# MULTIPLE SOLITON SOLUTIONS FOR TWO INTEGRABLE COUPLINGS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION

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In this work, we use the algebra of coupled scalars to develop two kinds of nonlinear integrable couplings of the modified Korteweg-de Vries (mKdV) equation. One of the integrable couplings of the mKdV equation gives multiple soliton solutions of distinct amplitudes, whereas the second kind gives multiple singular soliton solutions of distinct amplitudes as well. The Bäcklund transformation and the simplified Hirota's method will be used for this study. We show that these couplings possess multiple soliton solutions the same as the multiple soliton solutions of the mKdV equation, but differ only in the coefficients of the Bäcklund transformation. This difference exhibits soliton solutions with distinct amplitudes.

*Key words*: integrable couplings, bäcklund transformations, modified Korteweg-de Vries equation, multiple soliton solutions.

### **1. INTRODUCTION**

The ubiquitous Korteweg-de Vries (KdV) equation [1–12] in dimensionless variables reads

$$u_t \pm 6uu_x + u_{xxx} = 0. \tag{1}$$

This equation models a variety of nonlinear wave phenomena such as shallow water waves, acoustic waves in a harmonic crystal, and ion-acoustic waves in plasmas. The KdV equation is completely integrable and gives rise to multiple-soliton solutions. This equation has been studied by a variety of methods such as the inverse scattering method and the Bäcklund transformation method. The KdV equation admits multiplesoliton solutions and exhibits an infinite number of conservation laws of energy.

The modified KdV (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{2}$$

is important in many areas of nonlinear science. The mKdV equation appears in acoustic waves in certain anharmonic lattices, models of traffic congestion, transmission lines in Schottky barrier, Alfvén waves in a collision less plasma, ion acoustic solitons, elastic media, and in other applications. It possesses many remarkable properties such as conservation laws, inverse scattering transformation, bilinear transformation, multiple soliton solutions, breather solutions, Painlevé integrability, and Darboux transformation.

The theory of nonlinear integrable couplings of ordinary soliton systems was presented in [2,3] and further studied in [1] and others. In [2, 3], Ma *et al.* proposed the perturbation method for establishing integrable couplings. Zhang *et al.* [4] presented the enlarged Lie algebra method to obtain integrable couplings. Particularly noteworthy are the constructions of integrable couplings based on the non-semi simple Lie algebras [1]. In fact, there are several methods adopted to construct integrable couplings, such as perturbations, enlarging the spectral problem, creating new loop algebras, and semi direct sums of Lie algebra. Lot of work has been done in this field and many integrable couplings were constructed [5,6]. It is now known that for an integrable system, we can construct a new integrable differential equation system, called integrable couplings, which includes the given integrable equation as a sub-system.

In [1], a very natural triangular nonlinear couplings of integrable systems were developed. The construction in [1] was made on the level of evolution equations by a modification of the algebra of dynamical fields. The algebra of coupled scalars was used to develop n-coupled KdV (nc-KdV), given in the form

$$\begin{aligned} &(u_{1})_{t} + (u_{1})_{xxx} + 6u_{1}(u_{1})_{x} = 0, \\ &(u_{2})_{t} + (u_{2})_{xxx} + 6u_{2}(u_{2})_{x} + 6(u_{1}u_{2})_{x} = 0, \\ &(u_{3})_{t} + (u_{3})_{xxx} + 6u_{3}(u_{3})_{x} + 6(u_{1}u_{3})_{x} + 6(u_{2}u_{3})_{x} = 0, \\ &(u_{4})_{t} + (u_{4})_{xxx} + 6u_{4}(u_{4})_{x} + 6(u_{1}u_{4})_{x} + 6(u_{2}u_{4})_{x} + 6(u_{3}u_{4})_{x} = 0, \\ &\vdots, \end{aligned}$$
(3)  
$$\begin{aligned} &\vdots, \\ &(u_{n})_{t} + (u_{n})_{xxx} + 6u_{n}(u_{n})_{x} + 6\sum_{k=1}^{n-1} (u_{k}u_{n})_{x} = 0. \end{aligned}$$

The algebra of coupled scalars was introduced in [1] and was shown to be unital, commutative and associative. Moreover, the integrability of the couplings (3) was examined in [1].

Many reliable methods are used in the solitary waves theory to investigate solitons, and in particular multiple soliton solutions of completely integrable equations. The algebraic-geometric method, the inverse scattering method, the Bäcklund transformation method, the Darboux transformation method, the Hirota bilinear method, and other methods are used to make progress and new developments in this filed. In this work we aim to apply the Bäcklund transformations and the simplified Hirota's method [13–20] for a reliable study.

Our aim from this work is two fold. The first goal is to employ the developed algebra of coupled scalars [1] to derive two forms of nonlinear integrable couplings of the modified KdV equation. We aim second to study these couplings and show that it possess multiple soliton solutions and multiple singular soliton solutions the same as the mKdV equation, but differ only in the coefficients of the Bäcklund transformations that result in distinct amplitudes for each equation of the system.

