THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 13, Number 1/2012, pp. 11–18

SUFFICIENT OPTIMALITY CONDITIONS IN PROBLEMS WITH $(h,\varphi)-(p,r)$-INVEX FUNCTIONS

Karen PETROSYAN

Faculty of Mathematics and Informatics, University of Bucharest, Bucharest, Romania
E-mail: kemrik@gmail.com

The notion of p-invex sets with respect to η is extended for generalized algebraic operations of Ben-Tal to $h-p$-invex sets. A class of real valued functions, called $(h,\varphi)-(p,r)$-pre-invex functions (without differentiability) or $(h,\varphi)-(p,r)$ -invex functions with respect to η (in the differentiable case) is introduced. Sufficient optimality conditions are obtained for a nonlinear programming problem involving $(h,\varphi)-(p,r)$-invex functions with respect to η.

Key words: $h-p$-invex set with respect to η, $(h,\varphi)-(p,r)$-pre-invex functions with respect to η, $(h,\varphi)-(p,r)$-invex functions with respect to η.

1. INTRODUCTION

In this article we introduce $h-p$-invex sets with respect to a function $\eta: X\times X\to R^n$. Based on this definition, we define new classes of (nonconvex) functions which we call $(h,\varphi)-(p,r)$-pre-invex functions with respect to η, and their equivalents in the case of (h,φ)-differentiability – $(h,\varphi)-(p,r)$-invex functions with respect to η. The $(h,\varphi)-(p,r)$-pre-invex functions are a generalization of the (p,r)-pre-invex functions if we take $h(x)\equiv x$, and $\varphi(\alpha)\equiv\alpha$, which, are a generalization of pre-invex functions with respect to η introduced by Ben-Israel and Mond [2], whereas the $(h,\varphi)-(p,r)$-invex functions are a generalization of (p,r)-invex functions, which contain invex functions with respect to η introduced by Hanson [3].

The next part of considerations is developed for optimality conditions in nondifferentiable and differentiable mathematical programming problems. The sufficient optimality conditions are obtained in (h,φ)-mathematical programming problems with inequality constraints in which the functions occurring belong to the class of functions introduced in this article. The results obtained here extend those well-known so far; see Hanson [3].

2. PRELIMINARIES

DEFINITION 2.1. Let $\alpha\in R^m$, $q\in R^m$ be vectors, whose coordinates are positive and nonnegative numbers, respectively, and let r be an arbitrary real number. If we assume that

$$\sum_{i=1}^{m} q_i = 1,$$

then the weighted r-mean is defined as

$$M_r(\alpha;q)=M_r(\alpha_1,\ldots\alpha_m;q)=\begin{cases}\left(\sum_{i=1}^{m} q_i a_i^r\right)^{\frac{1}{r}} & \forall r\neq 0\\ \prod_{i=1}^{m}\alpha_i^{q_i} & r=0.\end{cases}$$

Definition 2.2. We say $S\subset R^n$ is a $h-p$-invex set with respect to a vector function $\eta:S\times S\to R^n$ if, for any $x,u\in S$ and $q_1\geq 0, q_2\geq 0, q_1+q_2=1$, we have

$$\log\left(M_p\left(e^{\eta_1(x,u)+h_1(u)},\ e^{h_1(u)};\ q\right),\cdots M_p\left(e^{\eta_n(x,u)+h_n(u)},\ e^{h_n(u)};\ q\right)\right)\in S.$$

Here the logarithm is understood to be taken componentwise. If $q_1=\lambda$ (the fact that $q_1+q_2=1$ implies that $q_2=1-\lambda$ for any $\lambda\in[0;\ 1]$) then, using the definition of weighted p-mean, we may write down the above relations as

$$\log\left(\lambda e^{p\eta(x,u)+ph(u)}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\in S,\ \ p\neq 0,$$
$$h(u)+\lambda\eta(x,u)\in S,\qquad\qquad p=0.$$

Again, the logarithm and exponentials are understood to be taken componentwise.

