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STRONGLY J-CLEAN GROUP RINGS

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An element of a ring is called strongly J-clean provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is called strongly J-clean if each of its element is strongly J-clean. In this article, we investigate some properties of strongly J-clean group rings.

Key words: clean rings, strongly J-clean rings, strongly J-clean group rings.

1. INTRODUCTION

Throughout this paper, R is an associative ring with identity. $J(R)$ will denote the Jacobson radical of R and $T_n(R)$ is the ring of all $n \times n$ upper triangular matrices over R.

An element is called clean if it can be written as the sum of an idempotent and a unit. A ring is called clean if each of its element is clean. Clean rings were firstly introduced by Nicholson [9]. Several peoples worked on this subject and investigate properties of clean rings, for example see [1], [2] and [6]. An element is called strongly clean if it can be written as the sum of an idempotent and a unit that commute. A ring is called strongly clean if each of its element is strongly clean. Also an element is called uniquely clean if it can be written uniquely as the sum of an idempotent and a unit. A ring is called uniquely clean if each of its element is uniquely clean. Uniquely clean rings were discussed by Anderson and Camillo [1] for the commutative case, and by Nicholson and Zhou [10, 11] for the noncommutative case.

An element of a ring is called strongly J-clean provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is called strongly J-clean if each of its element is strongly J-clean. Strongly J-clean rings introduced by Chen [3]. He investigated some propertices of strongly J-clean, also he [3, Proposition2.1] proved that every strongly J-clean element is strongly clean and showed that {uniquely clean rings} $\underset{\neq}{\subseteq}$ {strongly J-clean rings} $\underset{\neq}{\subseteq}$ {strongly clean rings}.

If G is a group and R is a ring we denote the group ring over R by RG. As every homomorphic image of a strongly J-clean ring is strongly J-clean, it is clear that if RG is strongly J-clean, then R is strongly J-clean. But it seems to be difficult to characterize R and G for which RG is strongly J-clean. Nicholson and Zhou [11, Proposition 24] proved that if R is a commutative uniquely clean ring, then RC_{2^k} is uniquely clean for all k (C_n is the cyclic group of order n) and if $n \geq 3$ is odd and R is a boolean ring, then RC_n is clean, but not uniquely clean. Chen, Nicholson and Zhou [4, Example 3] extended this result. They proved that if R is a ring and $G \neq 1$ is a locally finite group in which every finite subgroup has odd order, then RG is not uniquely clean. We prove in this case RG also is not strongly J-clean. Also Chen, Nicholson and Zhou [4, Theorem 5] proved that if R is a ring and G is a locally finite group, then RG is uniquely clean if and only if R is uniquely clean and G is a 2-group. We extend this result to strongly J-clean rings. Also we prove that if R is a boolean ring and G is a locally finite group, then RG is strongly J-clean if and only if G is a 2-group. Finally, we prove that if R is a commutative local strongly J-clean ring and G is an abelian 2-group, then we have the following statements:

(1) RG is local;
(2) $RG/J(RG)\cong Z_2$;
(3) $T_n(RG)$ is strongly J-clean for $n\geq 2$;
(4) $T_n(RG)$ is not uniquely clean unless $n=1$.

2. SOME PRELIMINARY RESULTS

Here we list a number of general results on strongly J-clean rings to be used in our result later.
The following results are clear by [3] and [11].

LEMMA 2.1. *Every factor ring of a strongly J-clean ring is strongly J-clean. In particular a homomorphic image of a strongly J-clean ring is strongly J-clean.*

LEMMA 2.2. *The following are equivalent for a ri*ng R:
(1) R *is local and strongly J-clean*;
(2) R *is strongly J-clean and 0 and 1 are the only idempotents in* R;
(3) $R/J(R)\cong Z_2$;
(4) R *is local and uniquely clean*;
(5) R *is uniquely clean and 0 and 1 are the only idempotents in* R.

