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PORTFOLIO OPTIMIZATION BASED ON VALUE-AT-RISK

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A widely reported and accepted measure of financial risk for industry segments and financial markets, by nature measuring the probability of worst case portfolio performance, is the Value at Risk (VaR) measure. We will consider an additional constraint based on the Expers' judgments that will help us to find a better solution.

Key words: investment analysis, stochastic optimization, Value at Risk, Conditional Value at Risk.

1. INTRODUCTION

The optimization models for portfolio selection have evolved from early mean-variance formats based on Markowitz's work (1952) to more recent scenario-based stochastic optimization forms (Hiller and Eckstein, 1993; Birge and Rosa, 1995; Vladimirou and Zenios, 1997; Cariño and Ziemba, 1998). The common idea in all model structures is the minimization of some measure of risk, while simultaneously maximizing some measure of performance.

The risk measure, for the most model frameworks, is a function of the entire range of possible portfolio returns. For example, the portfolio variance is used in a mean-variance framework, while concave utility functions are applied across the set of all possible outcomes in stochastic programming frameworks.

A common technique for measuring downside risk in a portfolio is Value at Risk (VaR). The formal VaR is defined as the α-quantile of the portfolio return distribution function, with $\alpha \in (0,\ 1)$. So, at the end of the time horizon and for low α values (e.g. 0.01, 0.05 or 0.1), it can be identified as the "worst case" outcome of portfolio performance. Unfortunately, VaR is a piecewise linear function whose graph can display many local minima and maxima.

Moreover, VaR as a practical measure of risk has been accepted by managers because it represents a tool easy to handle by the investor. Many international institutes as The Group of Thirty, the Derivatives Product Group, Bank of International Settlements, and European Union and others have recognized VaR at some level as the standard for risk assessment.

The decision makers have employed VaR as a tool for controlling enterprise risk. For example, the 1998 Basle Capital Accord proposes a bank's required set aside capital for market risk based on internal VaR estimates and the National Association of Insurance Commissioners (NAIC) also requires the reporting of VaR.

Uryasev and Rockafellar (1999) propose a scenario-based model for portfolio optimization using Conditional Value at Risk (CVaR), which is defined as the expected value of losses exceeding VaR. So, their optimization model minimizes CVaR while calculating VaR. In the case of normally distributed portfolio returns, the minimum-CVaR portfolio is equivalent to the minimum-VaR portfolio.

The practitioners are very much concerned about the transaction costs since these have significant effects on the investment strategy.

In order to purchase (invest) and/or sell (disinvest) assets, the investor has to pay certain fees. But, unfortunately, the transaction costs are often ignored because the precise treatment of transaction costs leads to a nonconvex minimization problem.

The selection problem of the rebalancing portfolio, under nonconvex transactional costs, was defined in Konno et al. [5, 6], where they considered a branch and bound algorithm for solving a portfolio optimization model under concave transaction costs. The [12, 13, 14] papers have a particularly influence on this context.

This paper is organized as follows. In Section 2, we review the theoretical background concerning the measure *Value-at-Risk* and the rebalancing problem. In Section 3, we state the maximum investment return with a minimum cost rebalancing under the mean-*Value-at-Risk* model. In Section 4, we model the programming problem under certain conditions about the cdf and we approach the case where we don't know anything about its shape. So, we develop some order estimators and approaches of VaR.

2. PRELIMINARIES

Let us consider a random variable R representing the vector of the rates of return of the *n* assets and let $F(.)$ be its distribution function. (Pr stands for probability)

***Definition* 1.** Let $\alpha \in (0,1)$ and R a random variable with $F_R(r) = \Pr(R \le r)$. The Value-at-Risk with threshold α of R is defined as

$$\mathrm{VaR}_{\alpha}(R) = \inf\left\{ r \,\middle|\, \Pr(R \le r) \ge \alpha \right\} = \inf\left\{ r \,\middle|\, \Pr(R > r) \ge 1 - \alpha \right\}.$$

***Observation* 1.** If the cumultive distribution function $F_R(.)$ is continuous and strictly increasing, then $VaR_{\alpha}(R) = F_R^{-1}(\alpha)$..

