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CNOIDAL WAVES, SOLITARY WAVES AND PAINLEVE ANALYSIS OF THE 5TH ORDER KDV EQUATION WITH DUAL-POWER LAW NONLINEARITY

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This paper studies the 5[th] order KdV equation with dual-power law nonlinearity. The non-topological as well as the singular soliton solution is obtained. Additionally the cnoidal wave solution is obtained for two values of the power law parameter. Finally, the Painleve analysis of the equation is also discussed.

Key words: 5[th] order KdV equation; cnoidal wave solution; Painleve analysis.

1. INTRODUCTION

There has been an overwhelming amount of research that has been going on in theory of nonlinear evolution equations (NLEEs). These NLEEs appear in various areas of scientific research, namely in Applied Mathematics, Theoretical and Applied Physics, Nonlinear Dynamics, Mathematical Biology, Mathematical Engineering as well as several other areas. One of the essential targets, in this area of research, is to carry out an integration of these NLEEs to extract several interesting solutions. These solutions will be very useful in various practial applications.

There are several tools that have been developed in the past couple of decades to integrate these NLEEs [1-20]. Some of these tools are the Adomian decomposition method, semi-inverse variational principle, homotopy perturbation method, exp-function method, simplest equation method, G'/G-expansion method and several others. These techniques of integration reveals several interesting and important solutions to various NLEEs. Some of these solutions are cnoidal waves, snoidal waves, shock waves, solitons and solitary waves, compactons, covatons, pekons, cuspons, stumpons, just to name a few.

This paper will focus on a particular NLEE, namely the 5th order Korteweg-de Vries (KdV) equation with dual-power law nonlinearity. The cnoidal waves and non-topological solitary waves solutions will be extracted for this NLEE, namely the 5th order KdV equation with dual-power law nonlineartity.

2. MATHEMATICAL ANALYSIS

Let us consider the equation [7]

$$q_t + (aq^n + bq^{2n})q_x + cq_{xxx} + kq_{xxxxx} = 0, \tag{1}$$

where $q = q(x,t)$ is the unknown function and a, b, c, k and $n > 0$ are constants with $ab > 0$. Without loss of generality we may suppose that $a = b = 1$. Indeed, the scaling

$$q(x,t) = \left(\frac{a}{b}\right)^{1/n} Q(\xi,\tau), \quad \text{where} \xi = \frac{b}{a^2}x \quad \text{and} \quad \tau = t. \tag{2}$$

gives us Equation (1) in the form

$$Q_\tau + (Q^n + Q^{2n})Q_\xi + \frac{b^3 c}{a^6} Q_{\xi\xi\xi} + \frac{b^5 k}{a^{10}} Q_{\xi\xi\xi\xi\xi} = 0. \tag{3}$$

In what follows, we will study following equation

$$q_t + (q^n + q^{2n})q_x + cq_{xxx} + kq_{xxxxx} = 0. \tag{4}$$

Let

$$q(x,t) = u^{2/n}(B(x - vt)) = u^{2/n}(\zeta), \quad \text{where} \zeta = B(x - vt). \tag{5}$$

Replacing (5) into (4) and simplifying we obtain following nonlinear ode :

$$\begin{aligned}
&-20B^4 kn\left(3n^3 - 11n^2 + 12n - 4\right)(u')^3 u'' u + 4B^4 k\left(6n^4 - 25n^3 + 35n^2 - 20n + 4\right)(u')^5 - \\
&-B^2(n-2)n^3\left(\left(5B^2 ku^{(4)} + 3cu''\right)u' + 10B^2 ku'' u'''\right)u^3 + \\
&+2B^2 n^2\left(n^2 - 3n + 2\right)\left(15B^2 k(u'')^2 + 10B^2 ku' u''' + c(u')^2\right)u' u^2 + \\
&+n^4\left(B^4 ku^{(5)} + B^2 cu''' - vu'\right)u^4 + n^4 u' u^8 + n^4 u' u^6 = 0.
\end{aligned} \tag{6}$$

