THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 14, Number 1/2013, pp. 20–27

OPTIMAL ESTIMATES IN LORENTZ SPACES OF SEQUENCES WITH AN INCREASING WEIGHT

Sorina BARZA [1], Anca-Nicoleta MARCOCI [2], Liviu-Gabriel MARCOCI [2], Lars-Erik PERSSON [3]

[1] Karlstad University, Department of Mathematics, SE-65188 Karlstad, Sweden
[2] Technical University of Civil Engineering Bucharest, Department of Mathematics and Computer Science, RO-020396 Bucharest, Romania
[3] Luleå University of Technology, Department of Mathematics, SE-97 187 Luleå, Sweden
Narvik University College, P.O. BOX 385, N-8505 NARVIK, Norway
E-mail: anca.marcoci@gmail.com

We study the weighted Lorentz spaces of sequences to find optimal estimates between different equivalent quasi-norms. We also obtain the best constant in the triangle inequality.

Key words: weighted Lorentz sequence spaces, normability, equivalent norms, decomposition norm, maximal norm, dual norm, sharp constants.

1. INTRODUCTION

Let $1<p<\infty$. For a sequence $x=(x_n)\in c_0$ (the space of null sequences) the decreasing rearrangement x^* of x is obtained by rearranging $(|x_n|)$ in decreasing order. A nonnegative sequence of real numbers $w=(w_n)$ will be called a weight sequence. Without loss of generality we may suppose that $w\notin l^1(\mathbb{N})$, see e.g. [6]. We recall the definition of the weighted Lorentz spaces of sequences

$$d(w,p)=\{x:\|x\|_{p,w}:=\left(\sum_{n=1}^{\infty}(x_n^*)^p w_n\right)^{\frac{1}{p}}<\infty\}.$$

It is proved in [6] that $\|\cdot\|_{p,w}$ is a norm if and only if w is a decreasing sequence. Also $d(w,p)$ is equivalently normable if and only if

$$\sum_{k=0}^{n}\left(\frac{1}{W_k}\right)^{1/p}\le C\frac{n+1}{W_n^{1/p}},\quad n=0,1,2,\ldots, \tag{1}$$

where $W_n=\sum_{k=0}^{n}w_k$. This is also the condition which characterizes the boundedness of the discrete Hardy operator from $d(w,p)$ to $l^p(w)$. Here $l^p(w)$ is the classical weighted Lebesgue space of sequences. In the recent papers [3] and [4] the authors considered estimates between a dual norm (defined in terms of Köthe duality), decomposition norm and the usual norm, in the continuous case. The main reason for these consideration was that the equivalent norm defined in terms of maximal function (or Hardy operator) does not give the best constant in the triangle inequality. In [3] it was considered the case of the classical Lorenz spaces $L^{p,s}$, with $p<s$, while in [4] similar results were proved for the weighted Lorentz spaces $\Gamma^p(w)$, where w is an increasing weight function. Using similar techniques, the same relations between norms on Lorentz spaces of sequences $l^{p,s}$ were proved in [5]. In this paper we extend the results proved in [5] to the case of more general Lorentz spaces of sequences with an increasing weight. Since we need the space

$d(p,w)$ to be normable we will necessarily have to assume that the weight satisfies the condition (1). Here and in the sequel we denote by $\tilde{w} = w^{1-p'}$. Also p' denotes the conjugate index of p, namely p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Observe that if w is an increasing sequence then $\tilde{w}$ is a decreasing sequence.

As a consequence of the fact that $\|\cdot\|_{p,w}$ is equivalent to a norm, it is easy to see that it is a quasi-norm satisfying the triangle inequality uniformly in the numbers of terms expressed as follows: there exists a constant $C_{p,w} > 0$ such that, for every finite collection $\{x^{(k)}\} \subset d(p,w)$, it yields that

$$\left\| \sum_{k=1}^{N} x^{(k)} \right\|_{p,w} \le C_{p,w} \sum_{k=1}^{N} \left\| x^{(k)} \right\|_{p,w}. \tag{2}$$

It can be proved also that the converse result holds. Moreover, an alternative equivalent norm is given by means of the following *decomposition norm*: $\|x\|_{(p,w)} := \inf\left\{ \sum_{k=1}^{N} \left\| x^{(k)} \right\|_{p,w} : x = \sum_{k=1}^{N} x^{(k)} \right\}..$

In what follows we use the notations $x \prec y$ if $\sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} y_i$ for all $n \ge 1$ and $x \le y$ if $x_n \le y_n$, for all $n \ge 1$, where $x = (x_n)_n$ and $y = (y_n)_n$.

