# TOPOLOGICAL SOLITONS AND LIE SYMMETRY ANALYSIS FOR THE KADOMTSEV-PETVIASHVILI-BURGERS EQUATION WITH POWER LAW NONLINEARITY IN DUSTY PLASMAS

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This paper studies the Kadomtsev-Petviashvili-Burgers equation with power law nonlinearity that arises in the study of dusty plasmas. The traveling wave hypothesis reveals the topological 1-soliton solution or the shock wave solution to the equation. Painlevé analysis is performed to check the Painlevé property and the Lie-group formalism is applied to investigate the symmetries. We derive the infinitesimals that admit the classical symmetry group. Partial differential equations are investigated by solving the corresponding characteristic equations. The Lie group formalism is again applied on investigated partial differential equations to deduce symmetries and the ordinary differential equations deduced from subalgebras are further studied and some exact solutions are obtained.

Key words: solitons, integrability, Lie symmetry analysis, Painlevé analysis.

### **1. INTRODUCTION**

The study of nonlinear evolution equations (NLEEs) appears in various areas of Applied Mathematics and Theoretical Physics as well as in Engineering Sciences. Many physical and applied phenomena are governed by various forms of NLEEs [1–21]. Therefore, it is imperative to take a close look into the study of these NLEEs from a very serious standpoint. One of the most important aspects of these NLEEs is the integrability issue. The integration of such equations to obtain the solutions lead to a better understanding of the physical phenomena that these equations model. While numerical simulation gives a visualization effect to picturize the solution, it is always very helpful to get a closed form analytical solution to carry out further analysis of the governing phenomena. For example, unless an analytical solution to a NLEE is known, it is not possible to analytically study the effect of stochastic perturbation of that NLEE, since the corresponding Langevin equation cannot be formulated. Therefore, this paper is going to address of one such NLEE that appears in the study of dusty plasmas. This is the Kadomstev-Petviashvili-Burgers (KP-Burgers) equation. In order to keep it on a generalized setting, this KP-Burgers equation will be studied with power law nonlinearity in this paper.

### 2. GOVERNING EQUATION

The dimensionless form of the KP-Burgers equation that is going to be studied in this paper is given by

$$\left(q_t + aq^n q_x + bq_{xx}\right)_x + cq_{yy} = 0 \tag{1}$$

Equation (1) arises in the study of dusty plasmas. This equation also models the two-dimensional propagation of fast and slow magnetosonic modes in warm collisional plasma [5]. Additionally, Eq. (1) governs nonlinear dust acoustic shock waves [20]. In Eq. (1), q(x, y, t) represents the wave profile. The independent variables are t, x and y where the first variable represents the temporal variable while the other two represents the spatial variables. Also, a, b and c are constants. The power law nonlinearity parameter is given by n.

In this paper, Eq. (1) is going to be first solved by the traveling wave hypothesis where a shock wave solution will be derived and the corresponding parameter constraints will fall out. Subsequently, the Lie symmetry approach will be used to derive a few additional solutions that will be useful in the study of this equation in plasmas.

## 2.1. Traveling Waves

In order to solve Eq. (1) by traveling waves, the starting hypothesis is going to be

$$q(x, y, t) = g(B_1 x + B_2 y - vt),$$
<sup>(2)</sup>

where in (2), g(s) represents the wave profile and

$$s = B_1 x + B_2 y - vt. \tag{3}$$

Here  $B_1$  and  $B_2$  are related to the direction ratios of the solitary wave profile and v is the velocity of the soliton. Substituting this hypothesis into (1) yields the corresponding Ordinary differential equation (ODE) as

$$-vB_1g'' + aB_1^2(g''g') + bB_1^3g'' + cB_2^2g'' = 0.$$
(4)

Integrating (4) twice with respect to s and choosing the integration constant to be zero, without any loss of generality, yields to

$$\frac{dg}{ds} = \frac{vB_1 - cB_2^2}{bB_1^3}g - \frac{a}{(n+1)bB_1}g^{n+1}.$$
(5)