Definition 2.3. Let $\eta:S\times S\to R^n$ be a vector function. A function $f:S\to R$ defined on a $h-p$-invex set $S\subset R^n$ with respect to η is called $(h,\varphi)-(p,r)$-pre-invex with respect to η at the point $u\in S$ on S if for any $x\in S$, $q_1\geq 0, q_2\geq 0, q_1+q_2=1$, the following inequality is satisfied

$$\hat{f}\left(\log M_p\left(e^{\eta(x,u)+h(u)},\ e^{h(u)};\ q\right)\right)\leq \log M_r\left(e^{\varphi(f(x))},\ e^{\varphi(f(u))};\ q\right),$$

where $\hat{f}(t)=\varphi\left(f\left(h^{-1}(t)\right)\right)$ is the function appearing in the definition of the (h,φ)-differential. If the mentioned inequality is satisfied at any point $u\in S$, then f is said to be $(h,\varphi)-(p,r)$-pre-invex with respect to η on S.

If $q_1=\lambda$ (the fact that $q_1+q_2=1$ implies that $q_2=1-\lambda$ for any $\lambda\in[0;\ 1]$) then, using the definition of weighted p-mean, we may write down inequality above as

$$\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right)\leq\log\left(\lambda e^{r\varphi(f(x))}+(1-\lambda)e^{r\varphi(f(u))}\right)^{\frac{1}{r}},\ \ p\neq 0, r\neq 0,$$
$$\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right)\leq\varphi\left(\lambda[\cdot]f(x)[+](1-\lambda)[\cdot]f(u)\right),\ \ p\neq 0, r=0,\qquad (1)$$
$$\hat{f}\left(\lambda\eta(x,u)+h(u)\right)\leq\log\left(\lambda e^{r\varphi(f(x))}+(1-\lambda)e^{r\varphi(f(u))}\right)^{\frac{1}{r}},\qquad p=0, r\neq 0,$$
$$\hat{f}\left(\lambda\eta(x,u)+h(u)\right)\leq\varphi\left(\lambda[\cdot]f(x)[+](1-\lambda)[\cdot]f(u)\right),\qquad p=0, r=0.$$

Remark. All classes of functions defined by (1) above according to **Definition 2.3** are called $(h,\varphi)-(p,r)$-pre-invex with respect to η. But one may use the terminology below.

- In case $p\neq 0, r=0,$ functions defined by (1) are called $(h,\varphi)-(p,0)$-pre-invex with respect to η (or shortly $(h,\varphi)-p$-pre-invex with respect to η);
- In case $p=0, r\neq 0,$ functions defined by (1) are called $(h,\varphi)-(0,r)$-pre-invex with respect to η (or shortly $(h,\varphi)-r$-pre-invex with respect to η);

- In case $p=0, r=0,$ functions defined by (1) are called $(h,\varphi)-(0,0)$-pre-invex with respect to η (or shortly (h,φ)-pre-invex with respect to η);

Definition 2.4. Let $f:S\to R$ be a differentiable function on an $h-p$-invex set $S\subset R^n$ with respect to η. If for all $x\in S$ one of the following relations holds:

$$\begin{aligned}
&\frac{1}{r}e^{r\varphi(f(x))}\geq\frac{1}{r}e^{r\varphi(f(u))}\left(1+\frac{r}{p}h\left(\nabla^* f(u)\right)\left(e^{p\eta(x,u)}-1\right)\right), && p\neq 0,\quad r\neq 0,\\
&\varphi\left(f(x)[-]f(u)\right)\geq\frac{1}{p}h\left(\nabla^* f(u)\right)\left(e^{p\eta(x,u)}-1\right), && p\neq 0,\quad r=0,\\
&\frac{1}{r}e^{rf(x)}\geq\frac{1}{r}e^{rf(u)}\left(1+rh\left(\nabla^* f(u)\right)\eta(x,u)\right), && p=0,\quad r\neq 0,\\
&\varphi\left(f(x)[-]f(u)\right)\geq h\left(\nabla^* f(u)\right)\eta(x,u), && p=0,\quad r=0.
\end{aligned}\tag{2}$$

then f is said to be $(h,\varphi)-(p,r)$-invex with respect to η at u on S.