2.1. Singular Solitons

We seek solutions to Equation (6) in the form

$$u(\zeta) = A\operatorname{csch}(\zeta). \tag{7}$$

Inserting ansatz (7) into (6) and after some algebra, we obtain following algebraic equation in the unknown $z = \log(\zeta)$:

$$\begin{aligned}
&\left(16B^4 k + 4B^2 cn^2 - n^4 v\right)z^8 + (4A^2 n^4 + 32B^4 kn^4 + 160B^4 kn^3 + 320B^4 kn^2 + 320B^4 kn + \\
&+64B^4 k + 8B^2 cn^4 + 24B^2 cn^3 + 4n^4 v)z^6 + (2(n^4\left(8A^4 - 4A^2 - 3v\right) + \\
&+16B^4 k\left(10n^4 + 40n^3 + 50n^2 + 20n + 3\right) - 4B^2 cn^2\left(2n^2 + 6n + 1\right))z^4 + \\
&+(4A^2 n^4 + 32B^4 kn^4 + 160B^4 kn^3 + 320B^4 kn^2 + 320B^4 kn + 64B^4 k + \\
&+8B^2 cn^4 + 24B^2 cn^3 + 4n^4 v)z^2 + (16B^4 k + 4B^2 cn^2 - n^4 v) = 0.
\end{aligned} \tag{8}$$

Equating to zero the coefficients of the different powers of z gives following algebraic system :

$$16B^4 k + 4B^2 cn^2 - n^4 v = 0.$$
$$4A^2 n^4 + 32B^4 kn^4 + 160B^4 kn^3 + 320B^4 kn^2 + 320B^4 kn + 64B^4 k + 8B^2 cn^4 + 24B^2 cn^3 + 4n^4 v = 0.$$
$$2(n^4\left(8A^4 - 4A^2 - 3v\right) + 16B^4 k\left(10n^4 + 40n^3 + 50n^2 + 20n + 3\right) - 4B^2 cn^2\left(2n^2 + 6n + 1\right) = 0.$$

Solving it with the aid of either Mathematica or Maple we obtain solutions. These are :

$$A = \pm\sqrt{\alpha}, \quad B = \pm\sqrt{\beta} \quad \text{and} \quad v = \frac{4B^2(4B^2 k + cn^2)}{n^4}, \tag{9}$$

where

$$\alpha = \pm\frac{\sqrt{R}ckn^2(n^4 + 5n^3 + 10n^2 + 10n + 4) \mp R}{2k^3 n^4(n^2 + 2n + 2)^3(n+1)(n+2)}, \tag{10}$$

$$\beta = \pm \frac{\sqrt{R} \mp ckn^2(n^4+5n^3+10n^2+10n+4)}{4k^2(n^2+2n+2)^2(n+1)(n+2)}, \tag{11}$$

$$R = -k^3 n^4 (n^2+2n+2)^2 \left(3n+2\right)\left(2n+1\right)\left(n+1\right)\left(n+2\right). \tag{12}$$

2.2 Non-Topological Solitons

The same procedure is valid for the ansatz

$$u(\zeta) = A\mathrm{sech}(\zeta). \tag{13}$$

In this case the corresponding system is

$$6B^4k + 4B^2cn^2 - n^4 v = 0$$
$$-A^2n^4 + 8B^4k\left(n^4+5n^3+10n^2+10n+4\right) + 2B^2cn^2\left(n^2+3n+2\right) = 0$$
$$A^2\left(4A^2+1\right)n^4 + 8B^4k\left(11n^4+45n^3+60n^2+30n+4\right) - 2B^2cn^2\left(n^2+3n+2\right) = 0,$$

and the solutions are

$$A = \pm\sqrt{\alpha'}, \quad B = \pm\sqrt{\beta'} \quad \text{and} \quad v = \frac{4B^2(B^2k+cn^2)}{n^4}, \tag{14}$$

where

$$\alpha' = \frac{\pm\sqrt{S}ckn^2\left(n^4+5n^3+10n^2+10n+4\right)+S}{2k^3n^4\left(n^2+2n+2\right)^3\left(n+1\right)\left(n+2\right)}, \tag{15}$$

$$\beta' = \frac{-ckn^2\left(n^4+5n^3+10n^2+10n+4\right)n^2 \mp \sqrt{S}}{4k^2\left(n^2+2n+2\right)^2\left(n+1\right)\left(n+2\right)}, \tag{16}$$

$$S = -k^3n^4\left(n^2+2n+2\right)^2\left(6n^4+25n^3+35n^2+20n+4\right). \tag{17}$$

3. CNOIDAL WAVES

Equation (1) admits cnoidal wave solutions in the special cases $n=1$, $n=2$, $n=\frac{1}{2}$ and $n=\frac{2}{3}$.

