THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 14, Number 3/2013, pp. 193–197

UPPER SEMICONTINUITY OF SOLUTION SETS FOR A CLASS OF VECTOR VARIATIONAL INEQUALITIES

Vasile PREDA, Diana-Elena STANCIU

University of Bucharest, Romania
E-mail: `preda@fmi.unibuc.ro, stanciu_diana79@yahoo.com`

We give some sufficient conditions for the compactness and the upper semicontinuity of the solution maps of a weak vector variational inequality involving a (v,η) -hemicontinuous and weakly (C,η) -pseudomonotone operator.

Key words: hemicontinuous functions, pseudomonotone functions, semicontinuity, vector variational inequality.

1. INTRODUCTION

In the last period the study of the stability of solution maps of optimization problems and variational inequalities with perturbing parameters represented an important topic for researchers. In the literature have been discussed many aspects of stability, such as semicontinuity, continuity, Lipschitz continuity or some kind of differentiability of the solution sets [1–25]. Among them, the continuity of the solution sets is required more often in applications.

The effect of small disturbances on the solution sets of variational inequalities have been investigated by many authors, both for scalar case [9, 11, 17] and vector case [2–4, 8, 13–15, 16–19]. Most of the results on the stability of solution sets of vector variational inequalities have been obtained assuming that the operator is continuous. However, Fang and Li [6] obtained results concerning the upper semicontinuity of solution sets for a weak vector variational inequality imposing weaker conditions on the operator.

In this paper we establish the upper semicontinuity of solution sets for a perturbed weak vector variational inequality if the operator is (v,η)-hemicontinuous and weakly (C,η)-pseudomonotone. Section 2 contains the general framework in which we will do the study and the main notions and results that are used here. Making certain assumptions on the operator, in Section 3 we will give sufficient conditions for the upper semicontinuity of the solution sets of a perturbed weak vector variational inequality.

2. PRELIMINARIES

Let X, Y and W be Banach spaces, K a nonempty subset of X, $C \subset Y$ a pointed closed and convex cone with nonempty interior and let Λ be a nonempty subset of W. Let $\eta : X \times X \to X$ be a function and $T : X \times \Lambda \to L(X,Y)$, where $L(X,Y)$ is the space of all linear continuous operators from X to Y. The value of a linear operator $t \in L(X,Y)$ at $x \in X$ is denoted by $\langle t, x \rangle$. Consider the parametric weak vector variational inequality corresponding to a parameter $\lambda_0 \in \Lambda$: ($PWVVI_\eta(\lambda_0)$) find $x \in K$ such that

$$\langle T(x,\lambda_0), \eta(y,x) \rangle \notin -\operatorname{int} C \quad , \quad \forall y \in K$$

Let $S_\eta(\lambda_0)$ denote the solution sets of ($PWVVI_\eta(\lambda_0)$), that is,

$$S_\eta(\lambda_0)=\{x\in K:\langle T(x,\lambda_0),\eta(y,x)\rangle\notin -\operatorname{int}C,\forall y\in K\}.$$

For $\eta(y,x)=y-x$, we obtain the inequality considered by Fang and Li [6]. Throughout our study, we assume that $S_\eta(\lambda)$ is nonempty for each λ in the neighborhood of λ_0.

Now, let us recall some basic definitions and their properties. For each $\varepsilon>0$ and $\lambda_0\in\Lambda$, denote by $B(\lambda_0,\varepsilon)$ the open ball with center λ_0 and radius ε, that is,

$$B(\lambda_0,\varepsilon):=\{\lambda\in\Lambda:\|\lambda_0-\lambda\|<\varepsilon\}.$$

***Definition* 1** [1, 12]. A set-valued map $F:\Lambda\to 2^X$ with $\operatorname{dom}F=\Lambda$ is said to be

i. upper semicontinuous in the sense of Berge (in short, B-usc) at $\lambda_0\in\Lambda$ if for every open set N satisfying $F(\lambda_0)\subset N$ there exists a $\delta>0$ such that for every $\lambda\in B(\lambda_0,\delta)$, $F(\lambda)\subset N$.

ii. lower semicontinuous in the sense of Berge (in short, B-lsc) at $\lambda_0\in\Lambda$ if for every open set N satisfying $F(\lambda_0)\cap N\neq\emptyset$ there exists a $\delta>0$ such that for every $\lambda\in B(\lambda_0,\delta)$, $F(\lambda)\cap N\neq\emptyset$.