THEOREM 2.1. *Let $S\subset R^n$ be a $h-p$-invex set with respect to η, and let $f:S\to R$ be a differentiable function. If f is $(h,\varphi)-(p,r)$-pre-invex with respect to η on S, then f is $(h,\varphi)-(p,r)$-invex with respect to η on S.*

Proof. Let $f:S\to R$ be defined on *a* $h-p$-invex set $S\subset R^n$ with respect to η. Moreover, we assume that f is a (h,φ)-differentiable $(h,\varphi)-(p,r)$-pre-invex function with respect to η on S. There are four possible cases.

- **Case 1:** $p\neq 0, r\neq 0,$
- **Case 2:** $p\neq 0, r=0,$
- **Case 3:** $p=0, r\neq 0,$
- **Case 4:** $p=0, r=0,$

We give the proof for each case.

- **Case 1:** $\boldsymbol{p\neq 0, r\neq 0}$. Assume $r>0$. By Definition 2.3 of $(h,\varphi)-(p,r)$-pre-invex functions we have

$$\begin{aligned}
&e^{\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right)}\leq\left(\lambda e^{r\varphi(f(x))}+(1-\lambda)e^{r\varphi(f(u))}\right)^{\frac{1}{r}}\Rightarrow\\
&\Rightarrow e^{r\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right)}\leq\lambda e^{r\varphi(f(x))}+(1-\lambda)e^{r\varphi(f(u))}\Rightarrow\\
&\Rightarrow\frac{e^{r\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right)}-e^{r\hat{f}(h(u))}}{\lambda}\leq e^{r\varphi(f(x))}-e^{r\varphi(f(u))}.
\end{aligned}\tag{3}$$

For simplicity, we define

$$\mu(\lambda)=\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}},$$

We now can easily check that $\mu(0)=h(u)$, and by letting $\lambda\to 0$ we get

$$e^{r\varphi(f(x))}-e^{r\varphi(f(u))}\geq\lim_{\lambda\to 0}\frac{e^{r\,\hat{f}(\mu(\lambda))}-e^{r\,\hat{f}(\mu(0))}}{\lambda}=\frac{r}{p}e^{r\varphi(f(u))}h\left(\nabla^* f(u)\right)\left(e^{p\eta(x,u)}-1\right),$$

which, after algebraic transformations, becomes

$$\frac{1}{r}e^{r\varphi(f(x))} \ge \frac{1}{r}e^{r\varphi(f(u))}\left(1+\frac{r}{p}h\left(\nabla^* f(u)\right)\left(e^{p\eta(x,u)}-1\right)\right), \tag{4}$$

Thus, according to Definition 2.4, f is $(h,\varphi)-(p,r)$-invex function with respect to η on S as it satisfies the first condition in (2). The proof of the case when $r<0$ is totally analogous, only the inequality direction changes at (3) and then changes back at (4).

- **Case 2:** $\boldsymbol{p\neq 0, r=0}$. By Definition 2.3 of $(h,\varphi)-(p,r)$-pre-invex functions we have

$$\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right) \le \varphi\left(\lambda[\cdot]f(x)[+](1-\lambda)[\cdot]f(u)\right) \Rightarrow$$

$$\frac{\hat{f}\left(\log\left(\lambda e^{p(\eta(x,u)+h(u))}+(1-\lambda)e^{ph(u)}\right)^{\frac{1}{p}}\right)-\hat{f}\left(h(u)\right)}{\lambda} \le \varphi\left(f(x)\right)+\varphi\left(f(u)\right)$$

and letting $\lambda\to 0$ we get

$$\varphi\left(f(x)[-]f(u)\right) \ge \frac{1}{p}h\left(\nabla^* f(u)\right)\left(e^{p\eta(x,u)}-1\right),$$