The paper is organized as follows: In Section 2 we state some lemmas, which will be used in the subsequent Sections. In Section 3 we prove our main results. An important result in Section 3 is Theorem 3.1, which gives us optimal constants between $\|x\|_{p,w}$ and $\|x^{\circ}\|_{p,w}$. Another result of this Section is that the dual norm of $x = (x_n) \in d(p,w)$ coincides with $\|x^{\circ}\|_{p,w}$, (see Theorem 3.3). In Theorem 3.4 we obtain that $\|x\|_{p,w} \le C_{p,w} \|x\|'_{p,w}$, where $C_{p,w}$ is the sharp constant . We prove also that the dual norm coincides with the decomposition norm (see Theorem 3.5). In the last Section we present some applications, e.g. the triangle inequality for the quasi-norm $\|x\|_{p,w}$.

2. PRELIMINARY RESULTS

We need the following statement related to the dual norm:

LEMMA 2.1. *Let $x = (x_n) \in d(p,w)$, where w is an arbitrary positive sequence which satisfies condition (1). Then $\|x\|'_{p,w} = \|x^*\|'_{p,w}$.*

A proof can be found e.g. in [2], p. 45-49.

LEMMA 2.2. *Let $x = (x_n)_n \in d(p,w)$ and w a weight which satisfies condition (1). Then the following statements hold:*

(a) *The equality*

$$\|x\|_{(p,w)} = \inf\left\{ \sum_{k} \left\| x^{(k)} \right\|_{p,w} \right\} \tag{3}$$

holds, where the infimum is taken over all finite non-negative sequences $|x_n| = \sum_k x_n^{(k)}$.

(b) *If $0 \le y \le x$, then $\|y\|_{(p,w)} \le \|x\|_{(p,w)}$.*

(c) *If $0 \le y^{(k)} \le x$ and $y^{(k)} \uparrow x$ when $k \to \infty$, then $\left\| y^{(k)} \right\|_{(p,w)} \to \|x\|_{(p,w)}$.*

Proof. The proof is similar with the proof of Lemma 2.7 in [3] so we omit the details.

LEMMA 2.3. *For each* $x \in d(p,w)$, *we have that*

$$\|x\|_{(p,w)} \leq \|x^*\|_{(p,w)}. \tag{4}$$

Proof. The proof is similar with the proof of Lemma 2.8 in [3].

The concept of level sequence with respect to another sequence was used in analogy with the level function for the study of similar problems in the framework of classical Lorentz spaces of sequences in [5] or [11]. The unique sequence $x^\circ = (x_n^\circ)$ in Theorem 3.3 in [5] is called the level sequence of $x = (x_n)$ with respect to $\varphi = (\varphi_n)$. This definition is analogous to the one given in the paper [3] for the continuous case. For more general cases of level functions see e.g. [11].

Let $p > 1$. For a weight sequence $w = (w_n)$ which satisfies

$$\frac{1}{n}\sum_{k=m}^{m+n-1} w_k \leq C\left(\frac{1}{n}\sum_{k=m}^{m+n-1} w_k^{1-p'}\right)^{1-p}, \tag{5}$$

for some constant C and any $m, n \in \mathbb{N}$ we denote by $C_{p,w} = \sup_{m,n\in\mathbb{N}}\left(\frac{1}{n}\sum_{k=m}^{m+n-1} w_k\right)\left(\frac{1}{n}\sum_{k=m}^{n+m-1} w_n^{1-p'}\right)^{p-1}$,

the optimal constant in the above inequality.

The following Lemma will be important in the proof of Theorem 3.1.

