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# φ – APPROXIMATE BIFLAT AND φ – AMENABLE BANACH ALGEBRAS

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In this paper we introduce and study the concept of a $\varphi$-approximate biflat and $\varphi$-pseudo contractible Banach algebra $A$, where $\varphi$ is a continuous homomorphism on $A$. We show that $A$ is $\varphi$- pseudo contractible if and only if $A$ is $\varphi$- approximate biflat and has a central approximate identity. We also introduce the notion of $\varphi$-amenability of a locally compact group $G$, where $\varphi$ is a continuous homomorphism on $G$. We prove that if the group algebra $L^1(G)$ is $\tilde{\varphi}$-amenable then $G$ is $\varphi$-amenable, where $\tilde{\varphi}$ is the extension of $\varphi$ to $M(G)$. In the case where $\varphi$ is an isomorphism on $G$ it is shown that the converse is also valid. Indeed, we have generalized a well-known result due to B. E. Johnson.

*Key words*: Banach algebra, $\varphi$– biprojective, $\varphi$– amenable.

## 1. INTRODUCTION AND PRELIMINARIES

A Banach algebra $A$ is called *amenable* if for each Banach $A$-module $X$, every bounded derivation from $A$ into the dual $A$-module $X^*$ is an inner derivation. The Banach algebra $A$ is called *biprojective* if there exists a bounded $A$-bimodule map $\theta: A \to A\hat{\otimes}A$ such that $\pi\circ\theta = id_A$, where $\pi$ denots the product morphism from $A\hat{\otimes}A$ into $A$ given by $\pi(a\otimes b) = ab$ for all $a, b \in A$. The notion of a *biprojective* Banach algebra was introduced by Helemskii [9]. Recently, some authors have added a kind of twist to the amenability definition. Given a continuous homomorphism $\varphi$ from $A$ into $A$, they defined and studied $\varphi$*-derivations* and $\varphi$*-amenability* (see [5],[15] and [17]).

Suppose that $A$ is a Banach algebra, let $Hom(A)$ denote the set of all continuous homomorphisms from A into itself. Also let X be a Banach A-bimodule. A linear operator $D: A \to X$ is called a $\varphi$*-derivation* if $D(ab) = D(a)\varphi(b) + \varphi(a)D(b)$ for all $a, b \in A$. A $\varphi$-derivation $D$ is called $\varphi$*-inner derivation* if there is $x \in X$ such that $D(a) = \varphi(a)x - x\varphi(a)$ for all $a \in A$. Let $Z^1_\varphi(A, X)$ denote the set of all continuous $\varphi$-derivations and $N^1_\varphi(A, X)$ be the set of all $\varphi$-inner derivations from $A$ into $X$. The first cohomology group $H^1_\varphi(A, X)$ is defined to be the quotient space $Z^1_\varphi(A, X)/N^1_\varphi(A, X)$. A Banach algebra $A$ is called $\varphi$*-amenable* if $H^1_\varphi(A, X^*) = \{0\}$ for all $A$-bimodules $X$. Note that every derivation of a Banach algebra $A$ into an $A$-bimodule $X$ is an $id_A$-derivation, where $id_A$ is the identity operator on $A$. Let $G$ be a locally compact group and $M(G)$ be the Banach space of complex-valued, regular Borel measure on $G$. Recall that the space $M(G)$ is a unital Banach algebra with the convolution multiplication and $L^1(G)$, the group algebra on $G$, is a closed ideal in $M(G)$. We write $\delta_g$ for the point mass at $g \in G$, the element $\delta_e$ is the identity of $M(G)$.

The aim of the present paper is to introduce and investigate $\varphi-$approximate biflat Banach algebras $A$ with $\varphi\in Hom(A)$. In particular, we prove that if the group algebra $L^1(G)$ is $\tilde{\varphi}$-amenable then $G$ is $\varphi$-amenable.

In the case where $\varphi$ is an isomorphism on $G$ the converse is also valid.

