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A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS BY USE OF HYPERBOLIC REGULAR STAR POLYGONS

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In this paper we present a new characterization of Möbius transformations by use of hyperbolic regular star polygons.

Key words: Möbius transformations, hyperbolic kite, hyperbolic regular star polygons.

1. INTRODUCTION

Möbius transformations are well known and fundamental in complex analysis, and they have many beautiful properties. For example, a map is Möbius if, and only if, it preserves cross ratios. As for the geometric aspect, circle preserving is the most well known characterization of Möbius transformations. In addition to this, in literature, there are many characterizations of Möbius transformations via some hyperbolic polygons, [1, 2, 5, 13, 14, 15]. For other characterizations of hyperbolic isometries (in fact, these are Möbius transformations), we refer the readers to [3,4].

The main purpose of this paper is to present a new characterization of Möbius transformations by use of hyperbolic regular star polygons. Our proofs are based on a geometric approach.

2. MÖBIUS TRANSFORMATIONS OF THE DISC

The most general Möbius transformation of the complex open unit disc $\mathcal{D}=\{z\in\mathbb{C}: |z|<1\}$ in the complex $z-$plane

$$z \to e^{i\theta}\frac{z_0+z}{1+\overline{z_0}z}=e^{i\theta}(z_0\oplus z)$$

defines the Möbius addition "$\oplus$" in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*

$$z \to (z_0\oplus z)=\frac{z_0+z}{1+\overline{z_0}z}$$

followed by a rotation. Here θ is a real number, $z_0\in\mathcal{D}$, and $\overline{z_0}$ is the complex conjugate of z_0. Möbius subtraction "$\ominus$" is given by $a\ominus z=a\oplus(-z)$, clearly $a\ominus a=0$ and $\ominus a=-a$. Möbius addition $\oplus$ is a binary operation in the disc $\mathcal{D}$, but clearly it is neither commutative nor associative. Möbius addition $\oplus$ gives rise to the groupoid $(\mathcal{D}, \oplus)$ studied by A.A. Ungar in several books and articles including [6, 7, 8, 9, 10, 11, 12]. Möbius addition is analogous to the common vector addition + in Euclidean plane geometry. Since Möbius addition $\oplus$ is not associative, the groupoid $(\mathcal{D}, \oplus)$ is not a group. However, it has a group-like structure that we present below.

The breakdown of commutativity in Möbius addition is "repaired" by the introduction of the gyrator (see [12], Def. 2.7, p.17),

$$gyr : \mathcal{D}\times\mathcal{D} \to Aut\,(\mathcal{D},\oplus),$$

which gives rise to gyrations,

$$gyr[a,b] = \frac{a\oplus b}{b\oplus a} = \frac{1+a\bar{b}}{1+b\bar{a}}, \tag{1}$$

where $Aut\,(\mathcal{D},\oplus)$ is the automorphism group of the groupoid $(\mathcal{D},\oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (1)

$$a\oplus b = gyr[a,b](b\oplus a)\,. \tag{2}$$

Coincidentally, the gyration $gyr[a,b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$a\oplus(b\oplus c) = (a\oplus b)\oplus gyr[a,b]c$$
$$(a\oplus b)\oplus c = a\oplus(b\oplus gyr[b,a]c)$$

for all $a,b,c\in\mathcal{D}$.

Definition 2.1. A groupoid $(\mathcal{G},\oplus)$ is a gyrogroup if its binary operation satisfies the following axioms

(G1) $0\oplus a = a$ (left identity property)

(G2) $\ominus a\oplus a = 0$ (left inverse property)

(G3) $a\oplus(b\oplus c) = (a\oplus b)\oplus gyr[a,b]c$ (left gyroassociative property)

(G4) $gyr[a,b]\in Aut\,(\mathcal{G},\oplus)$ (gyroautomorphism)

(G5) $gyr[a,b] = gyr[a\oplus b,b]$ (left loop property)

for all $a,b,c\in\mathcal{G}$.

Additionaly, if the binary operation $\oplus$ obeys the gyrocommutative law

(G6) $(a\oplus b) = gyr[a,b](b\oplus a))$ (gyrocommutative law)

for all $a,b,c\in\mathcal{G}$, , then $(\mathcal{G},\oplus)$ is called a gyrocommutative gyrogroup. It is easy to see that $\ominus a = -a$ for all elements a of $\mathcal{G}$.