- **Case 3:** $\boldsymbol{p=0, r\neq 0}$. By Definition 2.3 of $(h,\varphi)-(p,r)$-pre-invex functions we have

$$\hat{f}\left(\lambda\eta(x,u)+h(u)\right) \le \log\left(\lambda e^{r\varphi(f(x))}+(1-\lambda)e^{r\varphi(f(u))}\right)^{\frac{1}{r}}.$$

Assuming $r>0$ we can transform the inequality as

$$\frac{e^{r\hat{f}(\lambda\eta(x,u)+h(u))}-e^{\hat{f}(h(u))}}{\lambda} \le e^{r\varphi(f(x))}-e^{r\varphi(f(u))}, \tag{5}$$

By letting $\lambda\to 0$ we get

$$e^{rf(x)}-e^{rf(u)} \ge re^{rf(u)}\left(1+rh\left(\nabla^* f(u)\right)\eta(x,u)\right),$$

which after algebraic transformation can be written as

$$\frac{1}{r}e^{rf(x)} \ge \frac{1}{r}e^{rf(u)}\left(1+rh\left(\nabla^* f(u)\right)\eta(x,u)\right), \tag{6}$$

Again, similarly to Case 1, the proof for the $r<0$ is the same, except that the inequality changes to opposite direction at (5), and then changes back at (6).

- **Case 4:** $\boldsymbol{p=0, r=0}$. By Definition 2.3 of $(h,\varphi)-(p,r)$-pre-invex functions we have

$$\hat{f}\left(\lambda\eta(x,u)+h(u)\right) \le \varphi\left(\lambda[\cdot]f(x)[+](1-\lambda)[\cdot]f(u)\right) \Rightarrow$$
$$\Rightarrow \hat{f}\left(\lambda\eta(x,u)+h(u)\right) \le \varphi\left(\lambda[\cdot]f(x)\right)+\varphi\left((1-\lambda)[\cdot]f(u)\right)=\lambda\varphi\left(f(x)\right)+(1-\lambda)\varphi\left(f(u)\right),$$

which, after some algebraic transformation can be written as

$$\frac{\hat{f}\left(\lambda\eta(x,u)+h(u)\right) \le \varphi\left(\lambda[\cdot]f(x)\right)+\varphi\left((1-\lambda)[\cdot]f(u)\right)-\hat{f}\left(h(u)\right)}{\lambda}=\varphi\left(f(x)\right)-\left(f(u)\right).$$

By letting $\lambda \to 0$ we get

$$h\left(\nabla^* f(u)\right)\eta(x,u) \le \varphi\left(f(x)\right) - \varphi\left(f(u)\right) = \varphi\left(f(x)[-]f(u)\right) \Rightarrow$$
$$\Rightarrow h\left(\nabla^* f(u)\right)\eta(x,u) \le \varphi\left(f(x)[-]f(u)\right).$$

3. OPTIMALITY CONDITIONS

In this section we shall deal with sufficient optimality conditions in problems with inequality constraints. As it is known, the Karush-Kuhn-Tucker conditions are necessary for optimality in mathematical programming problem, if a certain constraint holds. It is also well-known that if the objective function and the functions of constraints occurring in the optimization problem are convex, the Karush-Kuhn-Tucker conditions become sufficient for optimality [1, 4]. This fact also takes place in the case of wider classes of functions, namely, when the functions occurring in optimization problem are invex with respect to the same η [3].

We shall demonstrate below the similar result for $(h,\varphi)-$generalized case, when the occurring functions are a certain kind of $(h,\varphi)-(p,r)$-invex with respect to the same η. Below we consider that the functions h and φ satisfy the conditions $h(0)=0, \varphi(0)=0$ and that φ is a monotone nondecreasing function.

Consider an optimization problem with inequality constraints of the form

$$\begin{aligned} & f(x) \to \min \\ & g_i(x) \le 0,\ i = 1,\cdots m, \end{aligned} \qquad \text{(P)}$$

where $f, g_i : X_0 \to R, i = 1,2,...m$ are $(h,\varphi)-$differentiable functions on an open nonempty set $X_0 \subset R^n$.