LEMMA 2.4. *Let* $p > 1$ *and* $w = (w_n)$ *be an increasing weight sequence which satisfies (5) such that,* $\left(\frac{w_k}{w_{k+m-1}}\right)_k$ *is decreasing for any nonnegative integer* m. *Then*

$$C_{p,w} = \sup_{n\in\mathbb{N}}\left(\frac{1}{n}\sum_{k=1}^{n} w_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} w_k^{1-p'}\right)^{p-1}. \tag{6}$$

Proof. It is clear that $C_{p,w} \geq \sup_{n\in\mathbb{N}}\left(\frac{1}{n}\sum_{k=1}^{n} w_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} w_k^{1-p'}\right)^{p-1}$. We prove now the converse inequality.

Let us fix $m \geq 1$. Observe that for $m = 1$ we have equality so we may suppose that $m > 1$. If we denote by $x_k = w_k$, $z_k = w_{k+m-1}$ and $r = 1 - p' < 0$, we have to prove that $\frac{\sum_{k=1}^{n} x_k}{\left(\sum_{k=1}^{n} x_k^r\right)^{\frac{1}{r}}} \geq \frac{\sum_{k=1}^{n} z_k}{\left(\sum_{k=1}^{n} z_k^r\right)^{\frac{1}{r}}}$.

Observe that both $(x_k)_k$ and $(z_k)_k$ are decreasing sequences, by hypothesis. Denote by $X_n = \sum_{k=1}^{n} x_n$, $Z_n = \sum_{k=1}^{n} z_n$ and $C = \frac{X_n}{Z_n}$.

Since $\left(\frac{X_k}{Z_k}\right)_k$ is decreasing (see e.g. [8]) we have that $\frac{X_k}{Z_k} \geq \frac{X_n}{Z_n} = C$, for $k \leq n-1$. Applying the discrete version of the majorization principle, also known as Karamata's inequality (see e.g. [8]) for the decreasing sequences $(x_k)_k$ and $y_k = Cz_k$ and for the convex function $\Phi(t) = t^r$, $r < 0$, we get that

$$\sum_{k=1}^{n} x_k^r \geq \sum_{k=1}^{n} (Cz_k)^r \text{ or} \sum_{k=1}^{n} x_k^r \geq C^r \sum_{k=1}^{n} z_k^r.$$

The last inequality implies that $\left(\sum_{k=1}^{n} x_k^r\right)^{\frac{1}{r}} \le \frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} z_k}\left(\sum_{k=1}^{n} z_k^r\right)^{\frac{1}{r}}$, since $r<0$. Since m was arbitrary we get the desired inequality and the proof is complete.

Remark 2.5. The above Lemma remains true also in the case of decreasing sequences if we consider $\tilde{w}=w^{1-p'}$.

In the next Sections we assume that all of our weights w are increasing sequences which satisfies (1) and the hypotheses of Lemma 2.4.

3. MAIN RESULTS

THEOREM 3.1. *Let* $1<p<\infty$ *and* w *be an increasing weight. Let* $x=(x_n)_n \in d(p,w)$ *be a nonnegative and decreasing sequence and* $x^\circ=(x_n^\circ)_n$ *be the level sequence with respect to the sequence* $\varphi=(\varphi_n)_n$, *where* $\varphi_n=w_n^{1-p'}$.

Then we have that

$$\left\|x^\circ\right\|_{p,w} \le \|x\|_{p,w} \le C_{p,w}^{\frac{1}{p}}\left\|x^\circ\right\|_{p,w}, \tag{7}$$

where $C_{p,w}$ *is defined by (6). The constants in (7) are optimal.*

Proof. First we consider the left hand side in (7). By Theorem 3.3 in [5] we have that $\frac{x_j^\circ}{w_j^{1-p'}}=\lambda_k$ for all $j\in I_k$. Applying Hölder's inequality we obtain that

$$\begin{aligned}\sum_{m\in I_k}(x_m^\circ)^p w_m=\lambda_k^{p-1}\sum_{m\in I_k}x_m^\circ=\left(\sum_{m\in I_k}w_m^{1-p'}\right)^{1-p}\left(\sum_{m\in I_k}x_m\right)^{p-1}\left(\sum_{m\in I_k}x_m^\circ\right)=\\ =\left(\sum_{m\in I_k}w_m^{1-p'}\right)^{1-p}\left(\sum_{m\in I_k}x_m\right)^{p}\le\sum_{m\in I_k}x_m^p w_m\end{aligned} \tag{8}$$

