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APPROXIMATE SOLUTIONS FOR DIFFUSION EQUATIONS ON CANTOR SPACE-TIME

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In this paper we investigate diffusion equations on Cantor space-time and we obtain approximate solutions by using the local fractional Adomian decomposition method derived from the local fractional operators. Analytical solutions are given in terms of the Mittag-Leffler functions defined on Cantor sets.

Key words: diffusion equations, adomian decomposition method, local fractional operators, approximate solutions, Cantor sets.

1. INTRODUCTION

The *Cantor space-time physics* is still an important issue to be developed. The *diffusion process* in this kind of space-time is irreversible due to the non-existence of a straight shortest path connecting them [1–3]. We recall that the fractal path in fractal Cantor space-time is always $\langle d \rangle = 2$, while in the smooth classical space it is always $d = 1$ [1]. The Weinberg's result for testing the influence of non-linearity on quantum theory is negligible [1] because of the case at a non-critical point. However, the influence of non-linear terms is crucial at a point of bifurcation.

The Cantor space-time proposal maintains that the quantum mechanics is a very special kind of diffusion process [1]. However, the formal similarity between Schrödinger equation and that of classical diffusion was reported. Recently, the element of fractal arc length squared in fractal space-time was written in the form [4],

$$\left(d^{\alpha} s\right)^{2} = g_{ij}^{\alpha} \left(dx_i\right)^{\alpha} \left(dx_j\right)^{\alpha} \quad , \tag{1}$$

where the fractal metrics $g_{ij}^{\alpha} = \left(x_1, x_2, x_3, \cdots, x_N\right)$ are local fractional continuous functions of the fractal space-time coordinates and they are different from constants. In fractal time-space, the *local fractional Schrödinger equation* was reported as [5]

$$i^{\alpha} h_{\alpha} \frac{\partial^{\alpha} T_{\alpha}}{\partial t^{\alpha}} = \frac{h_{\alpha}^{2}}{2m} \nabla^{2\alpha} T_{\alpha} + V_{\alpha} T_{\alpha} \quad , \tag{2}$$

where $\nabla^{2\alpha}$ is the local fractional Laplace operator given by [4–8]

$$\nabla^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial z^{2\alpha}} \tag{3}$$

and the local fractional partial derivative of high order read as [4, 6–7]

$$\frac{\partial^{k\alpha}}{\partial x^{k\alpha}} T_\alpha(x,y,z,t) = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k\ times} T_\alpha(x,y,z,t). \tag{4}$$

As a result, we can write a local fractional Schrodinger equation in one-dimension fractal space-time as follows

$$\frac{\partial^\alpha T_\alpha(x,t)}{\partial t^\alpha} = a^{2\alpha} \frac{\partial^{2\alpha} T_\alpha(x,t)}{\partial x^{2\alpha}}, \tag{5}$$

where $a^{2a} = h_\alpha / 2mi^\alpha$. We notice that (5) is similar to the *diffusion equation* on Cantor sets, namely [9]

$$\frac{\partial u(x,t)}{\partial t} = c^* \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}}, \tag{6}$$

where c^* is the fractal thermal capacity of the material per unit volume. We can obtain the diffusion equation on Cantor time-space given as [10]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a^{2\alpha} \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}}, \tag{7}$$

where α is fractal dimension of a Cantor set, and $u(x,t)$ satisfies the local fractional continuous condition [4–7]

$$f(x) \in C_\alpha(a,b) \tag{8}$$

or

$$\left| f(x) - f(x_0) \right| < \varepsilon^\alpha, \tag{9}$$

with $\left| x - x_0 \right| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$. We notice that the result for diffusion equations on Cantor sets differs from the ones derived within the classical [11, 12] and the fractional calculus [13–29], respectively.

We stress on the fact that the methods reported in [32–55] can't be applied to handle the differential equations on Cantor sets. The alternative methods for dealing with these equations were reported in [30, 31, 56–58].

The present work deals with a compact solution to diffusion equation on Cantor space-time by using the local fractional Adomian decomposition method based on local fractional operators.

The paper is organized as follows. In Section 2, a short introduction to local fractional calculus theory is given. The analysis method is presented in Section 3. The approximate solution to diffusion equation in Cantor space-time is given in Section 4. Finally in Section 5, the conclusions are given.

2. PRELIMINARIES

In this section, we give a brief introduction to the local fractional calculus theory.