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathcal{G},\oplus)$ is a gyrocommutative gyrogroup.

The axioms in Defition 1 imply the right identity property, the right inverse property, the right gyyroassociative law and the right loop property. We refer readers to [12] and [9] for more details about gyrogroups.

3. MÖBIUS GYROGROUPS: FROM DISC TO THE BALL

Let us identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way

$$u = u_1 + u_2 = (u_1,u_2) = \mathrm{u}\in\mathbb{R}^2,$$

where $u\in\mathbb{C}$. Then the equations

$$\begin{aligned} \bar{u}v + u\bar{v} &= 2(\mathrm{u}\cdot \mathrm{v}) \\ |u| &= \|\mathrm{u}\| \end{aligned} \tag{3}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathcal{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc $\mathbb{R}^2_1 = \{\mathrm{u} \in \mathbb{R}^2 : \|\mathrm{u}\| < 1\}$ of $\mathbb{R}^2$. Indeed, we get from Eq.(3) that

$$\begin{aligned} u \oplus v &= \frac{\mathrm{u}+\mathrm{v}}{1+\bar{\mathrm{u}}\mathrm{v}} = \frac{\mathrm{u}+\mathrm{v}}{1+\bar{\mathrm{u}}\mathrm{v}}\left(\frac{1+\bar{\mathrm{v}}\mathrm{u}}{1+\bar{\mathrm{v}}\mathrm{u}}\right) = \\ &= \frac{\left(1+2(\mathrm{u}\cdot\mathrm{v})+\|\mathrm{v}\|^2\right)\mathrm{u} + (1-\|\mathrm{u}\|^2)\,\mathrm{v}}{1+2(\mathrm{u}\cdot\mathrm{v})+\|\mathrm{v}\|^2\|\mathrm{u}\|^2} = \mathrm{u}\oplus\mathrm{v} \end{aligned} \tag{4}$$

for all $u, v \in \mathcal{D}$ and all $\mathrm{u}, \mathrm{v} \in \mathbb{R}^2_1$.

4. MÖBIUS ADDITION IN THE BALL

Let $\mathcal{V}$ be any real inner-product space and

$$\mathcal{V}_s = \{ v \in \mathcal{V} : \|v\| < s \}$$

be the open ball of $\mathcal{V}$ with radius $s > 0$. Möbius addition in $\mathcal{V}_s$ is motivated by Eq.(4), and is given by

$$u \oplus v = \frac{(1+\left(\frac{2}{s^2}\right)u\cdot v + \left(\frac{1}{s^2}\right)\|v\|^2)u + (1-\left(\frac{1}{s^2}\right)\|u\|^2)v}{1+\left(\frac{2}{s^2}\right)u\cdot v + \|u\|^4\|v\|^4}, \tag{5}$$

where $\|\cdot\|$ is the norm that the ball $\mathcal{V}_s$ inherits from its space $\mathcal{V}$ and where, ambiguously, + denotes both addition of real numbers on the real line and addition of vectors in $\mathcal{V}$. Without loss of generality, we may assume that $s = 1$ in Eq.(5). We, however, prefer to keep s as a free positive parameter in order to exhibit the results in the limiting case as $s \to \infty$, when the ball $\mathcal{V}_s$ expands to the whole of its real inner-product space $\mathcal{V}$, and Möbius addition $\oplus$ reduces to vector addition + in $\mathcal{V}$, i.e.,

$$\lim_{s\to\infty} u \oplus v = u + v$$

and

$$\lim_{s\to\infty} \mathcal{V}_s = \mathcal{V}.$$

To see the definition of Möbius scalar multiplication "$\otimes$" and its properties, we refer [9].

Definition 4.1. (Möbius Gyrovector Spaces). Let $(\mathcal{V}_s, \oplus)$ be a Möbius gyrogroup equipped with scalar multiplication $\otimes$. The triple $(\mathcal{V}_s, \oplus, \otimes)$ is called a Möbius gyrovector space.

Definition 4.2. The gyrodistance function in $\mathcal{D}$, is given by

$$d(A,B) = |B \ominus A|$$

for all $A, B \in \mathcal{D}$.

The connection between the gyrodistance function and the standard hyperbolic distance function is described in [9].