## 2. THE RESULTS

We start this section by introducing the following:

**Definition 2.1.** Let $A$ be a Banach algebra and $\varphi\in Hom(A)$. We say that $A$ is $\varphi-$ pseudo amenable if $A$ admit a $\varphi-$ approximate virtual diagonal, i.e., there is a net $(m_\alpha)\subset A\hat{\otimes}A$ (not necessary bounded), such that $m_\alpha\cdot\varphi(a)-\varphi(a)\cdot m_\alpha\to 0$ and $\pi(m_\alpha)\cdot\varphi(a)\to\varphi(a)$ where $\pi$ denots the product homomorphism from $A\hat{\otimes}A$ into $A$ given by $\pi(a\otimes b)=ab$ for all $a,b\in A$. $(a\in A)$.

**Definition 2.2** Let $A$ be a Banach algebra and $\varphi\in Hom(A)$. We say that $A$ is $\varphi-$ approximate biflat if there is a net $\theta_\alpha : A\to(A\hat{\otimes}A)^{**}$ $(\alpha\in I)$ of bounded $A$-bimodule morphisms such that $\pi^{**}\circ\theta_\alpha\circ\varphi(a)\to\varphi(a)$.

**THEOREM 2.3.** *Let $A$ be a Banach algebra with an approximate identity. Then $A$ is $\varphi$- pseudo amenable ( $\varphi\in Hom(A)$ ) if and only if $A$ is $\varphi$- approximate biflat.*

*Proof.* Let $(e_\beta)_{\beta\in I}$ be an approximate identity for $A$ and suppose that $\theta_\alpha : A\to(A\hat{\otimes}A)^{**}$ $(\alpha\in\Delta)$ satisfies $\pi^{**}\circ\theta_\alpha\circ\varphi(a)\to\varphi(a)$ $(a\in A)$. Then for every $a\in A$ and $f\in(A\hat{\otimes}A)^*$

$$\lim_\beta\lim_\alpha\langle f,\theta_\alpha(\varphi(e_\beta))\cdot\varphi(a)-\varphi(a)\cdot\theta_\alpha(\varphi(e_\beta))\rangle=\lim_\beta\lim_\alpha\langle f,\theta_\alpha\left(\varphi(e_\beta)\varphi(a)-\varphi(a)\varphi(e_\beta)\right)\rangle=0$$

Also, for $a\in A$ and $\psi\in A^*$,

$$\lim_\beta\lim_\alpha\langle\psi,\varphi(a)\cdot\pi^{**}\circ\theta_\alpha(\varphi(e_\beta))\rangle=\lim_\beta\langle\psi,\varphi(a)e_\beta\rangle$$
$$=\varphi(a).$$

Let $E=I\times\Delta^I$ be directed by the product ordering and for each $\lambda=(\beta,\alpha)\in E$, define $m_\lambda=\theta_\alpha(\varphi(e_\beta))$. Using the iterated limit theorem [6, Theorem 2.4], the above calculation gives

$$w^*-\lim_\lambda\left(m_\lambda\cdot\varphi(a)-\varphi(a)\cdot m_\lambda\right)=0\quad(a\in A),$$

and

$$w^*-\lim_\lambda\varphi(a)\cdot\pi^{**}(m_\lambda)=\varphi(a)\quad(a\in A).$$

By Goldestine's theorem we can assume that $(m_\lambda)\subset A\hat{\otimes}A$ and we can replace weak$^*$ convergence in equations by weak convergence. Applying Mazur's theorem, we then obtain a net $(m'_\lambda)\subset A\hat{\otimes}A$ of convex combinations of $(m_\lambda)$ such that

$$m'_\lambda\cdot\varphi(a)-\varphi(a)\cdot m'_\lambda\to 0,$$

and

$$\varphi(a)\cdot\pi^{**}(m'_\lambda)\to\varphi(a)\quad(a\in A).$$

That is $A$ is $\varphi-$ pseudo amenable. Conversely, let $(m_\beta)$ be a $\varphi-$approximate virtual diagonal for $A$ and define $\theta_\beta : A\to(A\hat{\otimes}A)^{**}$ by $a\mapsto a\cdot m_\beta$. Then for every $a\in A$ we have

$$\pi^{**}\circ\theta_\beta\circ\varphi(a)=\pi^{**}\circ(\varphi(a)\cdot m_\beta)=\varphi(a)\pi^{**}(m_\beta)\to\varphi(a).$$

**PROPOSITION 2.4.** *Let $A$ be a $\varphi-$amenable Banach algebra and $\varphi$ be an idempotent homomorphism on $A$.*

