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ON BOOLEAN PRODUCT OF OPERATOR-VALUED LINEAR MAPS DEFINED ON *-ALGEBRAS

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We extend the proof of the main result in [10] to completely positive maps defined on *- algebras. Thus, we obtain a simpler new proof of the main result in [11]. This relies on a matrix Cauchy-Schwarz type inequality stated in [11], and uses an extension of a Paschke-Stormer positivity criterion to block- matrices over a C*-algebra established in [10].

Key words: universal free product (*-,C*-)algebra, Boolean product of linear maps, complete positivity, matrix Cauchy-Schwarz type inequality.

1. INTRODUCTION

The Boolean product of linear functionals on algebras, and the involved independence are basic in the so-called Boolean quantum probability theory, and related topics (see, e.g., [17, 14, 6, 1, 8]). This theory is one of the three noncommutative probability theories (the other being R.L.Hudson's Boson or Fermion probability theory and D.V. Voiculescu's free probability theory) arised from an associative product which does not depend on the order of its factors and fulfills a universal rule for mixed moments (due to R. Speicher's answer [16] to M. Schürmann's conjecture [15] on the universal products of *-algebraic probability spaces).

In [7], we considered the Boolean product for linear maps between algebras and directly proved (adapting Boca's ideas from [2]), it preserves the complete positivity in C*-algebraic setting. Further, in [10] we gave a new proof of this fact. Then, in [11] we generalized the proof of the main result in [7] to completely positive maps defined on *- algebras and valued in C*-algebras, through an adequate matrix Cauchy-Schwarz type inequality, and signalized that this inequality also permits another new proof of the mentioned result in [11].

In the present Note, we expose this second-simpler- proof through the method in [10], inspired by a technique due to M. Bozejko, M. Leinert, and R. Speicher from [5] concerning the positivity of the conditionally free product of positive linear functionals on involutive algebras.

2. COMPLETE POSITIVITY, MATRIX CAUCHY-SCHWARZ TYPE INEQUALITY AND BOOLEAN PRODUCT OF LINEAR MAPS

Let remind some preliminaries as in [7, 10, 11].

Let A be a (complex) *-algebra (i.e., a complex algebra endowed with a conjugate linear involution *, which is an anti-isomorphism). We consider the cone A_+ of positive elements in A consisting of finite sums $\sum a_i{}^* a_i$, with $a_i \in A$. Thus, A_+ determines a preorder structure on the real linear subspace of self-adjoint elements in A.

For any positive integer n, let $M_n(A)$ be the *-algebra of $n \times n$ matrices $[a_{ij}]$ with entries from A. When A is a C*-algebra, A_+ determines an order structure on the real linear subspace of self-adjoint elements in A, and $M_n(A)$ becomes a C*-algebra.

Let B be another *-algebra and $Q : A \longrightarrow B$ be a linear map. Denoting by $\widetilde{A}$ and $\widetilde{B}$ the unitizations of A and B [by **C**, the field of complex numbers], we denote by $\widetilde{Q} : \widetilde{A} \longrightarrow \widetilde{B}$ the unitization of Q given by $\widetilde{Q}(a \oplus \lambda 1) = Q(a) \oplus \lambda 1$ *(*$a \in A$, $\lambda \in$ **C**). We say Q is positive if $Q(A_+) \subset B_+$.

For any positive integer n, let $Q_n : M_n(A) \longrightarrow M_n(B)$ be the inflation map given by $Q_n([a_{ij}]) = [Q(a_{ij})]$, for $[a_{ij}] \in M_n(A)$. Then Q is called n-positive if the map Q_n induced by Q is positive. The map Q is completely positive if it is n-positive, for all positive integer n.

The following criterion of positivity for a matrix over a C*-algebra is due to W.L. Paschke and E. Stormer (see Prop. 6.1 in [12], and Th.2.2 in [18]).

PROPOSITION 2.1. *Let B be a C*-algebra and* $[b_{ij}]_{i,j=\overline{1,n}} \in M_n(B)$ *. Then* $[b_{ij}]_{i,j=\overline{1,n}} \in M_n(B)_+$ *if and only if* $\sum_{i,j=1}^{n} b_i^* b_{ij} b_j \in B_+$ *, for all* $b_1, \ldots, b_n \in B$.

An extension of this Paschke-Stormer positivity criterion to block-matices was stated in [10] (see Lemma 3.4 in [10]).