Separating variables and integrating gives

$$\frac{as}{(n+1)bB_1} = \int \frac{\mathrm{d}g}{g\left(\alpha - g^n\right)},\tag{6}$$

where

$$\alpha = \frac{(n+1)\left(vB_1 - cB_2^2\right)}{aB_1^2},\tag{7}$$

which leads to, after carrying out the integration,

$$q(x, y, t) = \left\{ \frac{(n+1)\left(vB_1 - cB_2^2\right)}{2aB_1^2} \right\}^{\frac{1}{n}} \left[ 1 + \tanh\left\{ \frac{n\left(vB_1 - cB_2^2\right)}{2bB_1^3} \left(B_1x + B_2y - vt\right)\right\} \right]^{\frac{1}{n}}.$$
(8)

This represents the *topological 1-soliton solution*, that is also known as domain wall in (1+2)-D or a shock wave solution. This solution stays valid provided

$$aB_{1}^{2}\left(vB_{1}-cB_{2}^{2}\right) > 0, \qquad (9)$$

for even n. However, if n is odd, so such restrictions are required.

### 2.2. Painlevé Analysis

Weiss et al. [18] have introduced the Painlevé test for partial differential equations (PDEs) and has shown that there exists a close relationship between Painlevé property (PP) and integrability. While extending the idea of connection between PP and its integrability in the case of ODE(s) or PDE(s), Weiss et al. [18] have required that the solutions be single-valued around movable singularity manifolds.

Equation (1) will have the PP if its solution q(x, y, t) can be represented as a single valued expansion about its moving singular manifold. More precisely, if q is solution of the PDE then there is a Painlevé expansion

$$q(x, y, t) = \phi^{\alpha}(x, y, t) \sum_{j=0}^{\infty} q_j(x, y, t) \phi^j(x, y, t),$$
(10)

where  $\phi(x, y, t)$  and the expansion coefficients  $q_j(x, y, t)$  are analytic functions of the independent variables.

Inserting expansion (9) into Eq. (1), a leading order analysis uniquely determines the possible values of  $\alpha$ . From the dominant behaviour analysis of Eq. (1), we get  $\alpha = -\frac{1}{n}$ , where n > 0. Substituting (9) with

 $\alpha = -\frac{1}{n}$  into (1) leads to

$$q_{0} = \left(\frac{\left(\frac{\partial}{\partial x}\phi(x,t)\right)b(n+1)}{an}\right)^{\frac{1}{n}}.$$
(11)

Substituting (9) with (10) into Eq. (1), it is found that resonances occur at

$$j = -1, \frac{n+1}{n}, \frac{1+2n}{n}.$$
 (12)

The resonance at j = -1 corresponds to the arbitrary function  $\phi$  defining the singularity manifold for the Eq. (1). Other resonance values should be positive integers. So the Eq. (1) for n > 1 does not pass Painlevé test.

Now we will consider the case when n=1. Corresponding resonance values are -1, 2, 3. After detailed calculation, we find that compatibility condition at j=3 is not satisfied identically. So the Eq. (1) for n > 1 and n = 1 does not posses PP.

To derive exact solution of Eq. (1) for n = 1, let us truncate the Laurent series (9) at the constant level term to give

$$q = q_0 \phi^{-1} + q_1. \tag{13}$$

Next we use the Kruskal's [6] form

$$\phi(x, y, t) = x + y - \gamma(t), \tag{14}$$

where  $\gamma(t)$  is an arbitrary function of t. Substituting (13) with (14) in (1), we get

$$q_0 = \frac{2b}{a}, \quad q_1 = -\frac{c + \gamma'(t)}{a},$$
 (15)

where () denotes derivative with respect to t. Thus for n = 1, we arrive at following solution of Eq. (1)

q =	<u> </u>	(16)
$a(x+y-\gamma(t))$	a	

#### 2.3. Symmetry Analysis

In this subsection, we apply Lie classical method to find symmetries of Eq. (1) and we find some exact solutions.

### 2.3.1. Lie Symmetry Approach

Lie method of infinitesimal transformation groups which essentially reduces the number of independent variables in PDE and reduces the order of ODE has been widely used in equations of mathematical physics, some recent and important contributions are in [7, 8, 11, 12]. The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations and the associated determining equations are an overdetermined linear system. We let the group of infinitesimal transformations be defined as

$$t^{*} = t + \varepsilon \tau(x, y, t, q) + O(\varepsilon^{2})$$

$$x^{*} = x + \varepsilon \xi(x, y, t, q) + O(\varepsilon^{2})$$

$$y^{*} = y + \varepsilon \phi(x, y, t, q) + O(\varepsilon^{2})$$

$$q^{*} = q + \varepsilon \eta(x, y, t, q) + O(\varepsilon^{2})$$
(17)

and impose the invariance condition on (1). The invariance under (16) means that if q is solution of Eq. (1), then  $q^*$  is also a solution of it.