*Then $A$ is $\varphi$-approximate biflat.*

*Proof.* By [8, Proposition 4.1] $A$ has a bounded approximate identity $(e_\alpha)_{\alpha\in I}$. Let $E$ be a $w^*-$cluster point of $(\varphi(e_\alpha)\otimes\varphi(e_\alpha))_\alpha$ in $(A\hat{\otimes}A)^{**}$. We define a $\varphi-$ derivation $D\colon A\to(A\hat{\otimes}A)^{**}$ by $D(a)=\varphi(a)\cdot E-E\cdot\varphi(a)$ $(a\in A)$. Then for every $a\in A$ $\pi^{**}(D(a))=0$ . Therefore $D(A)\subseteq ker(\pi^{**})=$ $=(ker\pi)^{**}$. Thus there exists $N\in(ker\pi)^{**}$ such that $D=ad_{\varphi;N}$. Put $M=E-N$. Then for every $a\in A$

$$\pi^{**}(M)\cdot\varphi(a)=\varphi(a).$$

Let $(m_\alpha)_\alpha$ be a net in $A\hat{\otimes}A$ such that $M=w^*-\lim_\alpha m_\alpha$. Then $w-\lim_\alpha\left(m_\alpha\cdot\varphi(a)-\varphi(a)\cdot m_\alpha\right)=0$ and $w-\lim_\alpha(\pi(m_\alpha)\cdot\varphi(a)-\varphi(a))=0$ $(a\in A)$. Following the argument given in the proof of [2, Lemma 2.9.64] we can show that there exists a net $(m_\beta)_\beta$ in $A\hat{\otimes}A$ such that each $m_\beta$ is a convex combination of $m_\alpha$'s with $m_\beta\cdot\varphi(a)-\varphi(a)\cdot m_\beta\to0$ and $\pi(m_\beta)\cdot\varphi(a)\to\varphi(a)$ $(a\in A)$. Thus $A$ is $\varphi-$ pseudo amenable and so by Theorem 2.3 $A$ is $\varphi$-approximate biflat.

From the proof of the above proposition we obtain the following corollaries.

**COROLLARY 2.5** *Let $A$ be a $\varphi-$amenable Banach algebra $(\varphi\in Hom(A))$ with a bounded approximate identity. Then $A$ is $\varphi$-approximate biflat.*

**COROLLARY 2.6** *Let $L^1(G)$ be a $\varphi-$amenable Banach algebra $(\varphi\in Hom(L^1(G)))$. Then $L^1(G)$ is $\varphi$-approximate biflat.*

**Definition 2.7** Let $A$ be a Banach algebra with the norm $\|.\|_A$ . Then a Banach algebra $B$ with the norm $\|.\|_B$ is said to be an abstract Segal algebra with respect to A if:

(i) $B$ is a dense left ideal in $A$;

(ii) there exists $M>0$ such that $\|b\|_A\le M\|b\|_A$ for all $b\in B$;

(iii) there exists $C>0$ such that $\|ab\|_B\le C\|a\|_A\ \|b\|_B$ for all $a,\ b\in B$.

**THEOREM 2.8.** *Let A be a Banach algebra and $B$ be an abstract Segal algebra with respect to $A$. Suppose that $\varphi\in Hom(A)$ and $B$ contains a net $(e_\alpha)_\alpha$ such that $(e_\alpha)^2$ is an approximate identity for $B$ and $a\varphi(e_\alpha)=\varphi(e_\alpha)a$ for all $a\in A$. If $\varphi(B)\subseteq B$ and $A$ is $\varphi$-approximate biflat then $B$ is $\varphi$-approximate biflat.*