This estimate and Theorem 3.3 in [5] yields the first inequality in (7). Let us now consider $\Psi=(\Psi_n)_n$, where $\Psi_n=x_n^{p-1}w_n$, for all $n\ge 1$ and let $\tilde{\Psi}=(\tilde{\Psi}_n)_n$ be the level sequence of $\Psi=(\Psi_n)_n$ with respect to $\varphi=(\varphi_n)_n$, $\varphi_n=1$ for all $n\ge 1$. By Theorem 3.3 in [5], Lemma 3.6 in [1] p.9 and Hölder inequality we obtain that

$$\|x\|_{p,w}^p=\sum_{n=1}^{\infty}x_n^p w_n=\sum_{n=1}^{\infty}x_n\Psi_n\le\sum_{n=1}^{\infty}x_n\tilde{\Psi}_n\le\sum_{n=1}^{\infty}x_n^\circ\tilde{\Psi}_n\le\left\|x^\circ\right\|_{p,w}\left\|\tilde{\Psi}\right\|_{p',w^{1-p'}}.$$

To obtain the second inequality in (7) it is sufficient to prove that

$$\left\|\tilde{\Psi}\right\|_{p',w^{1-p'}}\le C_{p,w}^{\frac{1}{p}}\|x\|_{p,w}^{p-1}, \tag{9}$$

where the constant $C_{p,w}$ is as in Lemma 2.4.

Let $E:=\{n\ge 1:\tilde{\Psi}_n=\Psi_n\}$. Then, we have that $\mathrm{Z}_+ - E=\cup_k I_k$, where $\{I_k\}$ are disjoint and such that

$$\tilde{\Psi}_n=\frac{1}{|I_k|+1}\sum_{m\in I_k}\Psi_m, \text{for all} \quad n\in I_k. \tag{10}$$

By Hölder's inequality we get

$$\sum_{m\in I_k}\Psi_m \le \left(\sum_{m\in I_k} w_m\right)^{\frac{1}{p}}\left(\sum_{m\in I_k} x_m^p w_m\right)^{\frac{1}{p'}}. \tag{11}$$

We also have that $\sum_{n\in E}(\tilde{\Psi}_n)^{p'} w_n^{1-p'} = \sum_{n\in E}(\Psi_n)^{p'} w_n^{1-p'} = \sum_{n\in E} x_n^p w_n$. Hence, by (10) and (11) it yields that

$$\sum_{n\in I_k}(\tilde{\Psi}_n)^{p'} w_n^{1-p'} \le \frac{1}{(|I_k|+1)^{p'}}\left(\sum_{m\in I_k} w_m\right)^{\frac{p'}{p}}\left(\sum_{m\in I_k} x_m^p w_m\right)\left(\sum_{n\in I_k} w_n^{1-p'}\right) \le C_{p,w}^{\frac{p'}{p}} \sum_{m\in I_k} x_m^p w_m .$$

From Lemma 2.4 it follows that (9) holds, which means that the right hand side inequality in (7) is proved. It only remains to prove the sharpness of the obtained inequalities.

We note that the left hand side inequality in (7) becomes equality if, for a fixed k_0 we take $x=(x_n)_n$, where $x_n = \begin{cases} \dfrac{k_0}{\sum_{i=1}^{k_0} w_i^{1-p'}} w_n^{1-p'}, & \text{if } n \le k_0, \\ 0, & \text{otherwise.} \end{cases}$

For the right hand side inequality in (7) we obtain equality for $x=(x_n)_n$ and $k_0 \in \mathrm{N}^*$ fixed, where $x_n = \begin{cases} 1 \text{ if } n \le k_0 \\ 0 \text{ otherwise} \end{cases}$, then we have that $\|x\|_{p,w} = \left(\sum_{n=1}^{k_0} w_n\right)^{\frac{1}{p}}$. It is easy to verify that $x^\circ = (x_n^\circ)_n$, where $x_n^\circ = \begin{cases} \dfrac{k_0}{\sum_{i=1}^{k_0} w_i^{1-p'}} w_n^{1-p'}, & \text{if } n \le k_0 \\ 0, & \text{otherwise,} \end{cases}$ and $\|x^\circ\|_{p,w} = k_0\left(\sum_{n=1}^{k_0} w_n^{1-p'}\right)^{-\frac{1}{p'}}$. Since k_0 is arbitrary we get that also the constant on the right hand-side inequality (7) is optimal.