The corresponding operator is defined as [4–7, 30, 31, 57, 58]

$$D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha)\Delta\left(f(x) - f(x_0)\right)}{\left(x - x_0\right)^\alpha}. \tag{10}$$

The local fractional integral operator, as inverse of local fractional differential operator, has the form [–7, 31]

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha , \quad (11) $$

where the partition of the interval $[a,b]$ obeys [4–7, 10, 30, 31, 57, 58]:

$$ \Delta t_j = t_{j+1} - t_j ,\ \Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\} ,\ j = 0,..., N-1 ,\ t_0 = a \text{ and } t_N = b . $$

The local fractional multiple integrals of $f(x)$ are defined as [4]

$$ {}_{x_0}I_x^{(k\alpha)} f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \cdots {}_{x_0}I_x^{(\alpha)}}^{k\ times} f(x). \quad (12) $$

The local fractional integration by parts reads as follows [4, 6, 7]

$$ {}_aI_x^{(\alpha)} f(x) g^{(\alpha)}(x) = [f(x)g(x)]\big|_a^x - {}_aI_x^{(\alpha)} f^{(\alpha)}(x) g(x). \quad (13) $$

We recall that the Fubini's formula in local fractional integral has the form [4]

$$ {}_aI_b^{(\alpha)} {}_cI_d^{(\alpha)} \psi(x,y) = {}_cI_d^{(\alpha)} {}_aI_b^{(\alpha)} \psi(x,y) . \quad (14) $$

Similarly, the replacement theorem in local fractional integral can be expressed as given below

$$ {}_aI_x^{(\alpha)} {}_aI_\tau^{(\alpha)} f(t) = {}_aI_x^{(\alpha)} \left[\frac{(x-t)^\alpha}{\Gamma(1+\alpha)} f(t) \right]. \quad (15) $$

Local fractional Leibniz product law has the following expression [4, 6, 7]

$$ D_x^{(\alpha)} [f(x)g(x)] = \left(D_x^{(\alpha)} f(x)\right) g(x) + f(x) \left(D_x^{(\alpha)} g(x)\right). \quad (16) $$

In this work we will use the sub-functions, namely [6, 7]

$$ E_\alpha(x^\alpha) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)} ,\quad \sin_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]} , $$

$$ \cos_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)} ,\quad \sinh_\alpha x^\alpha = \sum_{k=0}^{\infty} \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]} , \quad (17a,b,c,d,e) $$

$$ \cosh_\alpha x^\alpha = \sum_{k=0}^{\infty} \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)} . $$

3. THE DESCRIPTION OF THE METHOD

In this section we outline a *local fractional Adomian decomposition method* [31] for handling the solutions of differential equations on Cantor space-time derived from local fractional operators.

Equation (7) can be written in a local fractional operator form as

$$ a^{2\alpha} L_{xx}^{(2\alpha)} u(x,t) - L_t^{(\alpha)} u(x,t) = 0 , \quad (18) $$

where $L_{xx}^{(2\alpha)}$ is a $2\alpha^{\text{th}}$ local fractional differential operator, which reads

$$ L_{xx}^{(2\alpha)} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \quad (19) $$

and a α^{th} local fractional differential operator is given by

$$L_t^{(\alpha)} = \frac{\partial^\alpha}{\partial t^\alpha}, \tag{20}$$

subjected to the fractal initial conditions

$$u(x,0) = r(x), \quad 0 \le x \le l. \tag{21}$$

By defining the one-fold local fractional integral operator

$$L_t^{(-\alpha)} m(t) = {}_0I_t^{(\alpha)} m(s)$$

we obtain

$$L_t^{(-\alpha)} L_t^{(\alpha)} u(x,t) = a^{2\alpha} L_t^{(-\alpha)} L_{xx}^{(2\alpha)} u(x,t)$$

therefore

$$u(x,t) = r(x) + a^{2\alpha} L_t^{(-\alpha)} L_{xx}^{(2\alpha)} u(x,t), \tag{22}$$

where the term $r(x)$ is to be determined from the fractal initial conditions.

Therefore, we can rewrite

$$u(x,t) = u_0(x,t) + a^{2\alpha} L_t^{(-\alpha)} L_{xx}^{(2\alpha)} u(x,t), \tag{23}$$

with $u_0(x,t) = r(x)$.

Hence, for $n \ge 0$ we have the following recurrence relationship

$$\begin{cases} u_{n+1}(x,t) = a^{2\alpha} L_t^{(-\alpha)} L_{xx}^{(2\alpha)} u_n(x,t) \\ u_0(x,t) = r(x) \end{cases}. \tag{24}$$

Finally, the approximation expression can be constructed as

$$u(x,t) = \lim_{n\to\infty} \phi_n(x,t) = \lim_{n\to\infty} \sum_{i=1}^{\infty} u_i(x,t). \tag{25}$$