*Proof.* Let $T_\alpha: A\hat{\otimes}A\to B\hat{\otimes}B$ be defined by $a\otimes b\mapsto a\varphi(e_\alpha)\otimes b\varphi(e_\alpha)$. Since $A$ is $\varphi$-approximate biflat there is a net $(\theta_\beta)_\beta$ with $\theta_\beta: A\to(A\hat{\otimes}A)^{**}$ such that $\pi_A^{**}\circ\theta_\beta\circ\varphi(a)\to\varphi(a)$ $(a\in A)$. For $\lambda=(\alpha,\gamma)$ define $\theta_\lambda: B\to(B\hat{\otimes}B)^{**}$ by $\theta_\lambda:=T_\alpha^{**}\circ\theta_\beta\circ j$ where $j:B\to A$ is the inclusion map. Then $\theta_\lambda$ is a bounded $B-$bimodul map. Note that because $\varphi(e_\alpha)$ lies in the center of $A$, $\pi_B^{**}\circ T_\alpha^{**}=R_\alpha^{**}\circ\pi_A^{**}$, where $R_\alpha: A\to B$ is defined by $a\mapsto a\varphi(e_\alpha^2)$ $(a\in A)$. Let $b\in B,\ f\in B^*$. By the iterated limit theorem we have

$$\begin{aligned}\lim_\lambda\langle f,\pi_B^{**}\circ\theta_\lambda\circ\varphi(b)\rangle &=\lim_\alpha\lim_\beta\langle f,\pi_B^{**}\circ T_\alpha^{**}\circ\theta_\beta\circ j\circ\varphi(b)\rangle\\ &=\lim_\alpha\lim_\beta\langle f,R_\alpha^{**}\circ\pi_A^{**}\circ\theta_\beta(\varphi(b))\rangle\\ &=\lim_\alpha\langle f,R_\alpha^{**}(\varphi(b))\rangle\\ &=\lim_\alpha\langle f,\varphi(b)\varphi(e_\alpha^2)\rangle\\ &=\varphi(b).\end{aligned}$$

Hence $\pi_B^{**} \circ \theta_\lambda \circ \varphi(b) \to \varphi(b)$ $(b \in B)$.

**Definition 2.9.** Let $A$ be a Banach algebra and $\varphi \in Hom(A)$. We say that $A$ is $\varphi-$ pseudo contractible if it has a central $\varphi$-approximate diagonal, i.e., a $\varphi$-approximate diagonal $(m_\alpha)$ satisfying $\varphi(a)m_\alpha = m_\alpha \varphi(a)$ for all $a \in A$ and all $\alpha$.

**PROPOSITION 2.10.** *For a Banach algebra $A$ the following two statements are equivalent.*
i) *$A$ is $\varphi-$ pseudo contractible.*
ii) *$A$ is $\varphi-$ approximate biflat and has a central approximate identity.*

*Proof.* (i)$\Rightarrow$ (ii) Suppose that $(m_\alpha) \subset A \hat{\otimes} A$ is a central $\varphi$-approximate diagonal for $A$. Define $\theta_\alpha : A \to (A \hat{\otimes} A)^{**}$ by $\theta_\alpha(a) := a \cdot m_\alpha$. Then for every $a \in A$ we have

$$\begin{aligned}\lim_\alpha \pi^{**} \circ \theta_\alpha \circ \varphi(a) &= \lim_\alpha \pi^{**}(\varphi(a) \cdot m_\alpha) \\ &= \varphi(a).\end{aligned}$$

So $\pi(m_\alpha)$ is a central approximate identity for $A$.

(ii)$\Rightarrow$ (i) Since $A$ is $\varphi-$ approximate biflat, there is a net $\theta_\alpha : A \to (A \hat{\otimes} A)^{**}$ $(\alpha \in \Delta)$ such that $\lim_\alpha \pi^{**} \circ \theta_\alpha \circ \varphi(a) = \varphi(a)$ $(a \in A)$. Let $(e_\beta)_{\beta \in I}$ be a central approximate identity for $A$. Let $E = I \times \Delta^I$ be directed by the product ordering and for each $\lambda = (\beta, \alpha) \in E$ define $m_\lambda = \theta_\alpha(e_\beta)$. Then $(m_\lambda)$ is a central $\varphi$-approximate diagonal for $A$.

**Definition 2.11.** Let $A$ be a Banach algebra and $\varphi \in Hom(A)$. We say that $A$ is $\varphi-$ biprojective if there exists a bounded $A$-bimodule map $\theta : A \to A \hat{\otimes} A$ such that $\pi \circ \theta \circ \varphi = id_A$ where $id_A$ is the identity operator on $A$.

*Remark 2.12* (i) Let $A$ be a biprojective Banach algebra. Then $A$ is $id_A -$ biprojective.

