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MARKET CORES AND STABLE SETS FOR A REINSURANCE MARKET

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We propose a game-theoretic model for a reinsurance market of pure exchange. Some related sets on the core market are given.

Key words: reinsurance market, market core, optimal allocations.

1. INTRODUCTION

The most researches in (re)insurance mathematics are sustained by the theory of risk but there are some difficulties to put in practice results like competitive equilibrium, Pareto optimality, representative agent pricing and its implications for insurance premiums. For Pareto optimality relative to some optimization problems, see for examples [6, 7, 8]. In the last decades, many authors applied game theory ideas to the analysis of a reinsurance market, [4, 5], but recently Aase, [1, 2], create some connections between them finding the correspondences for "collective rationality", "social stability" and "individual rationality". We present in this paper an alternative model based on game theoretic approach. The solutions of risk allocation problem end up in the core. We also introduce some core catcher sets like market dominance core and stable market set.

2. PRELIMINARIES

Let $N = \{1, 2, ..., n\}$ be a group of n reinsures having preferences $\geq_i$, $i \in N$, over a suitable set of random variables denoted by R, or gambles with realizations (outcomes) in some $A \subseteq R$. We represent these preferences by von Neumann-Morgenstern expected utility [10], meaning that there is a set of continuous utility functions $u_i : R \to \mathbb{R}$ such that $X \geq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$, where by the symbol E we denoted the mean operator. We assume some properties: monotonic preferences and risk aversion, so that, we have $u_i'(\omega) > 0$, $u_i''(\omega) \leq 0$ for all ω in the relevant domains. In some of the cases we shall also require strict risk aversion, meaning strict concavity for some u_i. For a better understanding we presume that each agent is invested with a random variable payoff X_i called initial portfolio. More precisely, there exists a probability space (Ω, K, P) such that we have the payoff $X_i(\omega)$ when $\omega \in \Omega$ occurs and, moreover, both expected values and variances exist for all these initial portfolios, which means that all $X_i \in L^2(\Omega, K, P)$. Because every agent can treat any affordable contracts, we will have a new set of random variables Y_i, $i \in N$, representing the final portfolios.

The following notational convention will be used: if X and Y are two random variables, then by $X \leq Y$ we mean that $Y - X \geq 0$ $P - a.s.$, i.e., the random variable $Y - X$ is nonnegative a. s..

We also use the notation $X_S := \sum_{i \in S} X_i$, $\forall S \subseteq N$.

3. SOME CONNECTIONS BETWEEN GAME THEORY AND A REINSURANCE MARKET

Let $N=\{1,2,...,n\}$ be a set of agents and let S be an arbitrary subset of N. The characteristic function of the game $v:2^N \to \mathbb{R}$ gives the total payoff for the players who belong to the coalition S, $S \in 2^N$, payoff obtained by cooperating. Let z_i be the payoff to player i who cooperates in this game. So,

$$\sum_{i=1}^{n} z_i = v(N),$$

which represents the "collective rationality" and it means that the players who will obtain by cooperating the maximum total payoff.

This assumption corresponds to Pareto optimality in our reinsurance market, i.e., the optimal solution Y solving

$$\sum_{i=1}^{n} \lambda_i Eu_i(Y_i) = Eu_{\lambda_N}(X_N),$$

where $\lambda_N = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_i \in \mathbb{R}_+^*$, are the agent weights and X_N is the representative agent pricing.

In our approach, the "individual rationality" condition $z_i \geq v(\{i\})$ corresponds to $Eu_i(Y_i) \geq Eu_i(X_i)$, $i \in N$, which implies that no player will participate in the game if he can obtain more alone. This rationality assumption is natural to impose because it corresponds for any coalition of all players, i.e., for any $S \in 2^N$, so we can write

$$\sum_{i=1}^{n} z_i \geq v(S),\ \forall S \in 2^N.$$

We can call this condition: "social stability" and it corresponds in reinsurance market to a further restriction on the investor weights $\lambda \neq 0$ such that

