THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 14, Number 2/2013, pp. 106–110

WEIGHTED L^p-SPACE AS A SEGAL ALGEBRA

Fatemeh ABTAHI

University of Isfahan, Department of mathematics, Isfahan, Iran
E-mail: f.abtahi@sci.ui.ac.ir

Let G be a locally compact group, ω be a weight function on G and $1 < p < \infty$. In the present note, it is proved that $L^p(G,\omega)$ can be considered as a Segal algebra or an abstract Segal algebra with respect to $L^1(G)$, just when G is compact.

Key words: abstract segal algebra, segal algebra, weighted L^p -space.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper G is a locally compact group, all integrals are taken with respect to a fixed left Haar measure λ and $1 < p < \infty$. We call any positive Borel measurable function ω on G a weight function. It is called locally summable if $\omega \in L^1(A)$, for each compact subset A of G with positive measure. For $x \in G$, define the function θ_x on G by

$$\theta_x(y) = \frac{\omega(xy)}{\omega(y)} \quad (y \in G).$$

By a weight of moderate growth, we mean ω with $\theta_x \in L^\infty(G)$, for all $x \in G$; that is

$$L_x = \| \theta_x \|_\infty = \operatorname{ess\,sup}_{y\in G} \frac{\omega(xy)}{\omega(y)} < \infty.$$

The space $L^p(G, \omega)$ with respect to λ is the set of all complex valued measurable functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space as defined in [7]; we denote this space by $\ell^p(G, \omega)$, where the case G is discrete. Then $L^p(G, \omega)$ is a Banach space with the norm $\| \cdot \|_{p,\omega}$, defined by $\| f \|_{p,\omega} = \| f\omega \|_p$ for each $f \in L^p(G, \omega)$. The dual space of $L^p(G, \omega)$ is the Banach space $L^q(G, \omega^{-1})$ of all functions g on G with $g\omega^{-1} \in L^q(G)$ under duality

$$\langle f, g\rangle = \int_G f(x)g(x)\,\mathrm{d}\lambda(x),$$

for all $f \in L^p(G, \omega)$ and $g \in L^q(G, \omega^{-1})$, where q is the exponential conjugate of p defined by $p^{-1} + q^{-1} = 1$. For measurable functions f and g on G, the convolution multiplication

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,\mathrm{d}\lambda(y)$$

is defined at each point $x \in G$ for which this makes sense.

A linear subspace $S(G)$ of the convolution group algebra $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions:

(*i*) $S(G)$ is dense in $L^1(G)$;

(*ii*) $S(G)$ is a Banach space under some norm $\|.\|_{S(G)}$ and $\|f\|_1 \leq \|f\|_{S(G)}$, for each $f \in S(G)$;

(*iii*) $S(G)$ is left translation invariant; i.e. $\|_x f\|_{S(G)} = \|f\|_{S(G)}$ for all $x \in G$ and $f \in S(G)$, and the map $x \mapsto {}_x f$ from G into $S(G)$ is continuous, where ${}_x f(y) = f(xy)$ for all $y \in G$.

Let (A, $\|.\|_A$) be a Banach algebra. Then (B , $\|.\|_B$) is an abstract Segal algebra with respect to (A , $\|.\|_A$) if :

(*i*) B is a dense left ideal in A and B is a Banach algebra with respect to $\|.\|_B$.

(*ii*) There exists M > 0 such that $\|f\|_A \leq \mathrm{M}\|f\|_B$, for each $f \in B$.

(*iii*) There exists C > 0 such that $\|fg\|_B \leq \mathrm{C}\|f\|_A \|g\|_B$ for each $f \in A$ and $g \in B$.

We take it as known that, $L^p(G)$ is a Segal algebra and so an abstract Segal algebra with respect to $L^1(G)$, when G is compact. It has been studied a lot of topological and algebraic properties related to L^p-spaces and weighted L^p-spaces, as well; see [1, 10, 2] and [9]. Also, recently in [3], the author in a joint work with R. Nasr Isfahani and A. Rejali, have verified the convolution properties on $L^p(G, \omega)$.

The main purpose of this work is giving a necessary and sufficient condition for $L^p(G, \omega)$ to be a Segal algebra and also an abstract Segal algebra with respect to $L^1(G)$.

2. MAIN RESULTS

Before proceeding to the proof of the main theorem, we turn our attention to this fact that when $L^p(G,\omega)$ is a Banach left $L^1(G)$-module. It provides us with a useful tool to be used in our main result. We state here the following proposition, for later use. The way of the proof of [9, Theorem 3.1] helps us to prove it.

