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FARADAY WAVES IN ONE-DIMENSIONAL BOSE-EINSTEIN CONDENSATES

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Faraday waves in Bose-Einstein condensates are analysed in a one-dimensional setting by variational means. A model that describes the bulk part of a condensate through a Gaussian envelope and incorporates the Faraday wave as a surface wave grafted on the envelope is used to derive a set of equations for Faraday waves in one-dimensional Bose-Einstein condensates.

Key words: Bose-Einstein condensates, Faraday waves; Variational treatment.

1. INTRODUCTION

The theoretical treatment of the nonlinear dynamics exhibited by Bose-Einstein condensates (BECs) has fruitfully benefited from existing developments in distinct fields such as nonlinear mechanics (the stability analysis of the Mathieu equation, for example, being paramount in most studies on the stability of BECs), dynamics of electron plasma waves (that arise in non-magnetized or weakly magnetized plasmas), nonlinear optics and many others [1]. Variational methods for cubic Schrödinger equations, in particular, have been used extensively in nonlinear optics to describe quasi-monochromatic wave trains propagating in a weakly nonlinear dielectric (*e.g.*, a Kerr medium, where the dielectric constant depends on the square of the electric field). In the field of Bose-Einstein condensates variational methods have been used to describe the dynamics of a condensate subjected to a time-depended perturbation. Chief among these methods are those tailored around a set of Gaussian functions that have been used to describe the properties of the bulk of the condensates such as the frequency of the lowest-lying modes, parametric resonances, etc.

Motivated by the experiments of Engels *et al*. [2] and the recent ones of Pollack *et al.* [3], we develop in this brief study a one-dimensional variational treatment of the dynamics of Bose-Einstein condensates that incorporates both *i*) the bulk of the condensate and *ii*) Faraday waves of small amplitude and small wavelength. Following the original prediction of Staliunas *et al.* [4] that the periodic modulation of the nonlinear interaction in a BEC gives rise to Faraday waves, Engels *et al.* were the first to demonstrate experimentally the emergence of Faraday waves using highly-elongated condensates subjected to periodic modulation of the radial confinement. Their drive modulated the effective interaction through the density of the condensate and allowed for observation of robust Faraday waves over a wide range of frequencies. These experimental results have been reinforced theoretically in [5, 6] through analytical calculations and extensive numerical simulations. On a related vein, the experiments of Pollack *et al.* relied on the modulation of the atomic scattering length (via Feshbach resonances) to study the collective excitations of cigar-shaped condensates and paved the way for the observation of Faraday waves in experiments where the nonlinear interaction is modulated directly through the scattering length. For the (relatively small) modulation frequency used by Pollack *et al.* one can easily show that the wavelength of the Faraday wave is too large to be relevant experimentally; however, one expects experimentally observable Faraday waves at frequencies higher by some two orders of magnitude.

It is the purpose of this paper to develop by variational means a set of equations that describe the dynamics of Faraday waves in a dilute, one-dimensional Bose-Einstein condensate. This set of equations allows one to investigate theoretically the interplay between the modulation of the scattering length and that of the strength of the confinement.

2. VARIATIONAL METHOD

We build the variational equations starting from the standard one-dimensional Gross-Pitaevskii Lagrangian density (with $m=1$ and $\hbar=1$)

$$l(x,t)=\frac{\mathrm{i}}{2}\left(\psi\frac{\partial\psi^*}{\partial t}-\psi^*\frac{\partial\psi}{\partial t}\right)+\frac{1}{2}\left|\frac{\partial\psi}{\partial x}\right|^2+V(x,t)|\psi|^2+\frac{g(t)N}{2}|\psi|^4, \tag{1}$$

where $\Psi=\Psi(x,t)$ and $V(x,t)=m/2\cdot\Omega^2(t)x^2$, using the hybrid trial wave-function