4. THE APPROXIMATE SOLUTION

Let us consider (19), subject to the fractal initial boundary conditions

$$u(x,0) = E_\alpha(x^\alpha)(0 \le x \le l). \tag{26}$$

From (24) we obtain the recurrence relationship as given below

$$u_{n+1}(x,t) = a^{2\alpha} L_t^{(-\alpha)} L_{xx}^{(2\alpha)} u_n(x,t), \tag{27}$$

together with the fractal conditions

$$u_0(x,t) = E_\alpha(x^\alpha). \tag{28}$$

Assuming the initial approximation (27), we obtain

$$u_1(x,t) = a^{2\alpha} \frac{t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha), \; u_2(x,t) = a^{4\alpha} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha(x^\alpha),$$

$$u_3(x,t)=a^{6\alpha}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(x^\alpha),\ u_4(x,t)=a^{8\alpha}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}E_\alpha(x^\alpha),$$

$$\vdots$$

and so for the remaining components.

Finally, we can present the solution in local fractional series form as

$$\phi_n(x,t)=E_\alpha(x^\alpha)\sum_{i=0}^{n}a^{2i\alpha}\frac{t^{2i\alpha}}{\Gamma(1+2i\alpha)}. \tag{29}$$

Hence, we get the compact solution

$$\begin{aligned}u(x,t)&=\lim_{n\to\infty}\phi_n(x,t)=\lim_{n\to\infty}E_\alpha(x^\alpha)\sum_{i=0}^{n}a^{2i\alpha}\frac{t^{2i\alpha}}{\Gamma(1+2i\alpha)}=\\&=E_\alpha(x^\alpha)\cosh_\alpha(a^\alpha t^\alpha).\end{aligned} \tag{30}$$

According to the theory of local fractional continuity, we can arrive at

$$\left|u(x,t)-u(x_0,t_0)\right|\le\varepsilon^\alpha. \tag{31}$$

Namely, we have

$$\left|E_\alpha(x^\alpha)-E_\alpha(x_0^\alpha)\right|\le\varepsilon^\alpha \tag{32}$$

as well as

$$\left|\cosh_\alpha(a^\alpha t^\alpha)-\cosh_\alpha(a^\alpha x_0^\alpha)\right|\le\left|a^\alpha\sinh_\alpha(a^\alpha x_0^\alpha)\right|\left|t-t_0\right|^\alpha<\varepsilon^\alpha, \tag{33}$$

where α is the fractal dimension of Cantor space-time. These results are not derived from fractional calculus [26–29].

The diffusion equation on a Cantor set is written in the form

$$L_t^{(\alpha)}u(x,t)=a^{2\alpha}L_{xx}^{(2\alpha)}u(x,t)-L_x^{(\alpha)}u(x,t) \tag{34}$$

An initial condition is described by

$$u(x,0)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}(0\le x\le l). \tag{35}$$

Therefore, we structure the recurrence formula as follows

$$\begin{cases}u_{n+1}(x,t)=a^{2\alpha}L_t^{(-\alpha)}L_{xx}^{(2\alpha)}u_n(x,t)-L_t^{(-\alpha)}L_x^{(\alpha)}u_n(x,t)\\u_0(x,t)=\dfrac{x^{2\alpha}}{\Gamma(1+2\alpha)}.\end{cases} \tag{36}$$

The approximations have the form

$$u_1(x,t)=\frac{a^{2\alpha}t^\alpha}{\Gamma(1+\alpha)}-\frac{t^\alpha}{\Gamma(1+\alpha)}\frac{x^\alpha}{\Gamma(1+\alpha)},\ u_2(x,t)=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{37a, b}$$

We recall that the other terms are zero. Hence, an analytical solution has the form

$$u(x,t)=\sum_{i=1}^{\infty}u_i(x,t)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{a^{2\alpha}t^\alpha}{\Gamma(1+\alpha)}-\frac{t^\alpha}{\Gamma(1+\alpha)}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{38}$$

From the local fractional set theory [4, 6], we can find that the solution (28) is a fractal one.

4. CONCLUSIONS

Local fractional calculus started to be a useful tool to model fractal complex systems because it reveals hidden aspects which cannot be observed by using other classical formalisms. Also, it gives the alternative description of quantum and high energy physics of Cantor space-time, namely, quantum and high energy physics of fractal space-time. This is a new research direction for physics of fractal space-time, which is devoted primarily to the integration of nonlinear dynamics and deterministic fractals into the foundation of quantum and high energy physics.

In this manuscript we analyzed the diffusion equation on Cantor space-time. We notice that the obtained results depend on the fractal dimension order of the differential equation on Cantor space-time. By using the local fractional Adomian decomposition method we obtained the approximation solutions of different types of partial differential equations on Cantor space-time. The results for handling the diffusion equation via the local fractional operators demonstrate reliability and efficiency of the new proposed method.

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