5. MÖBIUS GYROLINE AND GYROANGLE

As is well known from Euclidean geometry, the straight line passing through two given points A and B of a vector space $\mathbb{R}^n$ can be represented by

$$A+(-A+B)t$$

$t\in\mathbb{R}$. Obviously it passes through A when $t=0$, and through B when $t=1$.
In full analogy with Euclidean geometry, the unique Möbius geodesic passing through two given points A and B of a Möbius gyrovector space $(\mathcal{V}_s, \oplus, \otimes)$ is represented by the parametric gyrovector equation

$$L_{AB} = A \oplus (\ominus A \oplus B) \otimes t$$

with parameter $t\in\mathbb{R}$. Similarly, it passes through A when $t=0$, and through B when $t=1$. The gyroline L_{AB} turns out to be a circular arc that intersects the boundary of the ball $\mathcal{V}_s$ orthogonally, see [9].

The measure of a Möbius angle between two intersecting geodesic rays equals the measure of the Euclidean angle between corresponding intersecting tangent lines, as shown in Figure 1. The hyperbolic angle is invariant under left gyrotranslations and rotations, see [6].

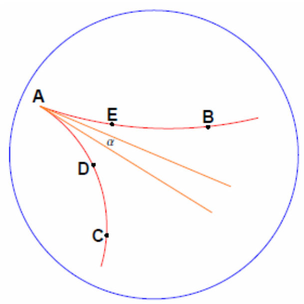

Fig. 1 – The unique geodesic that passes through two given points and the hyperbolic angle between two intersecting geodesic rays in $\mathcal{D}$.

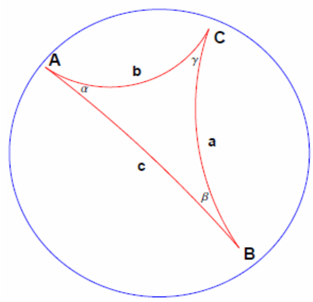

Fig. 2 – A hyperbolic triangle in $\mathcal{D}$.

6. MÖBIUS GYROTRIANGLE AND REGULAR STAR GYROPOLYGON

Definition 6.1. A gyrotriangle ΔABC in the complex unit disc $\mathcal{D}$ is a hyperbolic space object formed by the three points A, B and C, called the vertices of the gyrotriangle, and the hyperbolic vectors (that is gyrovectors) $|A\ominus B|$, $|B\ominus C|$ and $|C\ominus A|$ called the sides of the gyrotriangle. These are respectively the sides opposite to the vertices C, A and B. The gyrotriangle sides generate the three gyrotriangle gyroangles α, β and γ at the respective vertices A, B and C, as shown in Figure 2, see [9]. **Definition 6.2.** Let ΔABC be a gyrotriangle in $\mathcal{D}$ with vertices A, B and C, corresponding angles α, β and γ. The value $\delta := \pi - (\alpha+\beta+\gamma)$ is called the defect of the gyrotriangle ΔABC, see [9].

From now on, when we say angle, triangle, kite and regular star polygon, etc., we mean hyperbolic angle, hyperbolic triangle kite and hyperbolic regular star polygon etc., respectively.

Let ΔABC be a triangle in $\mathcal{D}$ with vertices A, B, C. The angle measures α, β and γ of the angles at the vertices A, B and C, are given by

$$\cos\alpha = ((\ominus A \oplus B)/|\ominus A \oplus B|)\cdot((\ominus A \oplus C)/|\ominus A \oplus C|),$$

$$\cos\beta = ((\ominus B \oplus C)/|\ominus B \oplus C|)\cdot((\ominus B \oplus A)/|\ominus B \oplus A|),$$

$$\cos\gamma = ((\ominus C \oplus A)/|\ominus C \oplus A|)\cdot((\ominus C \oplus B)/|\ominus C \oplus A|).$$

The following theorem plays a major role in hyperbolic geometry since the triangle angles determine uniquely its side lengths as follows:

THEOREM 6.1. *Let* ΔABC *be a triangle in* $\mathcal{D}$ *with vertices* A, B, C, *corresponding angles* α, β *and* γ, $0 < \alpha + \beta + \gamma < \pi$ *and side lengths* $|A \ominus B|$, $|B \ominus C|$ *and* $|C \ominus A|$. *The side lengths of the triangle* ΔABC *are determined by its angles according to the* ***AAA*** *to* ***SSS*** *conversion equations*

$$|\ominus B \oplus C|^2 = \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)},$$
$$|\ominus C \oplus A|^2 = \frac{\cos\beta + \cos(\alpha+\gamma)}{\cos\beta + \cos(\alpha-\gamma)}, \tag{6}$$
$$|\ominus A \oplus B|^2 = \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma + \cos(\alpha-\beta)}.$$

Definition 6.3. A regular star polygon is a self-intersecting, equilateral equiangular polygon, created by connecting one vertex of a simple, regular, $\rho-$sided polygon to another, non-adjacent vertex and continuing the process until the original vertex is reached again.

7. A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS BY USE OF HYPERBOLIC REGULAR STAR POLYGONS

Let us denote by X' the image of X under f, by $[A,B]$ the hyperbolic geodesic segment between points A and B, by AB the hyperbolic geodesic passing through point A and B, and by $\angle ABC$ the angle between $[A,B]$ and $[A,C]$.

LEMMA 7.1. *Let* $f:\mathcal{D}\to\mathcal{D}$ *be a continuous bijection which preserves kites. Then* f *preserves the measures of angles at the vertices of kites*.

Proof. Let $ABCD$ be a kite in $\mathcal{D}$ such that $|A\ominus B| = |A\ominus D|$ and $|B\ominus C| = |C\ominus D|$. Firstly, we have to show that $|A'\ominus B'| = |A'\ominus D'|$ holds true. Let us assume $|A'\ominus B'| \neq |A'\ominus D'|$. Because of the fact that f preserves the kites, we have $|A'\ominus B'| = |B'\ominus C'|$. Let P be the intersection of the geodesic AC with the geodesic BD and take an arbitrary point P_1 on $[P,C]$ such that ABP_1D is a kite. Since f is injective, the point P_1' is also lies on the segment $[P',C']$. Clearly, the quadrilateral $A'B'P_1'D'$ is not a kite and this implies that $|A'\ominus B'| = |A'\ominus D'|$ holds true. Similarly, one can easily show that $|B'\ominus C'| = |C'\ominus D'|$. Now, construct a sequence of kites $AB_iB_{i+1}B_{i+2}$ such that $|A\ominus B_i| = |A\ominus B_{i+2}|$ and $|B_i\ominus B_{i+1}| = |B_{i+1}\ominus B_{i+2}|$ and $\angle\, B_iAB_{i+2} = \frac{2\pi}{n}$ for $i = 1,3,5,\ldots,2n-1$, and n is an integer. Clearly, $n>2$ and by the assumption of lemma, each of the quadrilaterals $A'B_i'B_{i+1}'B_{i+2}'$ is also a kite for $i = 1,3,5,\ldots,2n-1$. Because of the fact that f is injective and $B_1 = B_{2n+1}$, we get $B_1' = B_{2n+1}'$. Hence, we must have $\angle B_i'A'B_{i+2}' = \frac{2\pi}{n}$ for $i = 1,3,5,\ldots,2n-1$. Accordingly, f preserves $\frac{m(2\pi)}{n}-$valued angles at the vertex A, where m is an integer. Since f is continuous and the set of rational numbers is dense in $\mathbb{R}$, we immediately get that f preserves the measures of all angles at the vertex A, i.e., $\angle DAB = D'A'B'$. In a similar way, one can show that $\angle BCD = B'C'D'$ holds true. Now, let us take $|C \ominus D| > |D \ominus A|$. In this situation, one can find a point M on $[C, D]$ such that $|M \ominus D| = |D \ominus A|$. Now, pick a point on $[A, M]$, say K, such that $\angle AKD = \pi/2$. Clearly we can easily construct a kite $MDAL$, where L is a point on DK. Finally, by following same process above, we get $\angle MDA = \angle CDA = \angle C'D'A' = M'D'A'$. In the case of $|C \ominus D| < |D \ominus A|$, one can easily show that $\angle CDA = \angle C'D'A'$ in a similar way. Thus, we proved that f preserves the measures of angles at the vertices of the kites.

The following result can be obtained by segments drawn from the vertices of the regular star polygon to its center.

COROLLARY 7.2. Let $f : \mathcal{D} \to \mathcal{D}$ be a continuous bijection which preserves kites. Then f preserves regular star polygons.