$$\psi(x,t)=\frac{\exp\left(-\frac{x^2}{2w^2(t)}+\mathrm{i}x^2\beta(t)\right)\left(1+\mathrm{i}u(t)\cos(kx)\right)}{\sqrt{\frac{1}{2}\pi^{1/2}\left(2+\left(1+\exp\left(-k^2w^2(t)\right)u^2(t)\right)w(t)\right)}}, \tag{2}$$

where u, β, and w are real, time-dependent functions. The wave function (normalized to 1), accounts for the bulk part of the condensates through the usual Gaussian functions [7] and incorporates the Faraday wave as a surface wave. For arbitrary values of k and $u(t)$ the Euler-Lagrange equations are not particularly insightful, but for Faraday waves of small amplitude and small wavelength, that is $u(t)<<1$ and $w(t)k>>1$, the Lagrangian is well approximated by

$$L(t)=\frac{k^2u^2(t)}{4}+\frac{1}{4w^2(t)}+\dot{u}(t)\exp\left(\frac{-k^2w^2(t)}{4}\right)+$$
$$+\frac{w^2(t)\left(\Omega^2(t)+4\beta^2(t)+2\dot{\beta}(t)\right)}{4}+\sqrt{\frac{2}{\pi}}\frac{Ng(t)}{w(t)} \tag{3}$$

and the Euler-Lagrange equations reduce to

$$\dot{w}(t)=2w(t)\beta(t), \tag{4}$$

$$\dot{\beta}(t)=\frac{2+\sqrt{2/\pi}Ng(t)w(t)}{4w^4(t)}-\frac{\Omega^2(t)+4\beta^2(t)}{2}+\frac{\exp\left(-k^2w^2(t)/4\right)k^2\dot{u}(t)}{2}, \tag{5}$$

and

$$u(t)=-2\exp\left(-\frac{k^2w^2(t)}{4}\right)w^2(t)\beta(t), \tag{6}$$

These equations can be further simplified to

$$\dot{w}(t)=2w(t)\beta(t), \tag{7}$$

and

$$\dot{\beta}(t)=\frac{2+\sqrt{2/\pi}Ng(t)w(t)}{4w^4(t)}-\frac{\Omega^2(t)+4\beta^2(t)}{2}+\exp\left(-\frac{k^2w^2(t)}{4}\right)k^4w^4(t)\beta^2(t), \tag{8}$$

which constitute the main result of our paper. Using that $\exp\left(k^2w^2(t)/2\right)>>k^2w^2(t)$ and $k^2w^2(t)>>1$ the previous equations can be further simplified to

$$\ddot{w}(t)+\Omega^2w(t)=\frac{1}{w^3(t)}+\frac{Ng(t)}{\sqrt{2\pi}w^2}+\frac{1}{2}\exp\left(-\frac{k^2w^2}{2}\right)k^4w^3(t)\dot{w}^2(t). \tag{9}$$

The numerical solution of these equations requires the $k(\omega)$ dispersion relation that follows easily from [4], namely

$$k=\sqrt{-\frac{gN}{2L}+\sqrt{\frac{g^2N^2}{4L^2}+\omega^2}}\,, \tag{10}$$

where $g=g(0)$, and $2L$ represents the spatial extent of the condensate (usually calculated within the Thomas-Fermi approximation). Notice, however, that the $k(\omega)$ dispersion relations (see [4–6] for a detailed discussion) are derived using a surface wave of the form $(1+(v(t)+\mathrm{i}u(t))\cos kx)$, *i.e.*, the Faraday wave admits an amplitude that has both a real and an imaginary part, while our surface wave has a purely imaginary amplitude (for reasons of analytical tractability of the underlying integrals of the Lagrangian).

3. CONCLUSIONS

We have derived an effective nonlinear differential equation that describes the dynamics of a Faraday wave in a one-dimensional condensate by means of a variational treatment. Our results can also be extended to the study of Faraday waves in a three-dimensional (cigar-shaped) Bose-Einstein condensate.

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