$$\sum_{i \in S} \lambda_i Eu_i(Y_i) \geq Eu_{\lambda_S}(X_S),$$

where

$$Eu_{\lambda_S}(X_S) := \sup_{Z_i, i \in S} \sum_{i \in S} \lambda_i Eu_i(Z_i) \text{ s.t. } \sum_{i \in S} Z_i \leq \sum_{i \in S} X_i := X_S,$$

and

$$\lambda_S = (\lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_s}) = (\lambda_i)_{i \in S},\ \lambda_i \in \mathbb{R}_+^*,\ S = \{i_1, ..., i_s\},\ \mathrm{S} \in 2^N,\ |\mathrm{S}| = s.$$

The set of vectors Z which satisfies the above equation is called the core of the game and represents a very attractive solution when it exists, but for a large class of games it is empty. This concept is very useful in economic applications.

4. THE REPRESENTATION OF A MODEL FOR A REINSURANCE MARKET

In this section we propose a structure for a game model applied in the case of a reinsurance market [6].

***Definition* 4.1**. A competitive reinsurance market is a pair denoted by $\mathsf{RM}\{u\} = \left\langle N,\ \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \right\rangle$ consisting of the agent set $N = \{1,2,...,n\}$ interpreted as (re)insurers where the function $u_\lambda(\cdot): R \to \mathbb{R}$ is the von Neumann-Morgenstern expected utility function and $Eu_{\lambda_\varnothing} = 0$.

Let $\mathsf{RM}(N,X)=\{\mathsf{RM}\{u\}\mid u\in\mathsf{U}\}$ denote the set of the all reinsurance markets where $N=\{1,2,...,n\}$ is the set of the players, $X=(X_1,...,X_n)$ the initial random vectors, $X_i\in L^2(\Omega,K,P)$ and $\mathsf{U}=\{u\mid u:R\to\mathbb{R}\}$ the set of utilities.

***Definition* 4.2**. A competitive reinsurance market is superadditive if

$$Eu_{\lambda_{S\cup T}}(X_{S\cup T})\geq Eu_{\lambda_S}(X_S)+Eu_{\lambda_T}(X_T) \text{ for all } S,T\subset N \text{ and } S\cap T=\varnothing .$$

We have the following result.

THEOREM 4.3. *For all* $S,T\subset N$ *and* $S\cap T=\varnothing$ *we have*

$$Eu_{\lambda_{S\cup T}}(X_{S\cup T})\geq Eu_{\lambda_S}(X_S)+Eu_{\lambda_T}(X_T).$$

Proof. We defined above the mean of utility for a representative agent like a supremum. So, we have

$$Eu_{\lambda_S}(X_S)=\sup_{\sum_{i\in S}Z_i\leq\sum_{i\in S}X_i,}\sum_{i\in S}\lambda_i Eu_i(Z_i)\ ,$$

$$Eu_{\lambda_T}(X_T)=\sup_{\sum_{i\in T}Z_i\leq\sum_{i\in T}X_i,}\sum_{i\in T}\lambda_i Eu_i(Z_i)$$

and

$$Eu_{\lambda_{S\cup T}}(X_{S\cup T})=\sup_{\sum_{i\in S\cup T}Z_i\leq\sum_{i\in S\cup T}X_i,}\sum_{i\in S\cup T}\lambda_i Eu_i(Z_i).$$

For proving this inequality we use some notation for a more convenience writing. We denote

$$\mathsf{Z}_1:=\{(Z_{i_1},...,Z_{i_s})\mid Z_{i_1}+...+Z_{i_s}\leq X_S\}$$
$$\mathsf{Z}_2:=\{(Z_{j_1},...,Z_{j_t})\mid Z_{j_1}+...+Z_{j_t}\leq X_T\}$$
$$\mathsf{Z}_3:=\{(Z_{i_1},...,Z_{i_s},Z_{j_1},...,Z_{j_t})\mid Z_{i_1}+...+Z_{i_s}+Z_{j_1}+...+Z_{j_t}\leq X_S+X_T\}$$
$$\mathsf{Z}_4:=\{(Z_{i_1},...,Z_{i_s},Z_{j_1},...,Z_{j_t})\mid Z_{i_1}+...+Z_{i_s}\leq X_S,\ Z_{j_1}+...+Z_{j_t}\leq X_T\}$$