The measure *Value-at-Risk* may evaluate the asymmetric risks, risks that cannot be satisfactory modeled with a classic measure as variance.

The future return of asset j, $j = 1,\ldots,n,$ is given by the random variable R_j. Let $x^0 = \left(x_1^0, x_2^0 \ldots, x_n^0\right)$ [5, 6] be the portfolio at time point 0 and $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ is the new portfolio. The transaction cost of the entire investment is $\sum_{j=1}^{n} c_j\left(y_j\right)$, where $c_j\left(y_j\right)$ is a nondecreasing nonconvex function up to certain point y_j [5].

Let us introduce the new portfolio at a certain later point $x = y + x^0$ made up by all the operations resulted from the rebalancing, a portfolio that has the following meaning:

– if $y_j > 0,\ j = 1,\ldots,n$, then $c_{ji}\left(y_j\right)$ is the associated cost with purchasing y_j units of the asset *j*, $i = 1,\ldots,k$;

– if $y_j < 0,\ j = 1,\ldots,n$, then $c_{ji}\left(y_j\right)$ is the associated cost with selling $\left|y_j\right|$ units of the asset *j*, $i = 1,\ldots,k$.

3. THE MEAN – VaR MODEL

A general portfolio optimization problem, with VaR instead of variance as risk measure, is formulated as the classic mean-variance approach. To formulate a general problem for a portfolio with n asset, and having as risk measure VaR, it is necessary that the decision maker fixes two parameters, the probability α_p and the return r_p.

Moreover, we will impose some restraints regarding the volatility of the temporary assets, and the fact that the investor wishes to withdraw at the moment $t = 0$ some of his investment to spend, an amount that is equal with b_0. To do that, the investor has to rebalance the portfolio by selling and buying assets from his initial portfolio.

In this model the objective function maximize the portfolio expected return at moment $t=1$ with the risk of the problem given by the constraint $\Pr\left(R^T x \le r_p\right) \le \alpha_p$.

With these preliminaries, the M-Var model is defined as follows:

$$\begin{aligned}
&\max \quad \left\{E\left[R^T x\right] - \sum_{j=1}^{n} c_j\left(y_j\right)\right\}, \quad \Pr\left(R^T x \le r_p\right) \le \alpha_p, \\
&b_0 = -e^T y - \sum_{j=1}^{n} c_j\left(y_j\right), \quad \sum_{j=1}^{n} y_j = 0, \ \sum_{j=1}^{n} x_j = 1, \\
&\varphi'_j \le x_j \le \varphi_j, \ j=1,\ldots,n, \quad 0 \le \varphi_j \le 1, \ j=1,\ldots,n, \quad 0 \le \varphi'_j \le 1, \ j=1,\ldots,n,
\end{aligned} \tag{1}$$

where the fees associated with $x=\left(x_1,\ldots,x_n\right)$ are named transaction cost, x_i represents the amount of investment (or disinvestment) of the asset *j (j = 1, ..., n)*.

The decision maker is willing to accept only portfolios for which the probability of return under any threshold fixed is less or equal to α_p, where this probability is given by the Expers' judgments. So, first restriction of the risk from the problem (1) is equivalent to $\mathrm{VaR}_{\alpha_p}\left(R^T x\right) \ge r_p$.