LEMMA 2.3. *The following are equivalent for a ring* R:
(1) R *is boolean*;
(2) R *is strongly J-clean and* $J(R)=0$;
(3) R *is von Neumann regular and strongly J-clean*;
(4) R *is uniquely clean and* $J(R)=0$;
(5) R *is clean, char(R) = 2 and 1 is the only unit in* R;
(6) R *is von Neumann regular and uniquely clean*.

LEMMA 2.4. *Let* R *be a commutative local ring. Then the following are equivalent*:
(1) $T_n(R)$ *is strongly J-clean for* $n\geq 2$;
(2) $R/J(R)\cong Z_2$;
(3) R *is strongly J-cle*an;
(4) R *is uniquely clean*.

Definition 2.5. A ring R (or a group G) is called locally finite if every finitely generated subring (subgroup) is finite.

In [8], Nicholson obtained sufficient conditions for a group ring to be local as follow:

LEMMA 2.6. *Let* R *be a local ring and let* G *be a locally finite* $p-$*group where* p *is a prime such that* $p\in J(R)$. *Then* RG *is local*.

Definition 2.7. An element $\overline{a}=a+J(R)\in R/J(R)$ is said to lift strongly modulo $J(R)$ in case there exist an idempotent $e\in R$ such that $ea=ae$ and $a-e\in J(R)$.

LEMMA 2.8. *A ring* R *is strongly J-clean if and only if for any* $a\in R$ *there exists an idempotent* $e\in R$ *such that* $ea=ae$ *and* $a-e\in J(R)$.

Proof. Let R be strongly J-clean and $a\in R$. Then $R/J(R)$ is boolean by [3, Theorem 2.3]. Hence $\overline{a}=a+J(R)\in R/J(R)$ is an idempotent. But each idempotent lifts strongly modulo $J(R)$. So there exists an idempotent $e\in R$ such that $ea=ae$ and $a-e\in J(R)$, as required.

Conversely, let $a \in R$. Then by hypothesis, there exists an idempotent $e \in R$ such that $ea = ae$ and $a - e \in J(R)$. But $a = e + (a - e)$ and $e(a-e) = (a-e)e$. So a is strongly J-clean. Hence R is strongly J-clean.

3. MAIN RESULTS

Chen, Nicholson and Zhou [4, Example 3] proved that if R is a ring and $G \neq 1$ is a locally finite group in which every finite subgroup has odd order, then RG is not uniquely clean. We prove, in this case, RG also is not strongly J-clean.

PROPOSITION 3.1. *Let R be a ring and let $G \neq 1$ be a locally finite group in which every finite subgroup has odd order. Then RG is not strongly J-clean.*

Proof. Suppose that RG is strongly J-clean and $\bar{R} = R/J(R)$. Since R is a homomorphic image of RG, it is strongly J-clean. Hence $\bar{R}$ is boolean by [3, Theorem 2.3]. Also since every finite subgroup has odd order, so each $n \in o(G)$ ($o(G)$ is the set of orders of all finite subgroups in G) is a unit in R. Further since $\bar{R}$ is von Neumann regular and G is locally finite, $\bar{R}G$ is von Neumann regular by [5, Theorem 3]. But $\bar{R}G$ is homomorphic image of RG, so $\bar{R}G$ is strongly J-clean. Hence $\bar{R}G$ is boolean and this is a contradiction because $G \neq 1$. Therefore RG is not strongly J-clean.

THEOREM 3.2. *Let R be a ring and G be a group. If RG is strongly J-clean, then R is strongly J-clean and G is a 2-group.*

Proof. It is clear that R is strongly J-clean. Hence $\bar{R} = R/J(R)$ is boolean. So Z_2 is a homomorphic image of $\bar{R}$ and therefore Z_2 is a homomorphic image of R. So Z_2G is a homomorphic image of RG. Therefore Z_2G is strongly J-clean and since Z_2 is local, so Z_2G is uniquely clean. Hence G is a 2-group by [4, Theorem 5].

Corollary 3.3. If R is a boolean ring and G is a locally finite group, then RG is strongly J-clean if and only if G is a 2-group.

Proof. If G is a 2-group, then RG is uniquely clean by [4, Lemma 9]. Hence RG is strongly J-clean. The converse follows from Theorem 3.2.