so, by using the above notation we have to prove that

$$\sup_{\mathsf{Z}_3}\sum_{i\in S\cup T}\lambda_i Eu_i(Z_i)\geq\sup_{\mathsf{Z}_1}\sum_{i\in S}\lambda_i Eu_i(Z_i)+\sup_{\mathsf{Z}_2}\sum_{i\in T}\lambda_i Eu_i(Z_i).$$

Since $S\cap T=\varnothing$ we have

$$\sup_{\mathsf{Z}_4}\sum_{i\in S\cup T}\lambda_i Eu_i(Z_i)=\sup_{\mathsf{Z}_4}\left(\sum_{i\in S}\lambda_i Eu_i(Z_i)+\sum_{i\in T}\lambda_i Eu_i(Z_i)\right)$$

Hence

$$\sup_{\mathsf{Z}_4}\sum_{i\in S}\lambda_i Eu_i(Z_i)+\sup_{\mathsf{Z}_4}\sum_{i\in T}\lambda_i Eu_i(Z_i)=\sup_{\mathsf{Z}_1}\sum_{i\in S}\lambda_i Eu_i(Z_i)+\sup_{\mathsf{Z}_2}\sum_{i\in T}\lambda_i Eu_i(Z_i)=Eu_{\lambda_S}(X_S)+Eu_{\lambda_T}(X_T).$$

The inequality from theorem is proved if we observe that $\mathsf{Z}_4\subseteq\mathsf{Z}_3$, and this imply that

$$\sup_{\mathsf{Z}_4}\sum_{i\in S\cup T}\lambda_i Eu_i(Z_i)\leq\sup_{\mathsf{Z}_3}\sum_{i\in S\cup T}\lambda_i Eu_i(Z_i).$$ □

In the following we denote the set of "investor weights" $\lambda_i,\ i \in N$, by

$$I^{**} = \left\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i Eu_i\left(Y_i\right) \le Eu_{\lambda_N}\left(X_N\right) \right\}$$

and by I^* the set of efficient "investor weights" vectors in the reinsurance market $\left\langle N,\ \left\{Eu_{\lambda_S}\left(X_S\right)\right\}_{S\subseteq N} \right\rangle$, i.e.,

$$I^{*} = \left\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i Eu_i\left(Y_i\right) = Eu_{\lambda_N}\left(X_N\right) \right\}.$$

Obviously we have $I^* \subset I^{**}$.

The "individual rationality" condition $Eu_i\left(Y_i\right) \ge Eu_i\left(X_i\right),\ i \in N,$ should hold in order that a weight vector λ has a real chance to be realized in the reinsurance market.

***Definition* 4.4**. A weight vector $\lambda \in \mathbb{R}^n$ is an imputation for the reinsurance market $\left\langle N,\ \left\{Eu_{\lambda_S}\left(X_S\right)\right\}_{S\subseteq N} \right\rangle$ if it is efficient and has the property of individual rationality, i.e.,

$$\mathit{1.}\ \sum_{i=1}^{n} \lambda_i Eu_i\left(Y_i\right) = Eu_{\lambda_N}\left(X_N\right);$$

$$\mathit{2.}\ Eu_i\left(Y_i\right) \ge Eu_i\left(X_i\right),\ i \in N.$$

We denote by I the set of imputations λ. Clearly, I is empty if and only if $\sum_{i=1}^{n} \lambda_i Eu_i\left(Y_i\right) > Eu_{\lambda_N}\left(X_N\right)$. Else, in the situation when $\sum_{i=1}^{n} \lambda_i Eu_i\left(Y_i\right) < Eu_{\lambda_N}\left(X_N\right)$, the reinsurance market is called N-essential and I is an infinite set [6].