The proof is complete.

Recall that for a sequence $x=(x_n)_n \in \ell^p(w)$, $1<p<\infty$ its dual norm is defined by $\|x\|'_{p,w} := \sup\left\{\sum_n x_n y_n : y=(y_n)_n, \|y\|_{p',w^{1-p'}} = 1\right\}$, where the supremum is taken over all sequences $y=(y_n)_n \in \ell^{p'}(w^{1-p'})$ with $\|y\|_{p',w^{1-p'}} = 1$.

From Lemma 2.1 and the Hardy-Littlewood inequality (see e.g. [2], p. 44), for any sequence $x=(x_n)_n \in d(p,w)$, $1<p<\infty$, we have that

$$\|x\|'_{p,w} = \sup\left\{\sum_{n=1}^{\infty} x_n^* y_n : \|y\|_{p',w^{1-p'}} = 1\right\}, \tag{12}$$

where the supremum is taken over all nonnegative and non-increasing sequences $y=(y_n)_n \in d(p', w^{1-p'})$ with $\|y\|_{p',w^{1-p'}} = 1$, and $x^* = (x_n^*)_n$ denotes the non-increasing rearrangement of the sequence $x=(x_n)_n$.

In the next Proposition, we summarize some well known results for $1<p<\infty$ and $x=(x_n)_n \in d(p,w)$. For the proof see e.g. [2] or [6].

PROPOSITION 3.2. *Let* $1<p<\infty$. *Then the following statements hold:*

(a) $\|x\|'_{p,w} \le \|x\|_{p,w}$.

(b) *If* w *is a decreasing sequence then*

$$\|x\|'_{p,w} = \|x\|_{p,w}. \tag{13}$$

(c) *If the sequence* $(x_n^* w_n)_n$ *is non-increasing, then (13) holds.*

(d) *If* $x = (x_n)_n$ *is an arbitrary nonnegative sequence in* $d(p,w)$*, then we have that*

$$\|x\|_{p,w} \le \inf_{x \prec z} \|z\|_{p,w}. \tag{14}$$

In the case of Lorentz spaces $L^{p,s}(R,\mu)$, it was proved by I. Halperin in [7] (see also [10], Theorem 3.6.5) that equality in (14) holds in the case of real functions defined on the interval $(0,1)$ and that the infimum is attained. For $p < s \le \infty$ a complete proof in the case of $L^{p,s}(R,\mu)$, where (R,μ) denotes a σ-finite nonatomic measure space, was given in the recent paper [3]. In the case when (R,μ) is a totally -finite measure space, completely atomic, with all atoms having the same measure the equality was proved in [5]. Our next result extends the result from [5].

THEOREM 3.3. *Let* $1 < p < \infty$. *Suppose that* $x = (x_n)_n \in d(p,w)$ *is a nonnegative and non-increasing sequence. Let* $\varphi = (\varphi_n)_n$, *where* $\varphi_n = w_n^{1-p'}$. *Then we have that*

$$\|x\|'_{p,w} = \inf_{x \prec z} \|z\|_{p,w} = \|x^\circ\|_{p,w},$$

where $x^\circ = (x_n^\circ)_n$ *is the level sequence of* $x = (x_n)_n$ *with respect to the sequence* $\varphi = (\varphi_n)_n$.