Replacing the upper relation in problem (1) we will have the following equivalent problem, where the risk is given by VaR:

$$\begin{aligned}
&\max \quad \left\{E\left[R^T x\right] - \sum_{j=1}^{n} c_j\left(y_j\right)\right\}, \quad \mathrm{VaR}_{\alpha_p}\left(R^T x\right) \ge r_p, \\
&b_0 = -e^T y - \sum_{j=1}^{n} c_j\left(y_j\right), \quad \sum_{j=1}^{n} y_j = 0, \ \sum_{j=1}^{n} x_j = 1, \\
&\varphi'_j \le x_j \le \varphi_j, \ j=1,\ldots,n; \quad 0 \le \varphi_j \le 1, \ j=1,\ldots,n; \quad 0 \le \varphi'_j \le 1, \ j=1,\ldots,n.
\end{aligned} \tag{2}$$

4. ESTIMATORS FOR *VaR* AND *CVaR*

However, *VaR* can be calculated if we know the form of the repartition function, or we can estimate it using various methods: order estimators [4] or simulation [2].

If we don't make any assumption about the shape of the cdf, then we can find an approximation of the distribution function.

4.1. Case of order estimators for VaR

Let the set of past observations independent and identically distributed $\left(R_1, R_2, \ldots, R_n\right)$ on the random variable R, and let the observations be ranked as $R_{1:n} \le R_{2:n} \le \ldots \le R_{n:n}$.

Then the estimated $\mathrm{VaR}_\alpha\left(R\right)$ is $R_{\alpha_n:n}$, where $\alpha_n = \min\limits_{i=1,\ldots,n}\left\{i \,\middle|\, \frac{i}{n} \ge \alpha\right\}$. So, the $\alpha-$quantile is estimated by the position of the observation $R_{\alpha_n:n}$ that has the $\alpha-$percent of the data on the left, for $\alpha \in \left(0,1\right)$.

See [4] for the properties of this estimator. Therefore, instead of problem (2) we consider the following problem:

$$\max \left\{E\left[R^T x\right]-\sum_{j=1}^{n} c_j\left(y_j\right)\right\}, \quad R_{\alpha_n : n} \geq r_p$$
$$b_0=-e^T y-\sum_{j=1}^{n} c_j\left(y_j\right), \quad \sum_{j=1}^{n} y_j=0, \sum_{j=1}^{n} x_j=1 \tag{3}$$
$$\varphi_j' \leq x_j \leq \varphi_j, j=1,\ldots,n; \quad 0 \leq \varphi_j \leq 1, j=1,\ldots,n; \quad 0 \leq \varphi_j' \leq 1, j=1,\ldots,n.$$

Another method of approximation is the statistics $R_{n\alpha F_R(r_0):n}$. It is an estimator of lower $\alpha-$quantile under threshold r_0 of the random variable R, given by $\inf\left\{r \middle| \Pr\left(R>r \middle| R \leq r_0\right) \leq 1-\alpha\right\}$. It follows that $R_{n\alpha F_R(r_0):n}$ is an estimator even for $\mathrm{VaR}_{\alpha F_R(r_0)}(R)$. From [9] we have that $R_{n\alpha F_R(r_0):n}$ is a consistent estimator.

THEOREM. *The lower* α *- quantile under threshold* r_0 *of the random variable* R *is* $\mathrm{VaR}_{\alpha F_R(r_0)}(R)$.

Proof. Because for any $r \geq r_0$ we have $\Pr\left(R>r \middle| R \leq r_0\right)=0$, it follows that $\Pr\left(R>r \middle| R \leq r_0\right) \leq 1-\alpha$. So, we have $\inf\left\{r \middle| \Pr\left(R>r \middle| R \leq r_0\right) \leq 1-\alpha\right\}<r_0$. It follows that we should analyze the inegality $\Pr\left(R>r \middle| R \leq r_0\right) \leq 1-\alpha$ only for $r<r_0$. In this case we have:

$$\Pr(R>r) \leq 1-\alpha F_R\left(r_0\right)$$

and $\inf\left\{r \middle| \Pr\left(R>r \middle| R \leq r_0\right) \leq 1-\alpha\right\}=\inf\left\{r \middle| \Pr(R \leq r) \geq \alpha F_R\left(r_0\right)\right\}$.

We will denote: $\inf\left\{r \middle| \Pr\left(R>r \middle| R \leq r_0\right) \leq 1-\alpha\right\}=\mathrm{VaR}_{\alpha, r_0}(R)$.