5. A CHARACTERIZATION FOR STABLE SETS AND MARKET DOMINANCE CORE

Baton and Lemaire have determined the special set for imputations [3], called the core, for negative exponential utilities in a reinsurance market and later, Aase [2], in the same situation determined the core renouncing at their independence assumption.

Because the set of imputations are too large for an essential reinsurance market, so, we need some criteria to single out those imputations that have the chance to appear. So, we could obtain some subsets of I as solution concepts. One of this solution concept is the core of a reinsurance market.

***Definition* 5.1**. The market core, denoted by MC, of a reinsurance market $\left\langle N,\ \left\{Eu_{\lambda_S}\left(X_S\right)\right\}_{S\subseteq N} \right\rangle$ is the set

$$MC = \left\{ \lambda \in I \mid \sum_{i \in S} \lambda_i Eu_i\left(Y_i\right) \ge Eu_{\lambda_S}\left(X_S\right),\ \forall S \subseteq N \right\}. \tag{1}$$

If $MC \ne \varnothing$ then the elements of MC can easily be obtained because the core is defined with the aid of a finite system of inequalities.

Other important subsets of imputations are the dominance market core (DM-core) and stable sets. These kinds of sets are defined in the following dominance relation over vectors in $\mathbb{R}^n$.

***Definition* 5.2**. Let $\left\langle N,\ \left\{Eu_{\lambda_S}\left(X_S\right)\right\}_{S\subseteq N} \right\rangle$ be a reinsurance market, and let $\lambda,\ \lambda' \in I,\ S \subseteq N$. We say that λ is better than λ' with respect to group of agents S, and denote it by $\lambda \operatorname{dom}_S \lambda',$ if

1. $\lambda_i > \lambda'_i,\ \forall i \in S$

2. $\sum_{i \in S} \lambda_i Eu_i(Y_i) \le Eu_{\lambda_S}(X_S),\ \forall S \subseteq N$.

We can interprete if first condition holds then the weights of vector λ are better than weight vector λ' for all agents of S; second condition guarantees that the weight vector λ is reachable for S.

***Definition* 5.3.** Let $\left\langle N,\ \left\{Eu_{\lambda_S}(X_S)\right\}_{S \subseteq N}\right\rangle$ be a reinsurance market with $\lambda,\ \lambda' \in I$.. We say that λ is better than λ', and denote it by $\lambda \mathrm{dom} \lambda'$, if there exists $S \subseteq N$ such that $\lambda \mathrm{dom}_S \lambda'$.

***Definition* 5.4.** The market dominance core MDC of a reinsurance market $\left\langle N,\ \left\{Eu_{\lambda_S}(X_S)\right\}_{S \subseteq N}\right\rangle$ consists of all undominated elements in I, i.e., $I \setminus \bigcup_{S \subseteq N} MD(S)$, where $MD(S)$ is the set of imputations which are dominated with respect to S.

We denote by $\mathrm{dom}A$ the set consisting of all imputations that are dominated by some element in A.

***Definition* 5.5.** For a reinsurance market $\left\langle N,\ \left\{Eu_{\lambda_S}(X_S)\right\}_{S \subseteq N}\right\rangle$ a subset K of I is called a stable set if

Internal stability: $K \cap \mathrm{dom}K = \varnothing$;
External stability: $I \setminus K \subset \mathrm{dom}K$.

We can note that K and $\mathrm{dom}K$ form a partition of I. The notation and the interpretation were given by Neumann and Morgenstern [10]. By internal stability all weights of agents in K are "equal" with respect to the dominance relation via group of agents and by external stability we understand that it exists a group of agents that prefers one of the achievable weight vector inside K.