The following result can be obtained by Lemma 7.1.

COROLLARY 7.3. Let $f : \mathcal{D} \to \mathcal{D}$ be a continuous bijection which preserves kites. Then f preserves the measures of angles at the vertices of all regular star polygons.

THEOREM 7.4. *Let f be a continuous bijection. Then f is Möbius if, and only if, f preserves kites.*

Proof. The "only if" part is clear since f is an isometry. Conversely, we may assume that f preserves kites and $f(0) = 0$ by composing with an isometry if necessary. Let x and y be two arbitrary points. Without loss of generality we may assume that $|x| < |y|$ holds.

Case 1. Let x, y and 0 be not collinear. Let k be the reflection of 0 in the ray xy. Then we see that the quadrilateral $0xky$ is a kite such that $|x| = |x \ominus k|$ and $|y| = |y \ominus k|$. By the assumption and Lemma 7.1, we must have that $0x'k'y'$ is also a kite such that $|x'| = |x' \ominus k'|$ and $|y'| = |y' \ominus k'|$ and holds true. Then, by Eq.(6), we immediately get $|x| = |x'|$, $|y| = |y'|$, $|x \ominus k| = |x' \ominus k'|$, $|y \ominus k| = |y' \ominus k'|$ and $|x \ominus y| = |x' \ominus y'|$. Clearly, the equality $|x \ominus y| = |x' \ominus y'|$ implies that

$$\bar{x}y + x\bar{y} = \overline{x'}y' + x'\overline{y'} \tag{7}$$

holds. As $\bar{x}y + x\bar{y} = \overline{x'}y' + x'\overline{y'}$, f preserves inner-products and then is the restriction on $\mathcal{D}$ of an unitary transformation, that is, f is Möbius.

Case 2. Let x, y and 0 be collinear points and L be a line passing through the origin. Now we pick two points p, q on L such that $|x| = |p|$, $|y| = |q|$ and $\angle x0p = \angle y0q$. Let k be the reflection of 0 in the ray xp and l be the reflection of 0 in the ray yq. Thus, we must have that the quadrilaterals $0pkx$ and $0qly$ are kites and following the same way above, we get that Eq.(7) holds true.

The proof is clear if we take $|x| = |y|$, so we omit it.

THEOREM 7.5. *Let $f : \mathcal{D} \to \mathcal{D}$ be a continuous bijection. Then f is Möbius if, and only if, f preserves regular star polygons.*

Proof. The "only if" part is clear since f is an isometry. Conversely, we may assume that f preserves regular star polygons and it suffices to to prove that f preserves the kites. Let $ABCD$ be a kite in $\mathcal{D}$ such that $|A \ominus B| = |A \ominus D|$ and $|B \ominus C| = |C \ominus D|$. Now, construct a sequence of kites $AB_iB_{i+1}B_{i+2}$ such that $|A \ominus B_i| = |A \ominus B_{i+2}|$ and $|B_i \ominus B_{i+1}| = |B_{i+1} \ominus B_{i+2}|$ and $\angle\, B_iAB_{i+2} = \dfrac{2\pi}{n}$, for $i = 1, 3, 5, \ldots, 2n-1$ and n is an integer. Since f is injective and $B_1 = B_{2n+1}$, we get $B_1^{'} = B_{2n+1}^{'}$. Obviously, $B_2 B_4 \cdots B_{2n}$ is a $n-$pointed regular star polygon centered at A, and by the assumption of the theorem $B_2^{'} B_4^{'} \cdots B_{2n}^{'}$ is also a $n-$pointed regular star polygon centered at $A^{'}$. Therefore, we must have $\angle B_i^{'} A^{'} B_{i+2}^{'} = \frac{2\pi}{n}$ $i = 1, 3, 5, \ldots, 2n-1$. Obviously, f preserves $\frac{m(2\pi)}{n}-$valued angles at the vertex A, where m is an integer. Because of the fact that f is continuous and the set of rational numbers is dense in $\mathbb{R}$, we get that f preserves the measures of angles at the vertex A. Similarly, one can easily show that f preserves the measures of angles at the other vertices of the kite $ABCD$. Indeed, this can be obtained by use of the method in Lemma 7.1. Therefore, we proved that f preserves the measures of angles with the vertices of the kites and this implies that f is Möbius.

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