3.1. Case I: $n=1$

Equation (1) takes the form

$$q_t + (q+q^2)q_x + cq_{xxx} + kq_{xxxxx} = 0. \tag{18}$$

After the traveling wave transformation (5) the corresponding nonlinear equation (6) converts to

$$(u^3+u^5)u' + 10B^4ku''u''' + (3cu''+5B^2ku^{(4)})B^2u' + (B^4ku^{(5)} + B^2cu''' - vu')u = 0. \tag{19}$$

We seek solutions to Equation (19) in the cnoidal wave form

$$u = u(\zeta) = A\mathrm{cn}(\zeta, m). \tag{20}$$

Inserting ansatz (20) into (20) gives following algebraic equation in the variable $z = \mathrm{sn}(\zeta, m)$:

$$(A^4 + 360B^4km^4)z^4 + (12B^2m^2(c - 20B^2k(m^2+1) - A^2 - 2A^4)z^2 + C = 0, \tag{21}$$

where $C = A^2 + A^4 - 4B^2c(1+m^2) + 8B^4k(2m^4 + 13m^2 + 2) - v.$

Equating the coefficients of z^4, z^2 and z^0 to zero and solving the obtained algebraic system gives following solutions :

$$A = \sqrt{\frac{3m^2(5k^2(1-2m^2) + \sqrt{-10k^3(2m^2-1)^2c})}{10k^2(2m^2-1)^2}}, \tag{22}$$

$$B = \pm\frac{1}{2}\sqrt{\frac{2ck(1-2m^2) \pm \sqrt{-10k^3(2m^2-1)^2}}{10k^2(2m^2-1)^2}}, \tag{23}$$

$$v = A^2(A^2+1) + 4B^2(2(2m^4 + 13m^2 + 2)B^2k - (m^2+1)c). \tag{24}$$

We must choose the parameters c, m and k adequatelly in order to get a real valued function u. For example, let us consider the choice

$$A = \sqrt{\frac{3m^2(5k^2(1-2m^2) + \sqrt{-10k^3(2m^2-1)^2c})}{10k^2(2m^2-1)^2}}$$

and (25)

$$B = \frac{1}{2}\sqrt{\frac{2ck(1-2m^2) - \sqrt{-10k^3(2m^2-1)^2}}{10k^2(2m^2-1)^2}}.$$

These numbers are real if

$$c > 0 \quad \text{and} \quad -\frac{2c^2}{5} < k < 0 \quad \text{and} \quad |m| > \frac{1}{2}. \tag{26}$$

On the other hand, observe that we obtain a topological soliton in the limit when $m \to 1$, since

$$\lim_{m\to 1}\text{cn}(\zeta, m) = \text{sech}(\zeta). \tag{27}$$

3.2. Case II: $n = 2$

Equation (1) takes the form

$$q_t + (q^2 + q^4)q_x + cq_{xxx} + kq_{xxxxx} = 0. \tag{28}$$

After the traveling wave transformation (5) the corresponding nonlinear equation (6) converts to

$$(u^4 + u^2 - v)u' + B^2cu''' + B^4u^{(5)} = 0. \tag{29}$$

Integrating equation (29) once with respect to z gives

$$15B^4ku^{(4)} + 15B^2cu'' + 3u^5 + 5u^3 - 15vu = 0, \tag{30}$$

with constant of integration equal to zero.