### 2. FIRST COUPLINGS OF THE MKDV EQUATION: MULTIPLE SOLITON SOLUTIONS

In a like manner to the approach presented in [1], where the algebra of coupled scalars was developed, we set a one field soliton system

$$u_{t} = K[u] = K[u, u_{x}, u_{xx}, u_{xxx}, \cdots],$$
(4)

that can be extended to the system of coupled PDEs of the form [1]

$$\vec{u}_{t} = K[\vec{u}] \equiv K[\vec{u}, \vec{u}_{x}, \vec{u}_{xx}, \vec{u}_{xxx}, \cdots],$$
(5)

where

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$
 (1)

Accordingly, the system (5) takes the form [1]

$$(u_{1})_{t} = K[u_{1}],$$

$$(u_{k})_{t} = K\left[\sum_{i=1}^{k} u_{i}\right] - K\left[\sum_{i=1}^{k-1} u_{i}\right], k = 2, 3, ...n.$$
(7)

Using (2), we can set

$$\vec{u}_t = -6\vec{u}^2\vec{u}_x - \vec{u}_{xxx}.$$
(8)

Inserting (8) into (7), we develop the n-coupled modified KdV (nc-mKdV), given in the form

$$(u_{1})_{t} = -(u_{1})_{xxx} - 6u_{1}^{2}(u_{1})_{x},$$

$$(u_{2})_{t} = -(u_{2})_{xxx} - 6u_{2}^{2}(u_{2})_{x} - 6(u_{1}^{2}u_{2})_{x} - 6(u_{1}u_{2}^{2})_{x},$$

$$(u_{3})_{t} = -(u_{3})_{xxx} - 6u_{3}^{2}(u_{3})_{x} - 6((u_{1} + u_{2})^{2}u_{3})_{x} - 6((u_{1} + u_{2})u_{3}^{2})_{x},$$

$$(u_{4})_{t} = -(u_{4})_{xxx} - 6u_{4}^{2}(u_{4})_{x} - 6((u_{1} + u_{2} + u_{3})^{2}u_{4})_{x} - 6((u_{1} + u_{2} + u_{3})u_{4}^{2})_{x}$$

$$\vdots$$

$$(u_{n})_{t} = -(u_{n})_{xxx} - 6u_{n}^{2}(u_{n})_{x} - 6\left[\left(\sum_{k=1}^{n-1}u_{k}\right)^{2}u_{n}\right]_{x} - 6\left[\left(\sum_{k=1}^{n-1}u_{k}\right)u_{n}^{2}\right]_{x}.$$
(9)

### 2.1. Multiple soliton solutions

Substituting

$$u_{i}(x,t) = e^{k_{i}x - \omega_{i}t}, 1 \le i \le n,$$
(10)

into the linear terms of each equation in (9) gives the dispersion relation by

$$\omega_i = k_i^3, \tag{11}$$

and as a result we obtain the following phase variables

$$\theta_i = k_i x - k_i^3 t. \tag{12}$$

The multiple soliton solutions of the couplings (9) are assumed to be

$$u_{i}(x,t) = R_{i} \left( \arctan \frac{F(x,t)}{G(x,t)} \right)_{x} = R_{i} \frac{F_{x}G - G_{x}F}{F^{2} + G^{2}},$$
(13)

where the auxiliary functions F(x,t) and G(x,t) for the single soliton solution are given by

$$F(x,t) = e^{\theta_1} = e^{k_1 x - k_1^3 t}$$

$$G(x,t) = 1$$
(14)

Substituting (13) into (9) and solving for  $R_i$  we obtain two distinct sets given by

$$R_i = (-1)^{i+1} 2, \quad 1 \le i \le n, \tag{15}$$

and

$$R_{j} = \begin{cases} 2 & \text{for } j = 1\\ (-1)^{j+1} 4 & \text{for } 2 \le i \le n. \end{cases}$$
(16)

Combining the obtained results gives two sets of single soliton solutions

$$u_{i}(x,t) = (-1)^{i+1} \frac{2k_{1}e^{k_{1}x-k_{1}^{3}t}}{1+e^{2k_{1}x-2k_{1}^{3}t}}, \quad 1 \le i \le n,$$
(17)

and

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$$u_{j}(x,t) = \begin{cases} \frac{2k_{1}e^{k_{1}x-k_{1}^{3}t}}{1+e^{2k_{1}x-2k_{1}^{3}t}} & \text{for } j=1, \\ (-1)^{j+1}\frac{4k_{1}e^{k_{1}x-k_{1}^{3}t}}{1+e^{2k_{1}x-2k_{1}^{3}t}} & \text{for } 2 \le j \le n. \end{cases}$$
(18)

In other words we obtain two sets of single soliton solutions with distinct amplitudes between the two sets for  $u_r, r \ge 2$ .