Herein, too, on invoking the invariance criterion as mentioned in [9], the following relation from the coefficients of the first order of  $\varepsilon$  is deduced:

$$\eta^{xt} + a(q^n \eta^{xx} + nq^{n-1}q_{xx}\eta + 2nq^{n-1}q_x\eta^x + n(n-1)q^{n-2}q_x^2\eta) + b\eta^{xxx} + c\eta^{yy} = 0,$$
(18)

where  $\eta^x, \eta^{xx}, \eta^{xt}, \eta^{yy}$  and  $\eta^{xxx}$  are extended (prolonged) infinitesimals acting on an enlarged space corresponding to  $q_x, q_{xx}, q_{xt}, q_{yy}$  and  $q_{xxx}$ , respectively. Using the expressions for  $\eta^x, \eta^{xx}, \eta^{xt}, \eta^{yy}$ , and  $\eta^{xxx}$  in equation (17) and  $q_{xt}$  must be replaced by Eq. (1). On substituting the coefficients of different differentials equal to zero lead to a number of PDEs in  $\tau, \xi, \phi$  and  $\eta$ , that need to be satisfied. The general solution of this large system helps us to obtain the infinitesimals  $\tau, \xi, \phi$  and  $\eta$ , as follow

$$\eta = 0$$
;  $\tau = C_1$ ,  $\xi = -C_2 \frac{y}{2c} + C_4$ ,  $\phi = C_2 t + C_3$ , (19)

where  $C_1, C_2, C_3$ , and  $C_4$  are arbitrary constants.

The corresponding vector fields are

$$V_1 = \frac{\partial}{\partial t}$$
;  $V_2 = -\frac{y}{2c}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}$ ;  $V_3 = \frac{\partial}{\partial y}$ ;  $V_4 = \frac{\partial}{\partial x}$ . (20)

# 2.3.2. Similarity Reductions and Exact Solutions

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In this section, we will use the symmetries calculated in the previous subsection to obtain exact solutions of (1).

2b

 $c + \gamma'(t)$ 

To obtain the symmetry reductions of Eq. (1), we have to solve the characteristic equation

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\phi} = \frac{\mathrm{d}q}{\eta},\tag{21}$$

where  $\tau, \xi, \phi$  and  $\eta$  are given by (18). To solve (20), we will consider two cases: (i)  $V_1 + V_2$ , (ii)  $\alpha_1 V_1 + \alpha_3 V_3 + \alpha_4 V_4$ 

**Case (i):**  $V_1 + V_2$ 

Corresponding similarity variables are

$$\xi^{1} = \frac{t^{2}}{2} - y, \ \xi^{2} = 2cx + \frac{t^{3}}{6} - (\frac{t^{2}}{2} - y)t,$$
(22)

$$q = F(\xi^1, \xi^2),$$
 (23)

where  $\xi^1, \xi^2$  are new independent variables and  $F(\xi^1, \xi^2)$  is a new dependent variable. Substituting (22) with (21) in (1), we have

$$-2\xi^{1}F_{\xi^{2}\xi^{2}} + 4ac(F^{n}F_{\xi^{2}})_{\xi^{2}} + 8bc^{2}F_{\xi^{2}\xi^{2}\xi^{2}} + F_{\xi^{1}\xi^{1}} = 0.$$
<sup>(24)</sup>

Again applying Lie symmetry method on Eq. (23), we get symmetries as follows:

$$\tau^{1} = 0 \quad ; \quad \tau^{2} = C \quad ; \quad \eta^{1} = 0,$$
 (25)

where  $\tau^1, \tau^2, \eta^1$  are infinitesimals corresponding to  $\xi^1, \xi^2, F$ , respectively. Solving the characteristic equation we have the following similarity variables of Eq. (23)

$$\zeta = \xi^1, F = G(\zeta), \tag{26}$$

where  $\zeta$  is a new independent variable and G is a new dependent variable. The corresponding solution of main Eq. (1) can be given as

$$q = k_1 \left(\frac{t^2}{2} - y\right) + k_2,$$
(27)

where  $k_1$  and  $k_2$  are arbitrary constants.