Let us denote by D the set of the feasible points of (P) i.e., the set of the form

$$D = \left\{x \in X_0 : g_i(x) \le 0,\ i = 1,2,...m\right\}.$$

Definition 3.1. A point $z \in R^n$ is called feasible for problem (P) if $z \in X_0$ and $g_i(z) \le 0, \quad i = 1,\cdots m$.

Now we redefine the Karush-Kuhn-Tucker conditions [1, 4] for the $(h,\varphi)-$case as follows.

Definition 3.2. We say that $(h,\varphi)-$Karush-Kuhn-Tucker conditions holds at a feasible point $\bar{x}$ for problem (P), if there exists $\overline{\xi_i}$, $i = 1,2,...m$ such that

$$\nabla^* f(\bar{x}) \oplus \bigoplus_{i=1}^{m} \xi_i \otimes \nabla^* g_i(\bar{x}) = 0, \tag{7}$$

$$\left[\sum_{i=1}^{m}\right]\left(\bar{\xi}_i [\cdot] g_i(\bar{x})\right) = 0, \tag{8}$$

$$\bar{\xi}_i \ge 0,\ i = 1,2,...m. \tag{9}$$

THEOREM 3.1. *Assume that the point $\bar{x} \in X_0$ is feasible for problem (P) and let $(h,\varphi)-$Karush-Kuhn-Tucker conditions be satisfied at the point $(\bar{x},\bar{\xi})$ if the objective function f and the constraint functions f are $g_i, i = 1,2,...m$-invex with respect to η at x on D, then $\bar{x}$ is a global minimum point in problem (P).*

Proof. Assume that x is an arbitrary feasible point for problem (P). By assumption, f and g_i are $(h,\varphi)-(p,r)$-invex with respect to the same function η at $\bar{x}$ on D. Therefore, for all $x, \bar{x} \in D$ the inequalities

$$\frac{1}{r}e^{r\varphi(f(x))} \geq \frac{1}{r}e^{r\varphi(f(\bar{x}))}\left(1+\frac{r}{p}h\left(\nabla^* f\left(\bar{x}\right)\right)\left(e^{p\eta(x,\bar{x})}-1\right)\right), \tag{10}$$

$$\frac{1}{r}e^{r\varphi(g_i(x))} \geq \frac{1}{r}e^{r\varphi(g_i(\bar{x}))}\left(1+\frac{r}{p}h\left(\nabla^* g_i\left(\bar{x}\right)\right)\left(e^{p\eta(x,\bar{x})}-1\right)\right), \tag{11}$$

are true. Denote $J=\left\{1\leq i\leq m;\ \bar{\xi}_i>0\right\}$, where $\bar{\xi}=\left(\bar{\xi}_1,\bar{\xi}_2,\ldots\bar{\xi}_m\right)$ is the Lagrange multiplier which appears in the (h,φ) – Karush-Kuhn-Tucker conditions (7–9). Since $\bar{\xi}_i>0$ for $i\in J$, we may write (11) as

$$\frac{\bar{\xi}_i}{r}e^{\frac{r}{\bar{\xi}_i}\bar{\xi}_i\ \varphi(g_i(x))} \geq \frac{\bar{\xi}_i}{r}e^{\frac{r}{\bar{\xi}_i}\bar{\xi}_i\ \varphi(g_i(\bar{x}))}\left(1+\frac{r}{\bar{\xi}_i p}\bar{\xi}_i h\left(\nabla^* g_i\left(\bar{x}\right)\right)\left(e^{p\eta(x,\bar{x})}-1\right)\right),\ i\in J. \tag{12}$$