PROPOSITION 2.2. *Let n,N be positive integers*, *B be a C*-algebra*; $T_{ij} \in M_N(B)$ *, for* $i,j = 1,\ldots,n$; *and* $b_1,\ldots,b_n \in B^{\oplus N}$ *. For* $i,j = 1,\ldots,n$*, define* $S_{ij} := T_{ij}$*, if* $i \ge j$*, and* $S_{ij} := S_{ji}{}^*$*, if* $i < j$*; and, also* $s_{ij} := b_i{}^* S_{ij} b_j$*, if* $i \ge j$*, and* $s_{ij} := s_{ji}{}^*$*, if* $i < j$.

Then $[S_{ij}]_{i,j} \in M_{nN}(B)_+ \Leftrightarrow [s_{ij}]_{i,j} \in M_n(B)_+$. □

In his analysis on the method of constructing irreducible finite index subfactors of Popa [4], Boca obtained the following Cauchy-Schwarz type inequality for completely positive maps defined on *-algebras, via the Kolmogorov decomposition theorem for operator-valued positive definite kernels (see Lemma 3.5 in [4]) .

LEMMA 2.3. *Let A be a unital *-algebra, L(H) be the bounded linear operators on a Hilbert space H, and* $Q : A \longrightarrow L(H)$ *be a unital completely positive map.*

Then Q is a Schwarz map, i.e.,

$$Q(a^*a) \ge Q(a)^*Q(a) \text{, for all } a \in A.$$

The next matrix version of the previous Cauchy-Schwarz type inequality was naturally derived in [11] (see Lemma 3.1 in [11]).

LEMMA 2.4. *Let A be a unital *-algebra, B be a unital C*-algebra, and* $Q : A \longrightarrow B$ *be a unital completely positive map.*

Then

$$[Q(a_i{}^* a_j)]_{i,j=\overline{1,n}} \ge [Q(a_i)^* Q(a_j)]_{i,j=\overline{1,n}} \text{ in } M_n(B),$$

for all $n \ge 1$, *and all* $a_1,\ldots,a_n \in A$.

The universal free product (*-, C*-)algebra is the direct sum in the category of (complex) (*-, C*-) algebras, non necessary unital [2, 3, 7, 10, 11, 15, 20].

As linear space, a realization of the universal free product corresponding to a family of (*-)algebras $(A_i)_{i\in I}$ is

$$A=\oplus_{n\geq 1}\oplus_{i_1\neq\cdots\neq i_n} A_{i_1}\otimes\ldots\otimes A_{i_n}.$$

In natural way, A is organized as a (*-)algebra.

Let B and A_i be (complex) algebras, and $Q_i : A_i \longrightarrow B$ be linear maps; $i\in I$.

In [7], we considered the Boolean product $Q=\bullet Q_i$ as the unique linear map defined on the universal free product A of the algebras A_i, $i\in I$, such that

$$Q(a_1...a_n)=Q_{i_1}(a_1)...Q_{i_n}(a_n)$$

for all $n\geq 1$, with $i_1\neq\ldots\neq i_n$, and $a_k\in A_{i_k}$, if $k=1,\ldots,n$; relatively to the natural embeddings of A_i into A emerging from the free product construction.

When B and A_i are (complex) *-algebras, and Q_i are Hermitian maps (in particular, e.g., when A_i and B are unital, and Q_i are unital positive maps), Q naturally becomes Hermitian.

The following main result of [11] (see Th. 3.2, or Cor. 3.7 in [11]) was proved by adapting Boca's method in [2], via the above Lemma 2.4.

THEOREM 2.5. *Let A_i be *-algebras, B be a C*-algebra, and $Q_i : A_i \longrightarrow B$ be linear maps, such that their unitizations are completely positive; $i\in I$.*

*Let $Q=\bullet Q_i$ be the Boolean product of $(Q_i)_{i\in I}$ defined on the *-algebraic free product A of $(A_i)_{i\in I}$.*

Then

$$[Q(a_i{}^*a_j)]_{i,j=\overline{1,n}}\geq[Q(a_i)^*Q(a_j)]_{i,j=\overline{1,n}} \text{ in } M_n(B),$$

for all $n\geq 1$, and all $a_1,\ldots,a_n\in A$..

Therefore [the unitization of] Q is completely positive.