Proof. In view of (14), in Proposition 3.2 and Theorem 3.3 (b) in [5] it is sufficient to prove that

$$\|x\|'_{p,w} \ge \|x^\circ\|_{p,w}. \tag{15}$$

We denote by $E = \{n \ge 1 : x_n = x_n^\circ\}$. According to Theorem in [5] it yields that $\mathrm{N}^* - E = \cup_k I_k$, where I_k are such that

$$\sum_{n\in I_k} x_n = \sum_{n \in I_k} x_n^\circ. \tag{16}$$

We first consider $\Psi = (\Psi_n)_n$, where $\Psi_n = (x_n^\circ)^{p-1} w_n$ for all $n \ge 1$. As before, we have that $\|\Psi\|_{p',w^{1-p'}}^{p'} = \sum_{n=1}^{\infty} \Psi_n^{p'} w_n^{1-p'} = \|x^\circ\|_{p,w}^{p}$. We choose $y = (y_n)_n$, where $y_n = \dfrac{\Psi_n}{\|x^\circ\|_{p,w}^{p-1}}$, which implies that $\|y\|_{p',w^{1-p'}} = 1$.

From Theorem in [5], for each k we have that for $n \in I_k$, $x_n^\circ = \lambda_k w_n^{1-p'}$, where λ_k is a constant. Thus, $\Psi_n = \lambda_k^{p-1}$ and $\|x^\circ\|_{p,w}^{p-1} \sum_{n\in I_k} x_n y_n = \lambda_k^{p-1} \sum_{n\in I_k} x_n = \lambda_k^{p-1} \sum_{n \in I_k} x_n^\circ = \sum_{n\in I_k} (x_n^\circ)^p w_n$.

Moreover, we have that $\|x^\circ\|_{p,w}^{p-1} \sum_{n\in E} x_n y_n = \sum_{n \in E} (x_n^\circ)^p w_n$. Thus, we obtain that $\sum_{n=1}^{\infty} x_n y_n = \|x^\circ\|_{p,w}$. This implies (15) and the proof is complete.

The final result in this Section gives the sharp estimate of the standard norm via the dual norm.

THEOREM 3.4. *Let* $1 < p < \infty$. *Then, for any sequence* $x = (x_n)_n \in d(p,w)$ *and for any increasing weight* w *which satisfies condition (1), it yields that*

$$\|x\|_{p,w} \le C_{p,w} \|x\|'_{p,w}, \tag{17}$$

where $C_{p,w}$ *is defined by (6). The constant is optimal.*

The proof follows immediately from Theorem 3.3 and Theorem 3.1.

The next Theorem shows the coincidence of the dual and the decomposition norms.

THEOREM 3.5. *Let* $1 < p < \infty$ *and* (w_n) *be an increasing weight. Then, for any sequence* $x = (x_n)_n \in d(p,w)$ *we have*

$$\|x\|'_{p,w} = \|x\|_{(p,w)} . \tag{18}$$

Proof. The proof is similar with Theorem 5.2 from [5].

COROLLARY 3.6. *Let* $x = (x_n)_n \in d(p,w)$, $1 \le p < \infty$. *Then*

$$\|x\|_{(p,w)} = \|x^*\|_{(p,w)} \tag{19}$$

Proof. The equality (19) follows immediately from (18) and Lemma 2.1.

4. APPLICATIONS

From Theorem 3.4 and Theorem 3.5, we get the following "triangle inequality":

THEOREM 4.1. *Let* $1 < p < \infty$, w *an increasing weight which satisfies (1) and suppose that* $x^{(k)} = (x_n^{(k)})_n \in d(p,w)$, $k = 1, \ldots, N$. *Then the following inequality holds*

$$\left\| \sum_{k=1}^{N} x^{(k)} \right\|_{p,w} \le C_{p,w} \sum_{k=1}^{N} \|x^{(k)}\|_{p,w} , \tag{20}$$

where $C_{p,w}$, *given by (6) is the optimal constant.*

Proof. Let $1 < p < \infty$. Let us note that the inequality (20) is equivalent to the inequality

$$\|x\|_{p,w} \le C_{p,w} \|x\|_{(p,w)} , \tag{21}$$

where $x = (x_n)_n$, is in $d(p,w)$. Inequality (21) follows directly from Theorem 3.4 and Theorem 3.5. Moreover, for $x = (x_n)_n$ with $x_n = \begin{cases} 1 & \text{if } n \le k_0, \\ 0 & \text{otherwise} \end{cases}$ for a fixed $k_0 \in \mathrm{N}^*$ we obtain that $\|x\|_{p,w} = \left(\sum_{n=1}^{k_0} w_n\right)^{\frac{1}{p}}$.