PROPOSITION 1.1. *Let G be a locally compact group, $1 < p < \infty$ and ω be a weight function on G. If $L^p(G,\omega)$ is a Banach left $L^1(G)$-module, then ω is of moderate growth.*

Proof. Since $L^p(G, \omega)$ is a Banach left $L^1(G)$-module, it follows that there exists a constant $K > 0$ such that for each $f \in L^1(G)$ and $g \in L^p(G, \omega)$ we have

$$\| f * g \|_{p,\omega} \leq K \| f \|_1 \| g \|_{p,\omega}.$$

Repeating argument of [9, Lemma 2.1] we conclude that ω^p is locally summable. So $L^p(G, \omega)$ contains characteristic functions χ_U, for each open neighborhood of the identity element of G with compact closure. We also have the following inequality, pointwise

$$\lambda(U)\chi_{xyV} \leq \chi_{xU} * \chi_{U^{-1}yV}, \tag{1.1}$$

for such sets U and V and arbitrary $x, y \in G$. Hence inequality (1.1) implies that

$$\lambda(U)\|\chi_{xyV}\|_{p,\omega} \leq \|\chi_{xU} * \chi_{U^{-1}yV}\|_{p,\omega} \leq \mathrm{K}\|\chi_{xU}\|_1 \|\chi_{U^{-1}yV}\|_{p,\omega}$$

and thus

$$\| \chi_{xyV} \|_{p,\omega} \leq K \| \chi_{U^{-1}yV} \|_{p,\omega}. \tag{1.2}$$

Let $x \in G$ be fixed. Since ω^p and ${}_x\omega^p$ are locally summable, it follows that there exists a family ν of sets of positive measures such that every $V \in \nu$ contains the identity and every neighbourhood of identity contains eventually all $V \in \nu$ and also the following equations hold:

$$\lim_{V\in\nu} \frac{1}{\lambda(V)} \int_{yV} \omega^p(r)\, d\lambda(r) = \omega^p(y)$$

and

$$\lim_{V\in\nu} \frac{1}{\lambda(V)} \int_{yV} \omega^p(xr)\, d\lambda(r) = \omega^p(xy),$$

for locally almost all $y \in G$; see [8, VIII, 1-2]. For such y and any $\varepsilon > 0$ for sufficiently small $V \in \nu$

$$\| \chi_{yV} \|_{p,\omega}^p = \int_{yV} \omega^p(r)\, d\lambda(r) < \lambda(V)\, \omega^p(y)\, (\varepsilon+1) \tag{1.3}$$

and

$$\| \chi_{xyV} \|_{p,\omega}^p = \int_{yV} \omega^p(xr)\, d\lambda(r) > \lambda(V)\, \omega^p(xy)\, (\varepsilon+1)^{-1}. \tag{1.4}$$

Moreover, there exists an open neighborhood U of the identity with compact closure such that

$$\| \chi_{U^{-1}yV} \|_{p,\omega} < (1+\varepsilon)\, \| \chi_{yV} \|_{p,\omega}.$$

Inequality (1.2) implies that

$$\| \chi_{xyV} \|_{p,\omega} \leq K \| \chi_{U^{-1}yV} \|_{p,\omega}.$$

Inequalities (1.3) and (1.4) with (1.2) yield the following inequality,

$$(1+\varepsilon)^{-1/p} \lambda(V)^{1/p}\, \omega(xy) < \mathrm{K}\, \lambda(V)^{1/p}\, \omega(y)\, (1+\varepsilon)^{1+1/p}$$

and hence

$$\omega(xy) < \mathrm{K}\, \omega(y) (1+\varepsilon)^{1+2/p}.$$

We conclude that

$$\frac{\omega(xy)}{\omega(y)} \leq K,$$

for locally almost every y. Therefore ω is of moderate growth.

Remark 1.2.

(1) Following, we certainly need to consider only those ω with $L^p(G,\omega) \subseteq L^1(G)$. That is, if f is a measurable function on G with $f\omega \in L^p(G)$, then we need that $f \in L^1(G)$. So, if $g \in L^p(G)$, then $f = g\,\omega^{-1} \in L^p(G,\omega)$, so we need that

$$\int_G \frac{|g(x)|}{\omega(x)} \mathrm{d}\lambda(x) < \infty .$$

It is standard that this is equivalent to $\omega^{-1} \in L^q(G)$. So, we shall henceforth assume that ω is a weight function on G with $\omega^{-1} \in L^q(G)$.

(2) Let us to an easier proof for Proposition 1.1, where $\omega^{-1} \in L^q(G)$. By part (1) of the present remark, $L^p(G,\omega)$ can be considered as a subspace of $L^1(G)$. It is known that ω is of moderate growth if and only if $L^p(G,\omega)$ is left translation-invaraint; and to prove the result, take $f \in L^p(G,\omega)$ and a bounded approximate unit $(e_\alpha)_{\alpha\in\Lambda}$ of $L^1(G)$. Then for any $x \in G$, $\| {}_x e_\alpha * f - {}_x f \|_1 \to 0$ (${}_x f$ means left translation of f by x). From the other hand, the net ${}_x e_\alpha * f$ has a subnet converging weakly to $g \in L^p(G,\omega)$. It follows that $g = {}_x f$ and so ${}_x f \in L^p(G,\omega)$. Consequently, ω is of moderate growth [4, Theorem 1.13].

(3) The preceding part confirms that for the case where $\omega^{-1} \in L^q(G)$, if $L^p(G,\omega)$ is a left ideal in $L^1(G)$, then ω is of moderate growth.

