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N-HOMOMORPHISM AMENABILITY

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In this paper, the notion of n-homomorphism amenability of Banach algebras is introduced. It is shown that n-homomorphism amenability for the group algebra $L^1(G)$ and the generalized Fourier algebra $A_p(G)$ ($p \in (1,+\infty)$) is equivalent to the amenability of the underlying group G.

Key words: amenability, Banach algebra, n-homomorphism.

1. INTRODUCTION

The concept of amenability for Banach algebras was introduced and studied for the first time by Johnson in [4]. He proved that $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable locally compact group. Since then several variants of this concept by using homomorphisms have appeared in the literature, each as a kind of cohomological triviality (for instance, see [1] and [8]).

Kaniuth, Lau and Pym [5] investigated the concept of φ-amenability of a Banach algebra A, where φ is a character on A (see also [6]). At that time and along with them, Monfared introduced the notion of character amenability for Banach algebras in [7]. By using results in [5], he characterized the structure of left (right) character amenable Banach algebras in several ways, and showed that for any locally compact group G, left (right) character amenability of the group algebra $L^1(G)$, is equivalent to the amenability of G.

Let A be a (Banach) algebra and let n be an arbitrary and fixed natural number. A linear map ϕ from A to the set of complex numbers C is called n-homomorphism if $\phi(a_1 a_2 ... a_n) = \phi(a_1)\phi(a_2)...\phi(a_n)$ for all $a_1, a_2, ..., a_n \in \mathsf{A}$. Obviously, every homomorphism is a n-homomorphism, but converse is false, in general.

For any Banach space X and Banach algebra A, each n-homomorphism ϕ on A induces a (innumerable) module structure(s) on X. In this paper, we employ this structure(s) and introduce the concept of n-ϕ-amenability for A and characterize it in terms of first Hochschild cohomology group of A with coefficients in X^*. We also define the notion n-homomorphism amenability for Banach algebras and show that for a locally compact group G, the group algebra $L^1(G)$ and the generalized Fourier algebra $A_p(G)$ ($p \in (1,+\infty)$) are n-homomorphism amenable if and only if G is amenable. Also, the measure algebra $M(G)$ is n-homomorphism amenable if and only if G is a discrete amenable group.

2. MAIN RESULTS

Let A be a Banach algebra, and let X be a Banach A-bimodule. A bounded linear map $D : \mathsf{A} \to \mathsf{X}$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathsf{A}).$$

For each $x \in \mathsf{X}$, we define a map $D_x : \mathsf{A} \to \mathsf{X}$ by

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathsf{A}).$$

It is easily seen that D_x is a derivation. Derivations of this form are called *inner derivations*. We use the notations $\mathsf{Z}^1(\mathsf{A},\mathsf{X})$ for the space of all continuous derivations from A into X and $\mathsf{N}^1(\mathsf{A},\mathsf{X})$ for the space of all inner derivations from A into X. The first Hochschild cohomology group of A with coefficients in X is the quotient space

$$\mathsf{H}^1(\mathsf{A},\mathsf{X}) = \mathsf{Z}^1(\mathsf{A},\mathsf{X})/\mathsf{N}^1(\mathsf{A},\mathsf{X}).$$

Let X be a A-bimodule. Then the dual space X^* of X is also a Banach A-bimodule by the following module actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle,\ \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \quad (a \in \mathsf{A}, x \in \mathsf{X}, f \in \mathsf{X}^*).$$

With the above notations, a Banach algebra A is called *amenable* if $\mathsf{H}^1(\mathsf{A},\mathsf{X}^*) = \{0\}$ for every Banach A-bimodule X.

Throughout this paper, n is a fixed natural number, A is a Banach algebra and $\sigma^{(n)}(\mathsf{A})$ is the set of all non-zero bounded linear n-homomorphisms from A to C. We denote $\sigma^{(2)}(\mathsf{A})$ by $\sigma(\mathsf{A})$.