From Theorem 3.3 and Theorem 3.5 we have that $\|x\|_{(p,w)} = k_0 / \left(\sum_{n=1}^{k_0} w_n^{1-p'} \right)^{\frac{1}{p'}}$

and therefore we get equality in (21). Thus, also the sharpness statement is proved and the proof is complete.

In the next Theorem we prove a discrete version of the Minkowski inequality:

THEOREM 4.2. *Let* $1 < p < \infty$ *and* $w = (w_n)$ *be a nonnegative increasing sequence of real numbers. Assume that* $x = (x_{n,m})_{n,m}$ *is a sequence with the property that for any* $m \in \mathrm{N}$ *the sequence* $x_m = (x_{n,m})_n$ *belongs to* $d(p,w)$ *and denote by* $X = (X_n)$ *the sequence given by* $X_n = \sum_{m=0}^{\infty} x_{n,m} v_m$, $n = 0,1,2\ldots,$, *where* v *is a nonnegative sequence such that* $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m} v_m < \infty$. *Then* $\|X\|_{p,w} \le C_{p,w} \sum_{m=0}^{\infty} \|x_m\|_{p,w} v_m$ *where* $C_{p,w}$ *is defined in (2.4) and it is the optimal constant in the inequality.*

Proof. By Theorem 3.4 we have

$$\|x_m\|_{p,w} \le C_{p,w} \|x_m\|'_{p,w} \tag{22}$$

where $C_{p,w}$ is defined in (6). Let now $y \in d(p', \tilde{w})$ such that $\|y\|_{p',\tilde{w}} = 1$. By Hölder's inequality we get

$$\sum_n X_n y_n = \sum_n \left(\sum_m x_{n,m} v_m \right) y_n = \sum_m \left(\sum_n x_{n,m} y_n \right) v_m \le \sum_m \|x_m\|_{p,w} \|y\|_{p',\tilde{w}} v_m = \sum_m \|x_m\|_{p,w} v_m .$$

By (22) we get our inequality. The inequality is sharp and this completes the proof of our theorem.

Remark 4.3. Inequality (20) was proved in [9] in a more general form but with different techniques.

ACKNOWLEDGMENTS

The first named author would like to thank the group of Mathematical Modelling of Karlstad University for support. The second and the third named authors were partially supported by CNCSIS-UEFISCSU project number 538/2009 PNII-IDEI code 1905/2008.

REFERENCES

1. G. BENNETT, *Factorizing the Classical Inequalities*, Mem. Amer. Math. Soc., **576**, 1996.
2. C. BENNETT, R. SHARPLEY, *Interpolation of Operators*, Academic Press, Boston, 1988.
3. S. BARZA, V. KOLYADA, J. SORIA, *Sharp constants related to the triangle inequality in Lorentz spaces*, Trans. Amer. Math. Soc., **361**, *10*, 555–5574 (2009).
4. S. BARZA, J. SORIA, *Sharp constants between equivalent norms in weighted Lorentz spaces*, J. Austral. Math. Soc., **88**, *1*, 19–27 (2010).
5. S. BARZA, A. N. MARCOCI, L.-E. PERSSON, *Best constants between equivalent norms in Lorentz sequence spaces*, J. Funct. Spaces Appl., Vol. 2012, Article ID 713534, 2012.
6. M. J. CARRO, J. A. RAPOSO, J. SORIA, *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Mem. Amer. Math. Soc., **187**, Providence, RI, 2007.
7. I. HALPERIN, *Function spaces*, Canad. J. Math., **5**, 273–288 (1953).
8. G. J. O. JAMESON, *The q-concavity constants of Lorentz sequence spaces and related inequalities*, Math. Z., **227**, 129–142 (1998).
9. A. KAMINSKA, A. M. PARRISH, *Covexity and concavity constants in Lorentz and Marcienkiewicz spaces*, J. Math. Anal. Appl., **343**, 337–351 (2008).
10. G. G. LORENTZ, *Bernstein polynomials*, Univ. of Toronto Press, Toronto, 1953.
11. G. SINNAMON, *Spaces defined by the level function and their duals*, Studia Math., **111**, *1*, 19–52 (1994).

Received December 28, 2011