(ii) Let $A$ be a $\varphi -$biprojective Banach algebra. Then $A^\#$ is $\varphi^\#$ $(\varphi^\# : A^\# \to A^\#)$ biprojective Banach algebra with $\varphi^\# |_A = \varphi$.

**PROPOSITION 2.12.** *Let $A$ be a $\varphi -$biprojective Banach algebra, and let $B$ be a $\psi -$biprojective Banach algebra with $\varphi \in Hom(A)$ and $\psi \in Hom(B)$. Then $A \hat{\otimes} B$ is $\varphi \otimes \psi$ biprojective.*

*Proof.* There exist an $A$-bimodule map $\theta_1 : A \to A \hat{\otimes} A$ with $\pi \circ \theta_1 \circ \varphi = id_A$ and $B$-bimodule map $\theta_2 : B \to (B \hat{\otimes} B)$ with $\pi \circ \theta_2 \circ \varphi = id_B$. Let $\theta_0 : (A \hat{\otimes} A) \hat{\otimes} (B \hat{\otimes} B) \to (A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B)$ be the isometric isomorphism given by $(a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)$ $(a_1, a_2 \in A,\ b_1, b_2 \in B)$. We let $\theta = \theta_0 \circ (\theta_1 \otimes \theta_2) : A \hat{\otimes} A \to (A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B)$. Then for $a \otimes b \in A \otimes B$ we have

$$\begin{aligned}\pi \circ \theta \circ (\varphi \otimes \psi)(a \otimes b) &= \pi \circ \theta_0 \circ (\theta_1 \otimes \theta_2) \circ (\varphi \otimes \psi)(a \otimes b) = \\ &= \pi \circ \theta_0 \circ (\theta_1 \otimes \theta_2)(\varphi(a) \otimes \psi(b)) = \\ &= \pi \circ \theta_0(\theta_1(\varphi(a)) \otimes \theta_2(\psi(b))) = \\ &= \pi \circ \theta_1 \circ \varphi(a) \otimes \pi \circ \theta_2 \circ (\psi(b)) = a \otimes b.\end{aligned}$$

Therefore $A \hat{\otimes} B$ is $\varphi \otimes \psi$ biprojective.
The proof of the following result is similar to that of Proposition 2.13.

**PROPOSITION 2.13.** *Let $A$ be a $\varphi$-biprojective Banach algebra, and let $B$ be a $\psi$-biprojective Banach algebra with $\varphi \in Hom(A)$ and $\psi \in Hom(B)$. Then $A \oplus B$ is $\varphi \oplus \psi$ biprojective.*

## 3. φ – AMENABLE BANACH ALGEBRAS

**PROPOSITION 3.1.** *Let $A$ be a Banach algebra with a bounded approximate identity which is a closed ideal of a Banach algebra $B$. Let $E$ be a pseudo-unital Banach $A$-bimodule, and $\tilde{\varphi}: B \to B$ be a continuous homomorphism such that $\varphi := \tilde{\varphi}|_A$ and $\varphi$ be a continuous epimorphism on $A$. Let $D \in Z^1_{\varphi}(A, E^*)$, then $E$ is a Banach $B$-bimodule in a canonical fashion, and there is a unique $\tilde{D} \in Z^1_{\tilde{\varphi}}(B, E^*)$ for which the following are valid.*

(i) $\tilde{D}|_A = D$;

(ii) $\tilde{D}$ is continuous with respect to the strict topology on $B$ and the $w^*$-topology on $E^*$.

*Proof.* For $x \in E$, let $\varphi(a) \in A$ and $y \in E$ be such that $x = \varphi(a) \cdot y$. For $b \in B$, define $b \cdot x := \tilde{\varphi}(ba) \cdot y$. We claim that $b \cdot x$ is well defined, i.e. independent of the choices $a$ and $y$. Let $\varphi(a_0) \in A$ and $y_0 \in E$ be such that $x = \varphi(a_0) \cdot y_0$, and let $(e_\alpha)_\alpha$ be a bounded approximate identity for $A$. Then

$$\tilde{\varphi}(ba) \cdot y = \tilde{\varphi}(b)\varphi(a) \cdot y = \lim_\alpha \tilde{\varphi}(b) e_\alpha \varphi(a) \cdot y = \lim_\alpha \tilde{\varphi}(b) e_\alpha \cdot x = \tilde{\varphi}(ba_0) \cdot y_0.$$