Let $\phi \in \sigma^{(n)}(\mathsf{A})$ and choose $u \in \mathsf{A}$ such that $\phi(u) = 1$. If X is a Banach space, then X can be viewed as a Banach left A-module by the following action:

$$a \cdot x = \phi(u^n \ \ a)x, \ (a \in \mathrm{A},\ x \in \mathsf{X}). \tag{2.1}$$

As it will be shown in Theorem 2.3, $a \cdot (b \cdot x) = ab \cdot x = \phi(a)\phi(b)x$ for all $a,b \in \mathsf{A}$ and $x \in \mathsf{X}$, and so this equality is independent from the choice of u. Similarly, we can define a right action A on X by $x \cdot a = \phi(au^n)x$ for all $a \in \mathsf{A}$ and $x \in \mathsf{X}$.

Suppose that the left action of A on X is given by (2.1). Then it is easy to check that $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ for all $a,b \in \mathsf{A}$ and $x \in \mathsf{X}$. Therefore, the Banach space X admits a A-bimodule structure dependent on an element $u \in \mathsf{A}$ with $\phi(u) = 1$. One can also verify that in this case the right action of A on the dual A-bimodule X^* will be $f \cdot a = \phi(u^n a) f$ for all $a \in \mathsf{A}$ and $f \in \mathsf{X}^*$. Let A be a Banach algebra, A^{**} be its second dual and $m \in \mathsf{A}^{**}$. Consider $\phi \in \sigma^{(n)}(\mathsf{A})$ such that $\phi(u) = 1$ for some $u \in \mathsf{A}$. Then m is said to be n-ϕ-*mean* on A^* (at u) if $m(\phi) = 1$ and $m(f \cdot a) = \phi(u^n a) m(f)$ for all $f \in \mathsf{A}^*$ and $a \in \mathsf{A}$. Also, A is called n-ϕ-*amenable* if there exists a n-ϕ-mean m on A^*. We say A is n-0-amenable if $\mathsf{H}^1(\mathsf{A},\mathsf{X}^*) = \{0\}$, for any Banach A-bimodule X for which the left action A on X is zero.

Note that if ϕ is a non-zero multiplicative linear functional on A, then the left module structure (2.1) and the definition of n-ϕ-mean (n-ϕ-amenability) will absolutely overlap with ϕ-amenability (character amenability) of A which has been introduced in [5] ([7]).

In the next theorem which is our main result in this paper, we characterize n-ϕ-amenability of a Banach algebra in terms of Hochschild cohomology groups.

THEOREM 2.1. *Let* A *be a Banach algebra and* $\phi \in \sigma^{(n)}(\mathsf{A})$ *such that* $\phi(u) = 1$. *Then the following are equivalent:*

(i) A *is* n-ϕ-*amenable (at* u*);*

(ii) *If* X *is a Banach* A-*bimodule in which the left action is given by* $a \cdot x = \phi(u^n a)x$ *for all* $a \in \mathsf{A}$ *and* $x \in \mathsf{X}$, *then* $\mathsf{H}^1(\mathsf{A},\mathsf{X}^*) = \{0\}$.

Proof. (i)$\Rightarrow$ (ii): Assume that m is a n-ϕ-mean in A^{**} and $D : \mathsf{A} \to \mathsf{X}^*$ is a module (n-ϕ-) derivation. Let $f = (D')^*(m) \in \mathsf{X}^*$, where D' is the restriction of D^* to X. For each $x \in \mathsf{X}, a,b \in \mathsf{A}$, we have

$$\langle D'(a\cdot x),b\rangle=\langle D(b),a\cdot x\rangle=\phi(u^n a)\langle D(b),x\rangle=\phi(u^n a)\langle D'(x),b\rangle,$$

and so $D'(a\cdot x)=\phi(u^n a)D'(x)$. Hence,

$$\langle f\cdot a,x\rangle=\langle f,a\cdot x\rangle=\langle m,D'(a\cdot x)\rangle=\phi(u^n a)\langle m,D'(x)\rangle=\phi(u^n a)\langle f,x\rangle,$$

and thus

$$f\cdot a=\phi(u^n a)f \quad (2.2)$$

On the other hand, for each $x\in \mathsf{X}, a,b\in \mathsf{A}$ we get

$$\begin{aligned}\langle D'(x\cdot a),b\rangle&=\langle D(b),x\cdot a\rangle=\langle a\cdot D(b),x\rangle=\langle D(ab),x\rangle-\langle D(a)\cdot b,x\rangle=\\&=\langle D'(x),ab\rangle-\langle D(a),b\cdot x\rangle=\langle D'(x)\cdot a,b\rangle-\phi(u^n b)\langle D(a),x\rangle=\\&=\langle D'(x)\cdot a,b\rangle-\phi(u)^{(n-2)}\phi(u^2)\phi(b)\langle D(a),x\rangle.\end{aligned}$$