Let $f_R(\,)$ be the probability density function for a random variable R and using [8] we have the following result.

REMARK 1. If the random variable R has a probability function in a convex combination of the number $\mathrm{VaR}_{\alpha, r_0}(R)$, and $f_R\left(\mathrm{VaR}_\alpha(R)\right)>0$, then we have the convergence in distribution:

$$\sqrt{n}\left(R_{n\alpha F_R(r_0):n}-\mathrm{VaR}_{\alpha, r_0}(R)\right) \underset{n \rightarrow \infty}{\rightarrow} \frac{\sqrt{\alpha(1-\alpha)}}{f_R\left(\mathrm{VaR}_{\alpha, r_0}(R)\right)} N(0,1).$$

4.2. The case of Some Estimators for CVaR

Using the Conditional Value-at-Risk as a measure of the risk gives an advantage to the users comparing to using VaR, because CVaR is a convex function.

Following the approach from [10,11] and using the Lemma 1 for $\alpha=\alpha F_R\left(r_0\right)$, it follows that the statistics

$$\mathrm{CVaR}_{\alpha F_R(r_0)}(R)=\inf\left\{\theta+\frac{1}{1-\alpha F_R\left(r_0\right)} E\left[(R-\theta)^{+}\right]\right\}$$

is an estimator of the measure $\mathrm{CVaR}_{\alpha, r_0}$.

Using the upper expression, we have

$$\begin{aligned}&\min \quad \left\{\theta + \frac{1}{1-\alpha F_R(r_0)} E\left[(R-\theta)^+\right]\right\}, \quad E\left[R^T x\right] - \sum_{j=1}^{n} c_j(y_j) \geq r_p\\ &b_0 = -e^T y - \sum_{j=1}^{n} c_j(y_j), \quad \sum_{j=1}^{n} y_j = 0, \sum_{j=1}^{n} x_j = 1\\ &\varphi'_j \leq x_j \leq \varphi_j,\ j = 1,\dots,n; \quad 0 \leq \varphi_j \leq 1,\ j = 1,\dots,n; \quad 0 \leq \varphi'_j \leq 1,\ j = 1,\dots,n.\end{aligned} \tag{4}$$

We denote the repartition empirical function by

$$F_n(r) = \frac{1}{n}\sum_{j=1}^{n} I_{\{R_j \leq r\}},$$

where $I_{\{R_j \leq r\}} = \begin{cases} 1, & R_j \leq r \\ 0, & \text{otherwise.}\end{cases}$

***Observation* 2**. Let $F_n^{-1}(F_R(r_0)) = \inf\{r | F_n(r) \geq F_R(r_0)\}$ be a $F_R(r_0)$ – the lower quantile of selection. Then $F_n^{-1}(F_R(r_0))$ is a consistent estimator of the measure $\text{VaR}_{\alpha, r_0}(R)$.

The relative cumulative frequency for a random variable R_j will be denoted by $R_{j:n}$.

But when we replace the order statistics with the relative cumulative frequency for a sample from R the left side of the repartition tail, we have the following result.

REMARK 2. Let $\alpha \in (0,1)$ be given. The random variable R is well defined ($E(R) < \infty$) and $(R_1, R_2, \dots, R_n)$ is a sample from the variable R. Then we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n\alpha F_R(r_0)} R_{j:n}}{n\alpha F_R(r_0)} = C\text{VaR}_{\alpha, r_0}(R).$$

5. CONCLUSIONS

VaR is a widely reported and accepted measure of risk across industry segments and market participants. In general, VaR optimal portfolios are more likely to incur large losses when losses occur. The more prudent approaches that incorporate the expected value of losses are more likely not to exceed VaR risk specifications when examined relative to holdout scenarios. But, using the Conditional Value-at-Risk as a measure of the risk, we have an huge advantage in comparing to using VaR, because CVaR is a convex function.

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