We seek solutions to Equation (30) in the cnoidal wave form

$$u = u(\zeta) = A\text{cn}(\zeta, m). \tag{31}$$

Inserting ansatz (31) into (30) gives following algebraic equation in the variable $z = \text{sn}(\zeta, m)$:

$$3(A^4+120B^4km^4)z^4+(30B^2m^2(c-2B^2k(2m^2+5))-5A^2-6A^4)z^2+D=0, \tag{32}$$

where $D=5A^2+3A^4-15B^2c+15B^4k(4m^2+1)-15v.$

Equating the coefficients of z^4, z^2 and z^0 to zero and solcving the obtained algebraic system gives following solutions :

$$A=\pm\sqrt{\frac{10k^2(m^2-2)+\sqrt{-30k^3(m^2-2)^2c}}{5k^2(m^2-2)^2}}, \tag{33}$$

$$B=\sqrt{\frac{3ck(m^2-2)\pm\sqrt{-30k^3(m^2-2)c}}{30k^2(m^2-2)^2}}, \tag{34}$$

$$v=\frac{1}{15}(5A^2+3A^4+15B^2(B^2k-c)+60B^4km^2). \tag{35}$$

We must choose the parameters c, m and k adequatelly in order to get a real valued function u. For example, let us consider the choice

$$A=B=\sqrt{\frac{3ck(m^2-2)-\sqrt{-30k^3(m^2-2)c}}{30k^2(m^2-2)^2}} \quad \text{and} \quad \sqrt{\frac{10k^2(m^2-2)+\sqrt{-30k^3(m^2-2)^2c}}{5k^2(m^2-2)^2}} \tag{36}$$

These numbers are real if

$$c>0 \text{ and } -\frac{3c^2}{10}<k<0 \text{ and } |m|<\sqrt{2}. \tag{37}$$

4. PAINLEVE ANALYSIS

The Painleve, as a test for integrability of partial differential equations (PDEs), was proposed by Weiss, Tabor and Carnevale in 1983 [17]. It is a generalization of the singular point analysis of ordinary differential equations (ODEs), which dates back to the work of S. Kovalevsky in 1888. A PDE is said to possess the Painleve property if solutions of the PDE are single-valued in the neighbourhood of non-characteristic, movable singularity manifolds. Using the standard Kruskal's simplified method, we expand the solution q about a singular manifold $\phi(x,t)=0$ in an infinite series

$$q=\phi^{\alpha}\sum_{j=0}^{\infty}q_j\phi^j, \tag{38}$$

where $\phi(x,t)=x+\psi(t)$, $q_0\neq 0$ and α is negative integer determined by balancing the powers of ϕ of dominant terms in the equation. ϕ is a non-characteristic manifold. Coefficients q_j are functions of x and t. There are basically three steps in the Painleve analysis, viz, dominant behaviour analysis, finding the resonances, and checking whether arbitrary coefficients enter at the resonance values [17].

From the dominant behaviour analysis of equation (1), we get $\alpha=-\frac{2}{n}$, where $n>0$. Substituting (38) with $\alpha=-\frac{2}{n}$ into (1) leads to

$$q_0 = \left(-\frac{4k\left(2+3n\right)\left(1+2n\right)\left(2+n\right)\left(1+n\right)}{n^4} \right)^{\frac{1}{2n}}. \tag{39}$$

Substituting (38) into (1), it is found that resonances occur at

$$j = -1, \frac{2(1+2n)}{n}, \frac{4(1+n)}{n}, \frac{\frac{3}{2n} + 2 \pm \frac{\sqrt{-15n^2 - 40n - 16}}{2}}{n}. \tag{40}$$

There exist complex resonance points, so we conclude that equation (1) is failed in the Painleve test.

5. CONCLUSIONS

This paper studied the 5th order KdV equation with dual-power law nonlinearity. The cnoidal wave solution as well as soliton solutions were obtained. The two types of soliton solutions obtained are the non-topological soliton and the singular solitons. Additionally, the Painleve analysis was carried out where the resonance values are also discussed.

In future, further analysis will be carried out for this equation. While the adiabatic parameter dynamics was already obtained [2,7], the quasi-stationary soliton solutions will also be obtained by the aid of multiple scale analysis. The semi-inverse variational principle will be applied to integrate the perturbed 5th order KdV equation. Such results will be reported in future.

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