3. THE SIMPLER NEW PROOF OF THE THEOREM 2.5

We present now the other- simpler - proof (announced in [11]) of the Theorem 2.5, through the method in [10], inspired by a Bozejko-Leinert-Speicher technique from [5] (see Section 3 in [10]).

If A is the algebraic free product of a family $(A_i)_{i\in I}$ of algebras, we denote by $W=\{a_1...a_n; n\geq 1, a_k\in A_{i_k}, i_1\neq\ldots\neq i_n\}$ the set of reduced words in A.

For $w=a_1...a_n\in W$, we call n the length of w and consider a_1 the first letter of w.

If $x=\sum_k w^{(k)}\in A$ call the length of x the maximal length in this representation of x.

The preliminary step to our actual aim is the next extension of Lemma 3.2 in [10].

Lemma 3.1. *Let A_i be *-algebras, B be a C *-algebra, and $Q_i : A_i \longrightarrow B$ be linear maps, such that their unitizations are completely positive; $i\in I$.*

*Let $Q=\bullet Q_i$ be the Boolean product of $(Q_i)_{i\in I}$ defined on the *-algebraic free product A of $(A_i)_{i\in I}$.*

Then [the unitization of] Q is a Schwarz map, i.e.,

$$Q(a^*a)\geq Q(a)^*Q(a), \text{ for all } a\in A.$$

In particular, Q is a positive map.

Proof. The definition of the Boolean product ensures us it is enough to prove the necessary inequality for every random variable $x(i)$ in A represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i ; if $i \in I$.

Assume that such a word $x(i)$ has p terms of length one**;** otherwise, the argument is similar**.**

Therefore, $x(i) = \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} y^{(k)}$, with $a^{(k)} \in A_i$ **;** and $y^{(k)} \in W$**,** but the first letter of $y^{(k)}$ does not belong to A_i .

Let $b_k = 1 \in \widetilde{B}$ **,** for $k = \overline{1, p}$ **;** and $b_k = Q(y^{(k)})$**,** for $k = \overline{p+1, N}$ **.**

Remark (as in [10]) that

$$Q(x(i) * x(i)) - Q(x(i)) * Q(x(i)) = \sum_{k,l=1}^{N} b_k^* b_{kl} b_l \ ,$$

where

$$b_{kl} := Q_i(a^{(k)} * a^{(l)}) - Q_i(a^{(k)}) * Q_i(a^{(l)}) \, .$$

Therefore, the above matrix Cauchy-Schwarz type inequality in Lemma 2.4, applied to $\widetilde{Q}_i$, implies the positivity of $[b_{kl}]_{k,l}$ in $M_N(B)$.

Consequently, it remains to conclude that $\sum_{k,l=1}^{N} b_k^* b_{kl} b_l \in B_+$, by the Paschke- Stormer criterion for the positivity of a matrix in $M_N(B)$ (i.e., Prop. 2.1); since Q is a Schwarz map if and only if $\widetilde{Q}$ is such a map. □

In particular, we get the following fact (see, e.g., [15] or Cor.3.4 in [11]).

Corollary 3.2. *Let φ_i be linear functionals defined on (complex) *-algebras, such that their unitizations are positive; $i \in I$. Let $\varphi = \bullet\varphi_i$ be the Boolean product defined on the *-algebraic free product A of $(A_i)_{i \in I}$.*

*Then $\varphi(a^* a) \geq |\varphi(a)|^2$, for all $a \in A$***.**

Thus the Boolean product $\varphi = \bullet\varphi_i$ is positive, too.

Let present now the other steps toward the announced goal.

Proof of the Theorem 2.5. In the light of the previous Lemma 3.1, it remains to consider in the sequel that $n \geq 2$.

Due to the definition of the Boolean product, it suffices to prove the inequality in Theorem 2.5 only for all words $x_s(i)$ in A, $s = 1,...,n$, represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i ; if $i \in I$; such that these representations contain the same number of terms, for all $s = 1,...,n$.

Assume that every such $x_s(i)$ with $i = \overline{1, p}$ has the length 1; every such $x_s(i)$ with $i = \overline{p+1, n}$ has the length greater or equal to 2, and p_s terms of length 1; otherwise, the argument is similar.