Dividing both sides of inequality (10) by $e^{r\varphi(f(\bar{x}))}$ and (12) by $e^{r\varphi(g_i(\bar{x}))}$, we get

$$\frac{1}{r}e^{r\left(\varphi(f(x))-\varphi(f(\bar{x}))\right)} \geq \frac{1}{r}\left(1+\frac{r}{p}h\left(\nabla^* f\left(\bar{x}\right)\right)\left(e^{p\eta(x,\bar{x})}-1\right)\right),$$

$$\frac{\bar{\xi}_i}{r}e^{\frac{r}{\bar{\xi}_i}\left(\bar{\xi}_i\ \varphi(g_i(x))-\bar{\xi}_i\ \varphi(g_i(\bar{x}))\right)} \geq \frac{\bar{\xi}_i}{r}\left(1+\frac{r}{\bar{\xi}_i p}\bar{\xi}_i h\left(\nabla^* g_i\left(\bar{x}\right)\right)\left(e^{p\eta(x,\bar{x})}-1\right)\right),\ i\in J.$$

After adding both sides of the above inequalities, we obtain

$$\frac{1}{r}\left(e^{r\left(\varphi(f(x))-\varphi(f(\bar{x}))\right)}+\sum_{i\in J}\bar{\xi}_i e^{\frac{r}{\bar{\xi}_i}\left(\bar{\xi}_i\ \varphi(g_i(x))-\bar{\xi}_i\ \varphi(g_i(\bar{x}))\right)}\right) \geq \frac{1}{r}\left(1+\sum_{i\in J}\bar{\xi}_i\right)+$$

$$+\frac{1}{p}\left(\left[h\left(\nabla^* f\left(\bar{x}\right)\right)+\sum_{i\in J}h\left(\nabla^* g_i\left(\bar{x}\right)\right)\right]\left(e^{p\eta(x,\bar{x})}-1\right)\right)=$$

$$=\frac{1}{r}\left(1+\sum_{i\in J}\bar{\xi}_i\right)+\frac{1}{p}\left(\left[h\left(\nabla^* f\left(\bar{x}\right)\oplus\bigoplus_{i=1}^{m}\bar{\xi}_i\otimes\nabla^* g_i\left(\bar{x}\right)\right)\right]\left(e^{p\eta(x,\bar{x})}-1\right)\right).$$

By (7) and the assumption $h(0)=0$ we obtain

$$\frac{1}{r}\left(e^{r\left(\varphi(f(x))-\varphi(f(\bar{x}))\right)}+\sum_{i\in J}\bar{\xi}_i e^{\frac{r}{\bar{\xi}_i}\left(\bar{\xi}_i\ \varphi(g_i(x))-\bar{\xi}_i\ \varphi(g_i(\bar{x}))\right)}\right) \geq \frac{1}{r}\left(1+\sum_{i\in J}\bar{\xi}_i\right).$$

Also using (8) and the assumption $\varphi(0)=0$ we get

$$\frac{1}{r}e^{r\left(\varphi(f(x))-\varphi(f(\bar{x}))\right)} \geq \frac{1}{r}\left(1+\sum_{i\in J}\bar{\xi}_i\left(1-e^{r\varphi(g_i(\bar{x}))}\right)\right),\ i\in J. \tag{13}$$

Let $r>0$ (in the case when $r<0$ the proof is analogous, one should change only the directions of inequalities below to the opposite one). Since x is a feasible point for problem (P), $g_i(x)\leq 0$ for each $i=1,2,\cdots m$. This, in turn, implies that $1-e^{r\varphi(g_i(\bar{x}))}\geq 0$ for each $i=1,2,\cdots m$. Taking this facts into account in (13) we get

$$e^{r\left(\varphi(f(x))-\varphi(f(\bar{x}))\right)} \geq 1+\sum_{i\in J}\bar{\xi}_i\left(1-e^{r\varphi(g_i(\bar{x}))}\right)\geq 1.$$

Hence, by assumption that φ is a nondecreasing function, we get

$$\varphi\left(f(x)\right)\geq\varphi\left(f(\bar{x})\right)\Rightarrow f(x)\geq f(\bar{x}),$$

which means that $\bar{x}$ is an optimal point in problem (P).

The assumption on functions in Theorem 3.1 could also be given in another form. It is enough to assume that the Lagrange function

$$L(x)=f(x)[+]\left[\sum_{i=1}^{m}\right]\left(\bar{\xi}_i[\cdot]g_i(x)\right)$$

is $(h,\varphi)-(p,r)$-invex with respect to η on D. And so, the following theorem is true.