Corollary 3.4. Let D be a division ring and let G be a locally finite group. Then DG is strongly J-clean if and only if $D \cong Z_2$ and G is a 2-group.

Proof. If DG is strongly J-clean, then D is strongly J-clean and G is a 2-group. But $D/J(D)$ is a division ring, so is local. Therefore $D/J(D) \cong Z_2$ and so $D \cong Z_2$.

Clearly, the converse follows from [4, Corollary 8].

THEOREM 3.5. *If R is a ring and G is a locally finite group, then RG is strongly J-clean if and only if R is strongly J-clean and G is a 2-group.*

Proof. If RG is strongly J-clean the result follows from Theorem 3.2.

Conversely, let R be strongly J-clean and G be a 2-group. We apply Lemma 2.8: if $a \in RG$ we must find an idempotent $f \in RG$ such that $af = fa$ and $a - f \in J(R)$. Let $w: RG \to R$ denote the augmentation map. Since $w(a) \in R$ and R is strongly J-clean, there is an idempotent $e \in R$ such that $ew(a) = w(a)e$ and $w(a) - e \in J(R)$ by Lemma 2.8. Hence $w(a-e) = w(a) - w(e) = w(a) - e \in J(R)$. As $R/J(R)$ is boolean thus $a - e \in J(RG)$ by [4, Lemma 10]. It remains to show that $ea = ae$. Suppose that $a = \sum_{g \in G} x_g . g$. Then, since $ew(a)=w(a)e$, so for every $r\in R$, $er=re$. Thus we have $ae=(\sum_{g\in G} x_g.g)(e.1_g)=\sum_{g\in G}(x_g.e).g=$

$=\sum_{g\in G}(e.x_g).g=e\sum_{g\in G}x_g.g=ea$.

Corollary 3.6. Let G *be a solvable group, and let* R *be a ring. Then* RG *is strongly J-clean if and only if* R *is strongly J-clean and* G *is a 2-group.*

Proof. If RG is strongly J-clean the result follows from Theorem 3.2.

Conversely, let R be strongly J-clean, G be a 2-group and $G=G^0\supseteq G^1\supseteq G^2\supseteq\ldots\supseteq G^n=1$ be the derived series for G. Then $G^i\lhd G$ and G^i/G^{i+1} is abelian for each i. Hence G^i/G^{i+1} is an abelian 2-group for each i, so G^i/G^{i+1} is locally finite for each i. But extensions of locally finite groups are again locally finite [7, Lemma 1.A.2]. So G is locally finite and therefore RG is strongly J-clean.

THEOREM 3.7. *Let* R *be a commutative local strongly J-clean ring and* G *be an abelian 2-group. Then we have the following statements*:

(1) RG *is local*;

(2) $RG/J(RG)\cong Z_2$;

(3) $T_n(RG)$ *is strongly J-clean for* $n\geq 2$;

(4) $T_n(RG)$ *is not uniquely clean unless* $n=1$.

Proof. To see (1) note that, since R is strongly J-clean, so $2\in J(R)$ by [3, Proposition 3.1]. But G is an abelian 2-group and R is local, thus RG is local and R is uniquely clean. So RG is uniquely clean. So RG is strongly J-clean. Obviously RG is commutative. Thus (2) and (3) follows from Lemma 2.4. Finally we know every idempotent in a uniquely clean ring is central. So $T_n(RG)$ is not uniquely clean unless $n=1$.

THEOREM 3.8. *Let* R *be a ring,* G *be a group and* $T_n(RG)$ *be strongly J-clean for* $n\geq 2$. *Then* R *is strongly J-clean*.

Proof. As $T_n(RG)$ is strongly J-clean, so RG is strongly J-clean. Hence R is strongly J-clean.

Example 3.9. Let n be a power of 2 and $D_n=\langle a,b:a^n=b^2=1,bab=a^{-1}\rangle$ be the dihedral group of order $2n$. Then D_n is a solvable 2-group. Thus if R is an strongly J-clean ring, then RD_n is strongly J-clean.

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