The application F is said to be B-lsc (respectively B-usc) on Λ if F is B-lsc (respectively B-usc) at each point $\lambda_0\in\Lambda$. F is said to be B-continuous on Λ if it is both B-lsc and B-usc on Λ.

Aubin and Ekeland [1] gave the following equivalent definition for a B-lower semicontinuous function: the set-valued map $F:\Lambda\to 2^X$ is B-lsc at $\lambda_0\in\Lambda$ if and only if for any sequence $\{\lambda_n\}\subset\Lambda$ with $\lambda_n\to\lambda_0$ and any $x_0\in F(\lambda_0)$ there exists a sequence $\{x_n\}\subset F(\lambda_n)$ such that $x_n\to x_0$.

PROPOSITION 1 [7]. *If F has compact values, then F is B-usc at $\lambda_0\in\Lambda$ if and only if for any sequence $\{\lambda_n\}\subset\Lambda$ with $\lambda_n\to\lambda_0$ and any $x_n\in F(\lambda_n)$ there exists $x_0\in F(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to x_0$.*

***Definition* 2.** Let K be a nonempty convex subset of X, $T:K\to L(X,Y)$ an operator and $\eta:K\times K\to X$ a function. We say that T is (v,η) -hemicontinuous if for any $x,y\in K$ and $t\in[0,1]$ the application $t\to\langle T(x+t(y-x)),\eta(y,x)\rangle$ is continuous at 0^+.

***Definition* 3.** Let K be a nonempty subset of X, $T:K\to L(X,Y)$ an operator and $\eta:K\times K\to X$ a function. We say that T is weakly (C,η)-pseudomonotone on K if for any $x,y\in K$ one has that relation $\langle T(x),\eta(y,x)\rangle\notin -\operatorname{int}C$ implies that $\langle T(y),\eta(y,x)\rangle\notin -\operatorname{int}C$.

In the following we give an equivalent result to the Generalized Linearization Lemma given by Yu and Yao [23] for the case of problem ($PWVVI_\eta(\lambda_0)$).

PROPOSITION 2. *Let K be a nonempty convex subset of X, $T:K\to L(X,Y)$ an operator and $\eta:K\times K\to X$ a function such that for any $x,y\in K$, $\eta(x,y)+\eta(y,x)=0$ and $\eta(y,y+\alpha\eta(x,y))=-\alpha\eta(x,y)$, $\forall\alpha\in[0,1]$. Consider the problems*

(P1) *find $x\in K$ such that for any $y\in K$, $\langle T(x),\eta(y,x)\rangle\notin -\operatorname{int}C$;*

(P2) *find $x\in K$ such that for any $y\in K$, $\langle T(y),\eta(y,x)\rangle\notin -\operatorname{int}C$.*

Then the following statements hold:

i. *problem* (P1) *implies problem* (P2) if T *is weakly* (C,η)*-pseudomonotone*;

ii. *problem* (P2) *implies problem* (P1) if T is (v,η)-*hemicontinuous*.

REMARK 1. Observe that if the hypotheses of the above proposition are satisfied and if T is weakly (C,η)-pseudomonotone and (v,η)-hemicontinuous, then problems (P1) and (P2) are equivalent. Also, assertion (i) of Proposition 2 holds even in the absence of the conditions regarding the application η.

3. UPPER SEMICONTINUITY OF SOLUTION SETS OF PROBLEM ($PWVVI_\eta(\lambda)$)

In this section we give sufficient conditions for the upper semicontinuity of the application $S_\eta : \Lambda \to 2^K$, where $S_\eta(\lambda)$ represents the solution sets of problem ($PWVVI_\eta(\lambda)$) defined in the previous section.

THEOREM 1. L*et K be a nonempty convex subset of X. If*

i. $T(\cdot,\lambda)$ *is* (v,η)-*hemicontinuous and weakly* (C,η)-*pseudomonotone on* K *for any* $\lambda \in \Lambda$;

ii. $T(x,\cdot)$ *is continuous on* Λ *for any* $x \in K$;

iii. η *is continuous in the second argument*;

iv. $\eta(x,y)+\eta(y,z)=\eta(x,z)$ *for any* $x,y,z \in K$,

v. $\eta(y, y+\alpha\eta(x,y)) = -\alpha\eta(x,y)$, $\forall x,y \in K$, $\forall \alpha \in [0,1]$,

then $S_\eta(\cdot)$ *has compact values and it is B-usc on* Λ.