Therefore, $x_s(i) = \sum_{k=1}^{N} a_s^{(k)}$ for $s = \overline{1, p}$ **;** and $x_s(i) = \sum_{k=1}^{p_s} a_s^{(k)} + \sum_{k=p_s+1}^{N} a_s^{(k)} y_s^k$, for $s = \overline{p+1, n}$ **;** with some $a_s^{(k)} \in A_i$ **;** and $y_s^{(k)} \in W$**,** but the first letter of $y_s^{(k)}$ does not belong to A_i .

Denote $b_s^{(k)} = 1 \in \widetilde{B}$, for all $k = \overline{1, N}$, if $s = \overline{1, p}$; but $b_s^{(k)} = 1 \in \widetilde{B}$**,** for $k = \overline{1, p_s}$, and $b_s^{(k)} = Q(y_s^{(k)})$**,** for $k = \overline{p_s + 1, N}$**,** if $s = \overline{p+1, n}$ **.**

Then, remark that we may write (as in [10])

$$Q(x_s(i)*x_t(i)) - Q(x_s(i))*Q(x_t(i)) = \sum_{k,l=1}^{N} b_s^{(k)}*b_{s,t}(k,l)b_t^{(l)} \quad \text{in } B,$$

where,

$$b_{s,t}(k,l) := Q_i(a_s^{(k)}*a_t^{(l)}) - Q_i(a_s^{(k)})*Q_i(a_t^{(l)}) \text{ for } k,l=\overline{1,N}, \text{ and } s,t=\overline{1,n}.$$

In consequence, by the matrix Cauchy-Schwarz type inequality in Lemma 2.4, applied to $\widetilde{Q}_i$ again, we deduce that $[b_{s,t}(k,l)]_{s,t=\overline{1,n}}^{k,l=\overline{1,N}}$ belongs to $M_{nN}(B)_+$.

Therefore, we may conclude, because the positivity of $[\sum_{k,l=1}^{N} b_s^{(k)}*b_{s,t}(k,l)b_t^{(l)}]_{s,t=\overline{1,n}}$ in $M_n(B)$ results via the extension of the Paschke-Stormer positivity criterion to block-matrices over a C*-algebra in Prop. 2.2, and the inequality in Theorem 2.5 is equivalent to the same kind of inequality corresponding to $\widetilde{Q}$. □

We denote by $*_0\, A_i$ the non-unital universal (or full) free product C*-algebra corresponding to a family $(A_i)_{i\in I}$ of C*-algebras.

After separation and completion of the corresponding universal free product *-algebra A in its enveloping C*-seminorm

$$\|a\| = \sup\{\|\pi(a)\|; \pi \ \ *\text{-representation of } A \textit{ as bounded operators} \text{ on a Hilbert space}\},$$

one can realize the universal (or full) free product $*_0\, A_i$ in the category of C*-algebras (see, e.g., [7, 10, 11, 20]).

The final fact is an extension in our actual framework of Cor. 3 in [7], being a Boolean analogue of Cor. 4 in [3], and a noncommutative analogue of Th. 10.8 in [13](or Prop. 4.23 in Chap. IV of [19]).

Corollary 3.3 *Let A_i be *-algebras, B_i be C*-algebras, and $Q_i : A_i \longrightarrow B_i$ be linear maps, such that their unitizations are completely positive; $i \in I$.*

*Let A and $B := *_0\, B_i$ be the *-algebraic free product of $(A_i)_{i\in I}$ and, respectively, the* C**-algebraic non-unital full free product of $(B_i)_{i\in I}$.*

Then there exists a common extension $Q : A \longrightarrow B$ such that:

1) $[Q(a_i * a_j)]_{i,j=\overline{1,n}} \geq [Q(a_i)*Q(a_j)]_{i,j=\overline{1,n}}$ *in* $M_n(B)$, *for all* $n \geq 1$, *and all* $a_1,...,a_n \in A$;
   and
2) $Q(a_1...a_n) = Q_{i_1}(a_1)...Q_{i_n}(a_n)$, *for all* $n \geq 1$, *with* $i_1 \neq ... \neq i_n$, *and* $a_k \in A_{i_k}$, *if* $k = 1,...,n$; *with respect to the natural embeddings of A_i into A and, respectively, of B_i into B, arising from the free product constructions.*

In particular, Q is completely positive.

In the same way as before, one can prove that the amalgamated Boolean (or, moreover, conditionally free) product of linear maps defined on *-algebras and valued in C*-algebras preserves the complete positivity (for these and other extensions, see, e.g., [8, 9]).

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