Case (ii):  $\alpha_1 V_1 + \alpha_3 V_3 + \alpha_4 V_4$ 

The corresponding similarity variables are

$$\xi^{1} = \alpha_{1}x - \alpha_{4}t, \quad \xi^{2} = \alpha_{3}t - \alpha_{1}y \tag{28}$$

$$q = F(\xi^1, \xi^2),$$
(29)

where  $\xi^1, \xi^2$  are new independent variables and F is a new dependent variable.

Using (28) with (27) in Eq. (1), yields the second type of similarity reduction

$$\alpha_{3}F_{\xi^{1}\xi^{2}} + a\alpha_{1}(F^{n}F_{\xi^{1}})_{\xi^{1}} - \alpha_{4}F_{\xi^{1}\xi^{1}} + \alpha_{1}^{2}bF_{\xi^{1}\xi^{1}\xi^{1}} + c\alpha_{1}F_{\xi^{2}\xi^{2}} = 0.$$
(30)

which is a nonlinear PDE in two independent variables. We further reduce (29) using its symmetries. The Eq. (29) has the following two translational symmetries:

$$\Gamma_1 = \frac{\partial}{\partial \xi^1} \quad ; \quad \Gamma_2 = \frac{\partial}{\partial \xi^2}.$$
 (31)

The combination  $\Gamma_1 + \alpha_5 \Gamma_2$ , where  $\alpha_5$  is a constant, of the two symmetries  $\Gamma_1$  and  $\Gamma_2$ , yields the following two invariants:

$$\tau^{1} = \xi^{2} - \alpha_{5}\xi^{1}, F = G(\tau^{1}), \tag{32}$$

which gives a group invariant solution  $G(\tau^1)$  and consequently using these invariants (1) is transformed into the third-order nonlinear ODE

$$(c\alpha_{1} - \alpha_{3}\alpha_{5} - \alpha_{1}\alpha_{5}^{2})G'' - \alpha_{1}\alpha_{5}^{3}bG''' + a\alpha_{1}\alpha_{5}^{2}(G^{n}G')' = 0,$$
(33)

where (') denotes derivative with respect to  $\tau^1$ .

Solving this equation and taking the first two constants of integration to be zero and reverting back to the original variables, we obtain the following group-invariant solutions of the KP-Burgers Eq. (1):

(i) 
$$q(x, y, t) = \left(\frac{A_1}{2A_3}\right)^{\frac{1}{n}} \left[-1 + \coth\left(k_1 + \frac{A_1}{2A_2}n((\alpha_3 + \alpha_4\alpha_5)t - \alpha_1\alpha_5x - \alpha_1y)\right)\right]^{\frac{1}{n}},$$
 (34)

(ii) 
$$q(x, y, t) = A_1^{\frac{1}{n}} \left[ -A_3 + k_1 A_1 e^{\frac{nA_1}{A_2} ((\alpha_3 + \alpha_4 \alpha_5)t - \alpha_1 \alpha_5 x - \alpha_1 y)} \right]^{\frac{-1}{n}},$$
 (35)

where  $A_1 = (c\alpha_1 - \alpha_3\alpha_5 - \alpha_1\alpha_5^2)$ ,  $A_2 = -\alpha_1\alpha_5^3b$ ,  $A_3 = \frac{a\alpha_1\alpha_5^2}{n+1}$ , and  $k_1$  is an arbitrary constant.

# **3. CONCLUSIONS**

This paper studied the KP-Burgers equation, with power law nonlinearity. Such an equation appears in the study of dusty plasmas. The traveling wave solutions reveals *the kink* or *the shock wave solution*. Then the Painlevé analysis gives a rational solution to the KP-Burgers equation. Finally, the Lie symmetry analysis also reveals a couple of other new solutions. These are singular solutions as well as kinks or shock waves. In particular these kinks being in (1+2) dimensions are known as *domain walls*.

These special solutions are going to be very useful in the study of dusty plasmas. In future, this equation will be studied further. There are several other approaches that will be used to integrate this equation. Additionally, the time-dependent coefficients are going to be taken into consideration. This is a situation that is much closer to physical reality. Such results will be reported in future publications.

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Received October 8, 2012