Therefore

$$D'(x\cdot a)=D'(x)\cdot a-\phi(u^2)\langle D(a),x\rangle\phi. \quad (2.3)$$

for all $x\in \mathsf{X}, a\in \mathsf{A}$. Note that $\phi(u^2)$ could not be zero. In other words, if $\phi(u^2)=0$, then $\phi(u^n a)=\phi(u)^{(n-2)}\phi(u^2)\phi(a)=0$ for all $a\in \mathsf{A}$. This contradicts our assumption that left action A on X is non-zero. Now, it follows from the definition of D' and (2.3) that

$$\begin{aligned}\langle a\cdot f,x\rangle&=\langle f,x\cdot a\rangle=\langle m,D'(x\cdot a)\rangle\\&=\langle m,D'(x)\cdot a\rangle-\langle D(a),x\rangle m(\phi)\phi(u^2)=\\&=\langle m,D'(x)\rangle\phi(u^n a)-\langle D(a),x\rangle\phi(u^2)=\\&=\langle f,x\rangle\phi(u^n a)-\langle D(a),x\rangle\phi(u^2).\end{aligned}$$

Then

$$D(a)\phi(u^2)=\phi(u^n a)f-a\cdot f. \quad (2.4)$$

Using (2.2) and (2.4), we obtain $D(a)=a\cdot g-g\cdot a$, where $g=-\frac{1}{\phi(u^2)}f$. Therefore D is an inner derivation.

(ii)$\Rightarrow$ (i): First we show that $\phi(u^n a)=\phi(au^n)$ for all $a\in \mathsf{A}$. Indeed,

$$\begin{aligned}\phi(u^n a)&=\phi(u)^{n-1}\phi(ua)=\phi(u)^{n-2}\phi(ua)\phi(u)\\&=\phi(u^{n-2}uau)=\phi(u)^{n-1}\phi(au)\\&=\phi(au)\phi(u)^{n-1}=\phi(au^n).\end{aligned}$$

Since $\phi(u^2)\neq 0$, $\phi(ab)=\phi(ba)$ for all $a,b\in \mathsf{A}$. Thus

$$a\cdot\phi=\phi\cdot a=\phi(u^n a)\phi \quad (a\in \mathsf{A}).$$

Therefore the set $\Omega=\{c\phi : c\in \mathsf{C}\}$ is a closed A-submodule of A^*. Put $\mathsf{X}=\mathsf{A}^*/\Omega$ and let $P:\mathsf{A}^*\to \mathsf{X}$ be the projection map. Take $\Psi\in \mathsf{A}^{**}$ such that $\Psi(\phi)=1$. Consider D_Ψ as the inner derivation from A into A^{**}. We have

$$\langle D_\Psi(a),\phi\rangle=\langle a\cdot\Psi-\Psi\cdot a,\phi\rangle=\langle \Psi,\phi\cdot a-a\cdot\phi\rangle=0 \quad (a\in \mathsf{A}).$$

Thus $D_\Psi(a)$ belongs to the range of P^*. Since P^* is monomorphism, there exists a unique element $D(a)\in \mathsf{X}^*$ such that $P^*(D(a))=D_\Psi(a)$. The map which is defined as above is a derivation on A. By hypotheses, there exists $\varphi\in \mathsf{X}^*$ such that

$$D(a)=a\cdot\varphi-\varphi\cdot a \quad (a\in \mathsf{A}).$$

Now, for each $a\in \mathsf{A}$

$$a \cdot P^*(\varphi) - P^*(\varphi) \cdot a = P^*(a \cdot \varphi - \varphi \cdot a) = P^*(D(a)) = D_\Psi(a) = a \cdot \Psi - \Psi \cdot a.$$

Put $m = \Psi - P^*(\varphi)$. Then it is easy to see that m is a n-ϕ-mean on A^*.