It is obvious that this operation of $B$ on $E$ makes $E$ into a left Banach $B$-module. Similarly, one defines a right Banach $B$-module structure on $E$, so that $E$ becomes a Banach $B$-bimodule. Now we define $\tilde{D}: B \to E^*$ by $b \mapsto w^* - \lim_\alpha D(be_\alpha)$. We claim that $\tilde{D}$ is well-defined, i.e., the $w^* - \lim_\alpha D(be_\alpha)$ does exist. Let $x \in E$, and let $\varphi(a) \in A$ and $y \in E$ be such that $x = y \cdot \varphi(a)$. Then

$$\langle D(be_\alpha), x\rangle = \langle D(be_\alpha), y \cdot \varphi(a)\rangle = \langle D(be_\alpha) \cdot \varphi(a), y\rangle = \langle D(be_\alpha a), y\rangle - \langle D(a), y \cdot \varphi(be_\alpha)\rangle \to$$
$$\to \langle D(ba), y\rangle - \langle D(a), y \cdot \tilde{\varphi}(b)\rangle \quad (b \in B).$$

So the $w^* - \lim_\alpha D(be_\alpha)$ exists. Moreover, for every $a \in A$,

$$\tilde{D}(a) = w^* - \lim_\alpha D(ae_\alpha) = D(a).$$

That is $\tilde{D}$ extends $D$. For every $b \in B$ and $a \in A$ we have

$$D(ba) = \tilde{\varphi}(b) \cdot Da + \tilde{D}(b) \cdot \varphi(a).$$

It is clear that for every $b \in B$, $\tilde{\varphi}(be_\alpha) \to \tilde{\varphi}(b)$ strictly. Let $b, c \in B$, Then

$$\tilde{D}(bc) = \tilde{\varphi}(b) \cdot \tilde{D}(c) + \tilde{D}(b) \cdot \tilde{\varphi}(c).$$

So $\tilde{D}$ is a derivation. Finally, for every $b \in B$ and $a \in A$

$$(\tilde{D}b) \cdot \varphi(a) = w^* - \lim_\alpha (D(be_\alpha) \cdot \varphi(a) = w^* - \lim_\alpha (D(be_\alpha a) - \varphi(be_\alpha) \cdot D(a) = D(ba) - \tilde{\varphi}(b) \cdot Da.$$

It follows that $\tilde{D}$ is continuous with respect to the strict topology on $B$ and the $w^*$-topology on $E^*$. To see this, let $(b_\alpha) \subset B$ such that $s - \lim b_\alpha = b$ (strict-limit). Then for every $a \in A$, $\varepsilon > 0$ there exists $\beta$ such that for every $\alpha \geq \beta$,

$$\|a(b_\alpha - b)\| + \|(b_\alpha - b)a\| < \varepsilon.$$

So $b_\alpha a \to ba$ $(a \in A)$. For $x \in E$, let $\varphi(a) \in A$ and $y \in E$ be such that $x = \varphi(a) \cdot y$. Then

$$(\tilde{D}b_\alpha)(x) = (\tilde{D}b_\alpha)(\varphi(a) \cdot y) = D(b_\alpha a)(y) - \tilde{\varphi}(b_\alpha) \cdot Da(y) \to$$
$$\to D(ba)(y) - \tilde{\varphi}(b) \cdot Da(y) = \tilde{D}(b)(\varphi(a) \cdot y) = \tilde{D}(b)(x).$$

Before turning our next result, we note that if $G$ is a locally compact group, and $\varphi$ is a continuous homomorphism on $G$ and $\tilde{\varphi}: M(G) \to M(G)$ is defined by $\langle \tilde{\varphi}(\mu), f\rangle = \int f \circ \varphi \, d\mu$ $\left(f \in C_0(G)\right)$ then $\tilde{\varphi}|_G = \varphi$ and $\tilde{\varphi}(L^1(G)) \subseteq L^1(G)$.

**Definition 3.2.** Let $G$ be a locally compact group, and $\varphi$ be a continuous homomorphism on $G$, and let $E$ be a subspace of $L^\infty(G)$ containing the constant functions. A $\varphi$-mean on $E$ is a functional $m \in E^*$ such that $(\tilde{\varphi}^{**}m)(1) = 1$.

