rho_{01}(x,y)h(x,y) + \rho_{02}(x,y)h(y,x). \end{aligned} \tag{20}$$

If we put $h_1(x,y) = \dfrac{h(x,y)}{h(y,x)} > 0$, we get

$$h_1(x,y)[f(x) - f(y)] \ge \langle \xi_2, \eta(x,y)\rangle + \rho_{01}(x,y)h_1(x,y) + \rho_{02}(x,y), \tag{21}$$

for any $\xi_2 \in \partial f(y)$, i.e.,

$$h_1(x,y)[f(x) - f(y)] \ge f^0(y, \eta(x,y)) + \rho_1(x,y), \tag{22}$$

where

$$\rho_1(x,y) = \rho_{01}(x,y)h_1(x,y) + \rho_{02}(x,y). \tag{23}$$

Now, we see that we can apply Theorem 2 with h_1 and ρ_1 defined as above.

Remark 3. In the case $\eta(x,y) + \eta(y,x) = 0$ or $\eta(x,y) = x - y$, the assumptions of the above theorems can be simplified.

4. (B,ρ,d)-PREINVEXITY

Now, we consider a more general class of B-preinvex functions type on a Banach space.

Let ρ be a real function on $D \times D$ and d a nonnegative real function on $D \times D$, where D is a invex set with respect to η.

Definition 1. We say that a real-valued function f defined on D is (B,ρ,d)-preinvex at $y\in D$ with respect to η if, for every $x\in D$ and $\lambda\in[0,1]$,

$$\begin{aligned} f\left(y+\lambda\eta(x,y)\right)\le b(x,y,\lambda)f(x)+\\ +\left[1-b(x,y,\lambda)\right]f(y)+\rho(x,y)b(x,y,\lambda)\cdot\left(1-b(x,y,\lambda)\right)d(x,y,\lambda). \end{aligned} \tag{24}$$

We say that f is (B,ρ,d)-preinvex on D with respect to η if it is (B,ρ,d)-preinvex at each $y\in D$ with respect to the same η.

Note that every (B,ρ,d)-preinvex function with respect to η is B-preinvex with respect to η with $\rho=0$. If $\rho\ge 0$ on $D\times D$, then f is weakly B-preinvex on D and if $\rho\le 0$ on $D\times D$, then f is strong (or approximatively) B-preinvex on D.

Using the classical ideas for B-preinvexity [10, 1, 2] and (F,ρ)-convexity [15] we obtain some interesting properties for this new class of functions.

Let D, f, η, b, ρ and d be defined as above.

Theorem 4. *Let f be a locally Lipschitz real-valued function on D, (B,ρ,d)-preinvex at $y\in D$. Also, assume that for each $x\in D$ and $\theta\in(0,1)$, the set-valued mapping $\lambda\to\partial f\left(y+\lambda\theta\eta(x,y)\right)$, $\lambda\in[0,1]$, is upper semicontinuous. Then there exists $\overline{\xi}\in\partial f(y)$ such that, for any $x\in D$,*

$$\overline{b}(x,y)\left[f(x)-f(y)\right]\ge\left\langle\overline{\xi},\eta(x,y)\right\rangle-\rho(x,y)\overline{b}(x,y)d(x,y), \tag{25}$$

where $\overline{b}(x,y)=\limsup_{\lambda\downarrow 0}\lambda^{-1}b(x,y,\lambda)$.

Remark 4. If f is continuously differentiable on D, then the mapping $\lambda\to\nabla f\left(y+\lambda\theta\eta(x,y)\right)$ is continuous and $\partial f(y)=\{\nabla f(y)\}$. Thus, the conclusion of Theorem 4 follows.

Remark 5. If $X=R^n$ and f is a locally Lipschitz real-valued function on D then, according to [4, Prop. 21.5], the set-valued mapping $\partial f(\cdot)$ is upper semicontinuous. Thus, the mapping $\lambda\to\partial f\left(y+\lambda\theta\eta(x,y)\right)$ is a upper semicontinuous mapping, and then the conclusion of Theorem 4 is valid.

Theorem 5. *Let f be a locally Lipschitz real-valued function on D, (B,ρ,d)-preinvex at $y\in D$. Further, assume that f is regular at y in Clarke's sense. Then, for every $\xi\in\partial f(y)$ and $x\in D$,*

$$\underline{b}(x,y)\left[f(x)-f(y)\right]\ge\left\langle\xi,\eta(x,y)\right\rangle-\rho(x,y)\underline{b}(x,y)d(x,y) \tag{25}$$

where $\underline{b}(x,y)=\liminf_{\lambda\downarrow 0}\lambda^{-1}b(x,y,\lambda)$.

Theorem 6. *Let f be a locally Lipschitz real-valued function on D, (B,ρ,d)-preinvex on D. Also, suppose that for each $x,y\in D$ and $\theta\in(0,1)$ the set-valued mapping $\lambda\to\partial f\left(y+\lambda\theta\eta(x,y)\right)$, $\lambda\in[0,1]$,* is *upper semicontinuous. Then there exists $\overline{\xi_1}\in\partial f(x)$ and $\overline{\xi_2}\in\partial f(y)$ such that*

$$\left\langle\overline{\xi_1},\eta(y,x)\right\rangle\overline{b}(x,y)+\left\langle\overline{\xi_2},\eta(x,y)\right\rangle\overline{b}(y,x)\le\left[\rho(x,y)d(x,y)+\rho(y,x)d(y,x)\right]\overline{b}(x,y)\overline{b}(y,x). \tag{27}$$

Remark 6. If in Theorem 6 we assume that f is also regular in Clarke's sense at x and y, then $\overline{b}$ can be substituted by $\underline{b}$.

Remark 7. As in Remarks 4 and 5, we can consider some special cases which will be omitted.

Remark 8. Relative to this new class of functions, we can establish some similar sufficient conditions for (B,ρ,d)-preinvexity.

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