The proving process of the above theorem shows that if we replace u by another element v in A such that $\phi(v) = 1$, even although the left module structure on X will be different, all assertions are still equivalent. Hence, n-ϕ-amenability of A is independent from choice of u and it is enough that Theorem 2.1 holds for some $u \in \mathsf{A}$ with $\phi(u) = 1$ and thus, we remove the word u if there is no risk of confusion. Now, this Theorem leads us to the following definition:

Definition 3.3. A Banach algebra A is said to be n-homomorphism amenable if Theorem 2.1 holds for all $\phi \in \sigma^{(n)}(A) \cup \{0\}$.

Recall that a Banach algebra A is right (left) character amenable if for all $\phi \in \sigma(\mathsf{A}) \cup \{0\}$ and all Banach A-bimodule X for which the left (right) module action is given by $a \cdot x = \phi(a)x$ ($x \cdot a = \phi(a)x$); $a \in \mathsf{A}, x \in \mathsf{X}$, every continuous derivation $D: \mathsf{A} \to \mathsf{X}^*$ is inner.

THEOREM 2.3. *Let $n \in \mathbb{N}$ and A be a Banach algebra. Then A is a right character amenable if and only if it is n-homomorphism amenable.*

Proof. Obviously, the zero map on A is n-homomorphism for all $n \in \mathbb{N}$. Now, let A be n- homomorphism amenable and ϕ be a character on A. Then ϕ is a n-homomorphism on A. Assume that A is n-ϕ-amenable (at u). We consider A-bimodule structure on a Banach space X by taking the left action to be $a \cdot x = \phi(a)x$; $a \in \mathsf{A}$, $x \in \mathsf{X}$ and the right action to be the natural one. Define a left action A on X as $a \bullet x = u^n a \cdot x$. Since ϕ is a character, $a \bullet x = \phi(u^n a)x = \phi(u)^n \phi(a)x = \phi(a)x$. These equalities show that both left actions A over X are equal. Thus A is ϕ-amenable, and so it is right character amenable. Conversly, suppose that A is a right character amenable and ϕ belongs to $\sigma^{(n)}(A) \cup \{0\}$. Define $\tilde{\phi}(a) := \phi(u^n a); a \in \mathsf{A}$ with $\phi(u) = 1$. Then

$$\begin{aligned}\tilde{\phi}(ab) &= \phi(u^n ab) = \phi(u^3)\phi(u)^{n-3}\phi(a)\phi(b) = \\ &= \phi(u)^{n-1}\phi(u^3)\phi(u)^{n-2}\phi(a)\phi(b) = \phi(u^{n+2})\phi(u^{n-2}ab) = \\ &= \phi(u^2)\phi(u)^{n-2}\phi(u^2)\phi(u)^{n-2}\phi(a)\phi(b) = \\ &= \phi(u^2)\phi(u)^{n-2}\phi(a)\phi(u^2)\phi(u)^{n-2}\phi(b) = \\ &= \phi(u^n a)\phi(u^n b) = \tilde{\phi}(a)\tilde{\phi}(b).\end{aligned}$$

The above equalities show that $\tilde{\phi}$ is a character on A. Since A is $\tilde{\phi}$-amenable, every derivation $D: \mathsf{A} \to \mathsf{X}^*$ in which $a \cdot x = \tilde{\phi}(a)x$, is inner. Therefore A is n-ϕ-amenable.

Let G be a locally compact group. Then, the Fourier algebra $A(G)$ and the generalized Fourier algebra $A_p(G)$, $p \in (1, +\infty)$, which were introduced in [2] and [3] are commutative Banach algebras with pointwise operations of addition and multiplication. It is proved in [7, Corollary 2.4] that $A_p(G)$ is character amenable if and only if G is amenable locally compact group and by Theorem 2.3 we have the following corollary.

COROLLARY 2.4. *Let $1 < p < \infty$, G be a locally compact group, and A be either Banach algebras $L^1(G)$ or $A_p(G)$. Then the following are equivalent:*

(i) A *is right character amenable;*
(ii) A *is n-homomorphism amenable;*
(iii) G *is amenable.*

Proof. The equivalence of (i) and (ii) follows from Theorem 2.3 and the equivalence of (i) and (iii) has been shown in [7, Corollary 2.4].

For a locally compact group G, n-homomorphism amenability of the measure algebra $M(G)$ is characterized in the next result.

COROLLARY 2.5. *The measure algebra* $M(G)$ *is* n*-homomorphism amenable if and only if* G *is a discrete amenable group.*

Proof. The result follows immediately from Theorem 2.3 and [7, Corollary 2.5].

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