**Definition 3.3.** A locally compact group $G$ is called $\varphi$-amenable if there is a $\varphi$-mean on $L^\infty(G)$ that is left invariant, i.e., for all $g \in G$ and $\sigma \in L^\infty(G)$ we have $m(\delta_g * \sigma) = m(\sigma)$ . (Note that the latter equation makes sense since $\delta_g * \sigma \in L^\infty(G)$).

**THEOREM 3.4.** *Let $G$ be a locally compact group and $\varphi$ be a homomorphism on $G$, if $L^1(G)$ is $\tilde{\varphi}$-amenable then $G$ is $\varphi$-amenable.*

*Proof.* Define $L^1(G)$-bimodule actions on $L^\infty(G)$ by

$$f \cdot \psi := \int_G f(\varphi(g))\psi(g)\mathrm{d}m_G(g) \text{ and } \psi \cdot f := \left(\int_G f(\varphi(g))\mathrm{d}m_G(g)\right)\psi \quad \left(f \in L^1(G), \psi \in L^\infty(G)\right).$$

Choose $n \in L^\infty(G)^*$ with $n(1) = 1$, and define $D : L^1(G) \to L^\infty(G)^*$ by $f \mapsto \tilde{\varphi}(f)\cdot n - n \cdot \tilde{\varphi}(f)$. Then $D(f)(1) = 0$. Let $E := L^\infty(G)/\mathrm{C}1$, then $D(L^1(G)) \subset E^*$. Since $L^1(G)$ is $\tilde{\varphi}$-amenable, there is $\tilde{n} \in E^*$ such that $D = ad_{\tilde{n}}$. Let $m := n - \tilde{n}$, and $p_\varphi(G) := \{f \in L^1(G),\ f \ge 0,\ \int_G f \cdot \varphi \mathrm{d}m_G = 1\}$. For every $\psi \in L^\infty(G)$ and $f \in p_\varphi(G)$, we have $m(f * \psi) == m(\psi)$. It is clear that if $f \in p_\varphi(G)$ then $f * \delta_g \in p_\varphi(G)$. We conclude that $m(\delta_g * \sigma) = m(f * \delta_g * \sigma) = m(\sigma)$ for $f \in p_\varphi(G),\ \sigma \in L^\infty(G)$. Thus $G$ is $\varphi$-amenable.

**THEOREM 3.5.** *Let $G$ be a locally compact group and $\varphi$ be an isomorphism on $G$. If $G$ is $\varphi$-amenable then, $L^1(G)$ is $\tilde{\varphi}$-amenable.*

*Proof.* Let $X$ be a Banach $L^1(G)$-bimodule. By [8, Proposition 4.5] there is no loss of generality if we suppose that $X$ is pseudo-unital. Let $D \in Z^1_{\tilde{\varphi}}(L^1(G), X^*)$, and by Proposition 3.1, let $\tilde{D} \in Z^1_{\tilde{\varphi}}(M(G), X^*)$ be the extension of $D$. For every $x \in X$ we define $\psi_x : G \to C$ by $g \mapsto (\delta_{\varphi(g)}\tilde{D}\delta_{g^{-1}})(x)$. Let $m$ be a $\varphi$- mean on $L^\infty(G)$ and let the functional $F$ be defined on $X$ by $x \mapsto (\tilde{\varphi}^{**}m)(\psi_x)$. It is obvious the $F$ is bounded. We prove that $\tilde{D} = ad_F$. To see this we first prove that

$$\tilde{D}\delta_g = \delta_g \cdot F - F \cdot \delta_g \quad (g \in G) \qquad (*).$$

Let $x \in X$, put $z = x \cdot \delta_{\varphi(g)} - \delta_{\varphi(g)} \cdot x$. For $h \in G$ we have

$$\psi_Z(h) = (\delta_{\varphi(h)}\tilde{D}\delta_{h^{-1}})(z) = \delta_{\varphi(h)}\tilde{D}\delta_{h^{-1}}(x \cdot \delta_{\varphi(g)} - \delta_{\varphi(g)} \cdot x).$$