THEOREM 3.2. *Assume that a point $\bar{x}\in R^n$ is feasible for problem (P) and let the $(h,\varphi)-$Karush-Kuhn-Tucker conditions be satisfied at the point $(\bar{x},\bar{\xi})$. If the function*

$$L(x)=f(x)[+]\left[\sum_{i=1}^{m}\right]\left(\bar{\xi}_i[\cdot]g_i(x)\right)$$

is a $(h,\varphi)-(p,r)$-invex function with respect to η at $\bar{x}$ on D, then $\bar{x}$ is a global minimum for problem (P).

Proof. As in Theorem 3.1, Theorem 3.2 will be proved only in case when $p\neq 0, r\neq 0,$. Let x be any other feasible point for problem (P). By assumption, the function

$$L(x)=f(x)[+]\left[\sum_{i=1}^{m}\right]\left(\bar{\xi}_i[\cdot]g_i(x)\right)$$

is $(h,\varphi)-(p,r)$-pre invex with respect to η at $\bar{x}$ on D. Hence the inequality

$$\frac{1}{r}e^{r\varphi(L(x))}\geq\frac{1}{r}e^{r\varphi(L(\bar{x}))}\left(1+\frac{r}{p}h\left(\nabla^{*}L(\bar{x})\right)\left(e^{p\eta(x,\bar{x})}-1\right)\right),\ i\in J, \tag{14}$$

holds for all $x\in D$. First of all we write the $(h,\varphi)-$differential of the function L as

$$\nabla^{*}L(\bar{x})=\nabla^{*}f(\bar{x})\oplus\bigoplus_{i=1}^{m}\bar{\xi}_i\otimes\nabla^{*}g_i(\bar{x}),$$

and now by condition (7) and from (14) we obtain the inequality

$$\frac{1}{r}e^{r\varphi(L(x))}\geq\frac{1}{r}e^{r\left(\sum_{i=1}^{m}\bar{\xi}_i\varphi(g_i(\bar{x}))-\sum_{i=1}^{m}\bar{\xi}_i\varphi(g_i(x))\right)},$$

which can be written as

$$f(x)-f(\bar{x})\geq\bar{\xi}\cdot\varphi\left(g(\bar{x})\right)-\bar{\xi}\cdot\varphi\left(g(x)\right).$$

Using (8), (9) and the fact that $x,$ is feasible point for problem (P), we get

$$f(x)\geq f(\bar{x}),$$

which means that the point $\bar{x},$ is a global minimum for problem (P).

REFERENCES

1. Bazaraa, M. S., Sherali H. D., Shetty C. M., *Nonlinear Programming: Theory and Algorithms*, Wiley, New York, 1991.
2. Ben-Israel, Mond B., *What is Invexity?*, J. Austral Math. Soc. Ser. B, **28**, pp. 1–9 (1986).
3. Hanson M. A., *On sufficiency of Kuhn-Tucker conditions*, J. Math. Anal. Appl., **80**, pp. 545–550 (1981).
4. Mangasarian O. L., *Nonlinear Programming*, McGraw-Hill, New York, 1969.
5. Martin D. H., *The essence of invexity*, J. Optim. Theory Appl., **42**, pp. 65–76 (1985).
6. Miculescu, R., *Some observations on generalized Lipschitz functions*. Rocky Mountain J. of Math., **37**, *3*, pp. 893–903 (2007).
7. Preda V., Diana-Elena S., *New Sufficient Conditions For B-Preinvexity and Some Extensions*, Proceedings of the Romanian Academy, Series A, **12**, *3*, pp. 197–202 (2011).
8. Preda, V., *On Efficiency and Duality for Multiobjective Programs*, J. Math. Anal. Appl., **166**, pp. 365–377 (1992).
9. Weir T., Jeyakumar V.,:*A class of nonconvex functions and mathematical programming*, Bull, Austral. Math. Soc., **38**, pp. 177–189 (1988).

Received November 11, 2011