Proof. Let $\lambda \in \Lambda$. We first prove the compactness of the set $S_\eta(\lambda)$. Consider a sequence $x_n \in S_\eta(\lambda)$ with $x_n \to x$. As $x_n \in S_\eta(\lambda)$, from Proposition 2 we have

$$\langle T(y,\lambda), \eta(y,x_n)\rangle \in Y \setminus (-\operatorname{int} C),\ \forall y \in K. \tag{1}$$

As $T(y,\lambda) \in L(X,Y)$ and η is continuous in the second argument, we have $\lim_{n\to\infty} \langle T(y,\lambda), \eta(y,x_n)\rangle = \langle T(y,\lambda), \eta(y,x)\rangle$ and, since $Y \setminus (-\operatorname{int} C)$ is a closed set, from (1) we get

$$\langle T(y,\lambda), \eta(y,x)\rangle \in Y \setminus (-\operatorname{int} C),\ \forall y \in K. \tag{2}$$

We apply once again Proposition 2 and obtain

$$\langle T(x,\lambda), \eta(y,x)\rangle \in Y \setminus (-\operatorname{int} C),\ \forall y \in K, \tag{3}$$

that is $x \in S_\eta(\lambda)$. Therefore $S_\eta(\lambda)$ is a closed set. As $S_\eta(\lambda) \subset K$ and K is a compact set, $S_\eta(\lambda)$ is a compact set.

By Proposition 1 and the compactness of $S_\eta(\lambda)$, to prove that $S_\eta(\lambda)$ is B-usc at λ it is enough to verify that for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \lambda$ and for any $x_n \in S_\eta(\lambda_n)$, there exist $x_0 \in S_\eta(\lambda)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Let $\{\lambda_n\} \subset \Lambda$ be a sequence with $\lambda_n \to \lambda$. For any $x_n \in S_\eta(\lambda_n)$, we have $x_n \in K$ and

$$\langle T(x_n,\lambda_n), \eta(y,x_n)\rangle \in Y \setminus (-\operatorname{int} C),\ \forall y \in K. \tag{4}$$

Since K is a compact set, there exist $x_0 \in K$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. From (4) we get

$$\left\langle T\left(x_{n_k},\lambda_{n_k}\right),\eta\left(y,x_{n_k}\right)\right\rangle\in Y\setminus\left(-\operatorname{int}C\right),\ \forall y\in K \tag{5}$$

and, by Proposition 2, we obtain

$$\left\langle T\left(y,\lambda_{n_k}\right),\eta\left(y,x_{n_k}\right)\right\rangle\in Y\setminus\left(-\operatorname{int}C\right),\ \forall y\in K\ . \tag{6}$$

As $T(y,\cdot)$ and $\eta(y,\cdot)$ are continuous and

$$\begin{aligned}
&\left\|\left\langle T\left(y,\lambda_{n_k}\right),\eta\left(y,x_{n_k}\right)\right\rangle-\left\langle T(y,\lambda),\eta(y,x_0)\right\rangle\right\|\le\\
&\le\left\|\left\langle T\left(y,\lambda_{n_k}\right),\eta\left(y,x_{n_k}\right)\right\rangle-\left\langle T(y,\lambda),\eta\left(y,x_{n_k}\right)\right\rangle\right\|+\\
&+\left\|\left\langle T(y,\lambda),\eta\left(y,x_{n_k}\right)\right\rangle-\left\langle T(y,\lambda),\eta(y,x_0)\right\rangle\right\|\le\\
&\le\left\|T\left(y,\lambda_{n_k}\right)-T(y,\lambda)\right\|\left\|\eta\left(y,x_{n_k}\right)\right\|+\left\|T(y,\lambda)\right\|\left\|\eta\left(x_0,x_{n_k}\right)\right\|,
\end{aligned} \tag{7}$$

we get $\left\langle T\left(y,\lambda_{n_k}\right),\eta\left(y,x_{n_k}\right)\right\rangle\to\left\langle T(y,\lambda),\eta(y,x_0)\right\rangle$ as $n_k\to\infty$. From the closedness of $Y\setminus(-\operatorname{int}C)$ and from (6) it follows that

$$\left\langle T(y,\lambda),\eta(y,x_0)\right\rangle\in Y\setminus\left(-\operatorname{int}C\right),\ \forall y\in K\ . \tag{8}$$

Also, by Proposition 2, we have

$$\left\langle T(x_0,\lambda),\eta(y,x_0)\right\rangle\in Y\setminus\left(-\operatorname{int}C\right),\ \forall y\in K\ , \tag{9}$$

that is, $x_0\in S_\eta(\lambda)$.