THEOREM 1.3. *Let G be a locally compact group, $1 < p < \infty$ and ω be a weight function on G such that $\omega^{-1} \in L^q(G)$. Then the following assertions are equivalent*:

(*i*) $L^p(G,\omega)$ *is a left ideal in* $L^1(G)$.

(*ii*) G is *compact*, ω is *locally summable and of moderate growth*.

Proof. (i) $\Rightarrow$(ii). Let $g \in L^p(G,\omega)$ be fixed. For a bounded net (f_i) in $L^1(G)$, $(f_i * g)_i$ is a bounded net in $L^p(G,\omega)$ and since $L^p(G,\omega)$ is reflexive, it follows that there exists a subnet (f_{i_j}) of (f_i) and $h \in L^p(G,\omega)$ such that $f_{i_j} * g \to h$, in the weak topology of $L^p(G,\omega)$. Since $L^\infty(G) \subseteq L^q(G,\omega^{-1})$, thus $f_{i_j} * g \to h$ in the weak topology of $L^1(G)$. So multiplier $T : L^1(G) \to L^1(G)$ defined by $f \to f * g$ is weakly compact and hence G is compact by [6, Theorem 3.1]. Thus for each $f \in L^p(G,\omega)$, $\chi_G * f \in L^p(G,\omega)$. Since

$$\chi_G * f(x) = \int_G f(t)\,\mathrm{d}\lambda(t) \qquad (x \in G),$$

hence $\omega \in L^p(G)$ and so ω and ω^p are locally summable by compactness of G. Thus Remark 1.2 implies that ω is of moderate growth. For (ii) $\Rightarrow$(i), recall that every weight function that is locally summable and of moderate growth, is equivalent to a continuous function; see [5]. So ω being nonzero is bounded below away from zero and bounded above on G. Hence $L^p(G,\omega) = L^p(G)$ and consequently the result follows.

By the elementary definitions of abstract Segal algebras, if $L^p(G,\omega)$ is an abstract Segal algebra with respect to $L^1(G)$, then it is a left ideal in $L^1(G)$ and so G is compact, ω is locally summable and of moderate growth, by Theorem 1.3.

We state here the following equivalences, that is interesting in its own right.

COROLLARY 1.4. *Let G be a locally compact group, $1 < p < \infty$ and ω be a weight function on G such that $\omega^{-1} \in L^q(G)$. Then the following assertions are equivalent*:

(i) $L^p(G,\omega)$ is a left ideal in $L^1(G)$.

(ii) G is compact, ω is locally summable and of moderate growth.

(iii) G is compact and ω is equivalent to a continuous function.

(iv) $L^p(G,\omega)$ is a Segal algebra.

(v) $L^p(G,\omega)$ is an abstract Segal algebra with respect to $L^1(G)$.

(vi) $L^p(G,\omega)$ is the usual algebra $L^p(G)$.

ACKNOWLEDGEMENTS

The author would like to thank the referee of the paper for invaluable comments. The author would also like to thank the Banach Algebra Center of Excellence for Mathematics, University of Isfahan.

REFERENCES

1. F. ABTAHI, R. NASR ISFAHANI and A. REJALI, *On the L^p-conjecture for locally compact groups,* Arch. Math. (Basel), **89**, pp. 237–242, 2007.
2. F. ABTAHI, R. NASR ISFAHANI and A. REJALI, *On the weighted ℓ^p-space of a discrete group,* Publ. Math. Debrecen., **75**, *3–4*, pp. 365–374, 2009.
3. F. ABTAHI, R. NASR ISFAHANI and A. REJALI, *Convolution on weighted L^p-spaces of locally compact groups,* Proceedings of the Romanian Academy, Series A, **13**, *2*, pp. 97–102, 2012.
4. R. E. EDWARDS, *The stability of weighted Lebesgue spaces,* Trans, Amer, Math. Soc., **93**, pp. 369–394, 1959.
5. H. G. FEICHTINGER, *Gewichtsfunktionen auf, lokalkompakten Gruppen,* Sitzber. Österr. Akad. Wiss. Abt. II., **188**, *8–10*, pp. 451–471, 1979.
6. F. GHAHRAMANI, *Weighted group algebra as an ideal in its second dual space,* Proc. Amer. Math. Soc. Phil. Soc., **90/1**, pp. 71–76, 1984.
7. E. HEWITT and B. ROSS, *Abstract harmonic analysis I*, *II*, Springer Verlag, New York, 1970.
8. A. IONESCU TULCEA and C. IONESCU TULCEA, *Topics in the theory of lifting*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 48, Springer Verlag, 1969.
9. YU. N. KUZNETSOVA, *Invariant weighted algebras $L^p(G,\omega)$*, Math. Notes., **84,** *3–4*, pp. 529–537, 2008.
10. S. SAEKI, *The L^p-conjecture and Young's inequality*, Illinois J. Math., **34**, pp. 615–627, 1990.

Received March 12, 2012