THEOREM 5.6. *Let* $\left\langle N,\ \left\{Eu_{\lambda_S}(X_S)\right\}_{S \subseteq N}\right\rangle$ *be a reinsurance market, and* K *a stable set for the set of weight vectors. Then*

1. $MC \subset MDC \subset K$,
2. *If the reinsurance market is superadditive, then* $MC = MDC$,
3. *If* MDC *is a stable set, then there is no other stable set.*

Proof. 1. To prove the first implication $MC \subset MDC$, we suppose that it exists $\lambda \in MC$ such that $\lambda \notin MDC$. In this case, there is $\lambda' \in I$ and a group of agents $S \in 2^N \setminus \{\varnothing\}$ such that $\lambda \mathrm{dom}_S \lambda'$. Then

$$Eu_{\lambda_S}(X_S) \ge \sum_{i \in S} \lambda'_i Eu_i(Y_i) > \sum_{i \in S} \lambda_i Eu_i(Y_i) \ge Eu_{\lambda_S}(X_S)$$

which implies that $\lambda \notin MC$, false.

To show the last implication $MDC \subset K$, it is sufficient to prove that $I \setminus K \subset I \setminus MDC$. Let be $\lambda \in I \setminus K$. From the definition of the external stability of K, we have $\lambda' \in K$ with $\lambda' \mathrm{dom} \lambda$. Because the elements from MDC are not dominated, we have $\lambda \notin MDC$, that imply that $\lambda \in I \setminus MDC$.

2. For a better understanding we divide this proof in two parts:

First time we show that for $\lambda \in I$ with $\sum_{i \in S} \lambda_i Eu_i(Y_i) < Eu_{\lambda_S}(X_S)$ for some $S \subseteq N,\ S \ne \{\varnothing\}$, there exists $\lambda' \in I$ such that $\lambda' \mathrm{dom}_S \lambda$. Let us define λ' as

$$\lambda'_i Eu_i(Y_i) := \lambda_i Eu_i(Y_i) + \frac{Eu_{\lambda_S}(X_S) - \sum_{i \in S} \lambda_i Eu_i(Y_i)}{|S|}, \text{ if } i \in S$$

$$\lambda_1' Eu_i\left(Y_i\right) := \lambda_i Eu_i\left(Y_i\right) + \frac{Eu_{\lambda_N}\left(X_N\right) - Eu_{\lambda_S}\left(X_S\right) - \sum_{i \in N \setminus S} \lambda_i Eu_i\left(Y_i\right)}{\left|N \setminus S\right|} \text{ if } i \notin S .$$

Then $\lambda' \in I$ and to demonstrate that $\lambda_1' Eu_i\left(Y_i\right) \geq \lambda_1 Eu_i\left(X_i\right)$ for $i \in N \setminus S$ we use the superadditivity of the reinsurance market. Also, we obtain that $\lambda \operatorname{dom}_S \lambda'$.

Second for proving $MC = MDC$, we already have from 1) that $MDC \subset MC$. Now, we just have to suppose $\lambda \in MDC$. In this situation there exists no $\lambda' \in I$ such that $\lambda' \operatorname{dom} \lambda$. If we consider the situation from the first part we have $\sum_{i \in S} \lambda_1 Eu_i\left(Y_i\right) \geq Eu_{\lambda_S}\left(X_S\right),\ \forall S \subseteq N,\ S \neq \varnothing$. So, $\lambda \in MC$.

3) We suppose that MDC is a stable set and, also, that K is another stable set. By 1) we have $MDC \subset K$. To show that, in fact, these two sets are equal we have just to prove that $K \setminus MDC = \varnothing$. By negation, we suppose that there exists $\lambda \in K \setminus MDC$. From the condition of external stability of MDC there exists $\lambda \in MDC(\subset K)$ such that $\lambda' \operatorname{dom} \lambda$. This is a contradiction to the condition of internal stability of the stable set K. So, $K \setminus MDC = \varnothing$.

Another market core-like solution concept that is based on the norm of equity [11], is the equal market division core $EMDC$. This is the set

$$\left\{ \lambda \in I \mid \exists S \subset N \text{ such that } \frac{Eu_{\lambda_S}\left(X_S\right)}{|S|} > \lambda_i Eu_i\left(Y_i\right),\ \forall i \subseteq S \right\}.$$

It contains those elements can be seen as efficient" investor weight" vectors for the grand group of agents which cannot be improved upon by the equal division allocation of any subgroup of agents.

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