REFERENCES

1. J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis* (Reprint of the 1984 original), Dover, Mineola (2006).
2. Y. H. Cheng and D. L. Zhu, *Global stability results for the weak vector variational inequality,* J. Global Optim., **32**, pp. 543–550, 2005.
3. T. D. Chuong, J. C. Yao, and N. D. Yen, *Further results on the lower semicontinuity of efficient point multifunctions,* Pacific Journal of Optimization, **6**, pp. 405–422, 2010.
4. T. D. Chuong, N. Q. Huy, and J. C. Yao, Stability *of semi-infinite vector optimization problems under functional perturbations,* Journal of Global Optimization. **45**, pp. 583–595, 2009.
5. S. Dafermos, *Sensitivity analysis in variational inequalities,* Math. Oper. Res., **13**, pp. 421-434, 1988,
6. Z. M. Fang and S. J. Li, *Upper semicontinuity of solution maps for a parametric weak vector variational inequality,* J. Ineq. Appl., Article ID 482726, doi:10.1155/2010/482726, 2010
7. F. Ferro, *A minimax theorem for vector-valued functions,* J. Optim. Theory Appl., **60**, pp. 19–31, 1989.
8. X. H. Gong, K. Kimura, and J.C. Yao, *Sensitivity analysis of strong vector equilibrium problems,* J. Nonlinear Convex Anal., **9**, pp. 83–94, 2008.
9. L. Gong, *Global stability result for the generalized quasivariational inequality problem,* J. Optim. Theory Appl.. **70**, pp. 365–375, 1991.
10. F. Jianghua and Z. Renyou, *Stability analysis for variational inequality in reflexive Banach space,* Nonlinear Anal., **69**, pp. 2566-2574, 2008.
11. P.Q. Khanh and L.M. Luu, *Lower semicontinuity and upper semicontinuity of the solution sets and approximate solution sets of parametric multivalued quasivariational inequalities,* J. Optim. Theory Appl., **133**, pp. 329–339, 2007.
12. B.T. Kien, *On the lower semicontinuity of optimal solution sets,* Optimization, **54**, pp. 123–130, 2005.
13. K. Kimura and J. C. Yao, *Semicontinuity of solution mappings of parametric generalized vector equilibrium problems,* J. Optim. Theory Appl., **138**, pp. 429–443, 2008.
14. K. Kimura and J. C. Yao, *Semicontinuity of solution mappings of parametric generalized strong vector equilibrium problems,* Journal of Industrial and Management Optimization, **4**, pp. 167–181, 2008.
15. K. Kimura and J. C. Yao, *Sensitivity analysis of solution mappings of parametric vector quasiequilibrium problems,* Journal of Global Optimization, **41**, pp. 187–202, 2008.

16. K. Kimura and J. C. Yao, *Sensitivity analysis of solution mappings of parametric generalized quasi-vector equilibrium problems,* Taiwanese Journal of Mathematics, **12**, pp. 2233–2268, 2008.
17. C.S. Lalitha and G. Bhatia, *Stability of Parametric Quasivariational Inequality of the Minty Type,* J. Optim. Theory Appl., **148**, pp. 281-300, 2011.
18. S. J. Li and C. R. Chen, *Stability of weak vector variational inequality,* Nonlinear Anal., **70**, pp. 1528–1535, 2009.
19. S. J. Li, G. Y. Chen, and K. L. Teo, *On the stability of generalized vector quasivariational inequality problems,* J. Optim. Theory Appl., **113**, pp. 283–295, 2002.
20. L.D. Muu, *Stability property of a class of variational inequalities,* Math. Operationsforsch. Stat. Ser. Optim., **15**, pp. 347–351, 1984.
21. V. Preda, M. Beldiman and A. Batatorescu, *On Variational-like Inequalities with Generalized Monotone Mappings, Generalized Convexity and Related Topics,* edited by I. Konnov, D.T. Luc and A. Rubinov, Lecture Notes in Economics and Mathematical Systems, **583**, pp. 415–432, 2006.
22. D.E. Stanciu, *Semicontinuity of solutions of Minty type quasivariational inequalities*, will appear in Revue Roumaine Math. Pures Appl.
23. R.L. Tobin, *Sensitivity analysis for variational inequalities,* J. Optim. Theory Appl., **48**, pp. 191–204, 1986.
24. S. J. Yu and J. C. Yao, *On vector variational inequalities,* J. Optim. Theory Appl., **89**, pp. 749–769, 1996.
25. J. Zhao, *The lower semicontinuity of optimal solution sets,* J. Math. Anal. Appl., **207**, pp. 240–254, 1997.

Received June 14, 2012