Since $\tilde{D}(\delta_{h^{-1}}\delta_g) = \tilde{D}(\delta_{h^{-1}}) \cdot \delta_{\varphi(g)} + \delta_{\varphi(h^{-1})} \cdot \tilde{D}(\delta_g)$, it follows that

$$-(\delta_{\varphi(h)}\tilde{D}\delta_{h^{-1}})(\delta_{\varphi(g)}) = \delta_h\delta_{h^{-1}}\tilde{D}\delta_g - \delta_{\varphi(h)}\tilde{D}(\delta_{h^{-1}}\delta_g) =$$

$$= \tilde{D}(\delta_g) - \delta_{\varphi(g)}(\delta_{\varphi(g^{-1})}\delta_{\varphi(h)}\tilde{D}(\delta_{g^{-1}}\delta_h)^{-1}).$$

Taking $y = x \cdot \delta_{\varphi(g)}$, we obtain

$$\psi_Z(h) = (\delta_{\varphi(h)} + \tilde{D}\delta_{h^{-1}})(y)\tilde{D}\delta_g(x) - (\delta_{\varphi(g^{-1}h)}\tilde{D}(\delta_{g^{-1}}\delta_h)^{-1})(y) =$$

$$= \psi_y(h) - \psi_y(g^{-1}h) + (\tilde{D}\delta_g)x.$$

So we have $\psi_Z = \psi_y - \delta_g * \psi_y + \tilde{D}\delta_g(x)$. Thus

$$m(\psi_Z) = m(\psi_y) - m(\delta_g * \psi_y) + m(\tilde{D}\delta_g)(x) = m(\psi_y) - m(\psi_y) + m(\tilde{D}\delta_g)(x) =$$
$$= (\tilde{D}\delta_g)(x)m(1),$$

and

$$(\tilde{\varphi}^{**}m)(\psi_Z) = \tilde{D}\delta_g(x)(\tilde{\varphi}^{**}m)(1) = \tilde{D}\delta_g(x).$$

Thus $\tilde{D}\delta_g(x) = F(z)$. So

$$(\tilde{D}\delta_g)(x) == (\delta_g \cdot F - F \cdot \delta_g)(x).$$

We now prove that $\tilde{\varphi}$ is an epimorphism on $L^1(G)$. Let $\mu \in L^1(G)$, define $\mu_\varphi(B) = \mu(\varphi^{-1}(B))$ ($B$ is borel set on $G$). It is clear that $\tilde{\varphi}(\mu_\varphi) = \mu$.

Since every measure $\mu$ in $M(G)$ is the s-lim (strict-lim) of a net $(\mu_i)$ such that each $\mu_i$ a linear combination of point masses [9, Exercise A.2.4 ], then by (*) we have

$$\tilde{D}(\mu) = \mu \cdot F - \mu \cdot F \quad (\mu \in M(G)),$$

as required.

**PROPOSITION 3.6.** *Let $A$ be a Banach algebra and $B$ an abstract Segal algebra with respect to $A$, and suppose that $\varphi : B \to B$ is an idempotent homomorphism such that $\overline{\varphi(B)} = B$. If $B$ is $\varphi$-amenable, then so is $A$.*

*Proof.* By [8, Proposition 4.1] $B$ has a bounded approximate identity $(e_\alpha)_{\alpha \in I}$, let $M = \sup\{\|e_\alpha\| : \alpha \in I\}$. Since $B$ is an abstract Segal algebra, there exists $C > 0$ such that

$$\|ab\|_B \le C\|a\|_A \; \|b\|_B$$

for all $a, b \in B$. So, for each $b \in B$,

$$\|b\|_B = \lim_\alpha \|be_\alpha\|_B \le C\|b\|_A \, M.$$

Thus the norms $\|.\|_A$ and $\|.\|_B$ are equivalent on $B$. Since $B$ is dense in $A$, it follows that $A = B$. Hence $A$ is $\varphi$-amenable. □

**COROLLARY 3.7.** *Let $S^1(G)$ be a Segal algebra on $G$ and $\varphi : S^1(G) \to S^1(G)$ be an idempotent homomorphism such that $\overline{\varphi(S^1(G))} = S^1(G)$. If $S^1(G)$ is $\varphi$-amenable, then so is $L^1(G)$.*

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