For the two soliton solutions we set the auxiliary functions by

$$F(x,t) = e^{\theta_1} + e^{\theta_2}$$

$$G(x,t) = 1 - a_{12}e^{\theta_1 + \theta_2}.$$
(19)

Using (19) in (13) and substituting the result in (9), we obtain the following phase shift coefficient

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2},\tag{20}$$

and hence we set

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le 3.$$
(21)

Using (15) and (16), and the previous results we also obtain two distinct sets of two soliton solutions.

It is well known that a two soliton solution [11] can degenerate into a resonant triad under the conditions

$$a_{12} = 0, or(a_{12})^{-1} = 0, \text{ for } |k_1| \neq |k_2|.$$
 (22)

Accordingly, the resonance phenomenon does not exist for this coupling because  $a_{12} \neq 0$  and  $(a_{12})^{-1} \neq 0$  for  $|k_1| \neq |k_2|$ .

For the three soliton solutions, we set

$$F(x,t) = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + b_{123}e^{\theta_1 + \theta_2 + \theta_3},$$
  

$$G(x,t) = 1 - a_{12}e^{\theta_1 + \theta_2} - a_{13}e^{\theta_1 + \theta_3} - a_{23}e^{\theta_2 + \theta_3}.$$
(23)

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{24}$$

Two sets of three soliton solutions, with distinct amplitudes are obtained by substituting (23) into (13) as presented earlier. This shows that each equation of the coupling (9) possess the same properties as the mKdV equation: the same phase variable, the same phase shift, and the non resonance phenomena. However, the only difference is that the amplitudes are distinct for distinct i.

## 3. SECOND COUPLINGS OF THE MKDV EQUATION: MULTIPLE SINGULAR SOLITON SOLUTIONS

Using the second form of the mKdV equation

$$\vec{u}_t = 6\vec{u}^2\vec{u}_x - \vec{u}_{xxx},$$
(25)

and using the analysis presented before for the derivation of the couplings of the mKdV equation, we obtain a second couplings of the mKdV equation given by In this section, we will examine multiple singular soliton solutions of the second kind of the couplings of the mKdV equation

$$\begin{aligned} (u_{1})_{t} &= -(u_{1})_{xxx} + 6u_{1}^{2}(u_{1})_{x}, \\ (u_{2})_{t} &= -(u_{2})_{xxx} + 6u_{2}^{2}(u_{2})_{x} + 6(u_{1}^{2}u_{2})_{x} + 6(u_{1}u_{2}^{2})_{x}, \\ (u_{3})_{t} &= -(u_{3})_{xxx} + 6u_{3}^{2}(u_{3})_{x} + 6((u_{1} + u_{2})^{2}u_{3})_{x} + 6((u_{1} + u_{2})u_{3}^{2})_{x}, \\ (u_{4})_{t} &= -(u_{4})_{xxx} + 6u_{4}^{2}(u_{4})_{x} + 6((u_{1} + u_{2} + u_{3})^{2}u_{4})_{x} + 6((u_{1} + u_{2} + u_{3})u_{4}^{2})_{x} \\ &\vdots \end{aligned}$$
(26)  
$$\begin{aligned} &\vdots \\ (u_{n})_{t} &= -(u_{n})_{xxx} + 6u_{n}^{2}(u_{n})_{x} + 6\left[\left(\sum_{k=1}^{n-1}u_{k}\right)^{2}u_{n}\right]_{x} + 6\left[\left(\sum_{k=1}^{n-1}u_{k}\right)u_{n}^{2}\right]_{x}. \end{aligned}$$

### 3.1 Multiple singular soliton solutions

Substituting

$$u_i(x,t) = e^{k_i x - \omega_i t}, \quad 1 \le i \le n,$$
 (27)

into the linear terms of each equation in (26) gives the dispersion relation by

$$\omega_i = k_i^3, \tag{28}$$

and as a result we obtain the following phase variables

$$\theta_i = k_i x - k_i^3 t. \tag{29}$$

The singular soliton solutions of the couplings (26) are assumed to be

$$u_i(x,t) = R_i \left( \ln \frac{f(x,t)}{g(x,t)} \right)_x = \frac{gf_x - fg_x}{gf},$$
(30)

where the auxiliary functions f(x,t) and g(x,t) for the single singular solution are given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - k_1^3 t},$$
  

$$g(x,t) = 1 - e^{\theta_1} = 1 + e^{k_1 x - k_1^3 t}.$$
(31)

Substituting (30) into (26) and solving for  $R_i$  we obtain two sets of solutions

$$R_i = (-1)^{i+1}, \quad 1 \le i \le n, \tag{32}$$

and

$$R_{j} = \begin{cases} 1 & \text{for } j = 1\\ (-1)^{j} 4 & \text{for } 2 \le i \le n. \end{cases}$$
(33)

Substituting (31) into (30) gives two sets of single singular soliton solutions given by

$$u_{i}(x,t) = (-i)^{i+1} \frac{2k_{1}e^{k_{1}x-k_{1}^{3}t}}{1-e^{2k_{1}x-2k_{1}^{3}t}}, \quad 1 \le i \le n,$$
(34)

and

$$u_{j}(x,t) = \begin{cases} \frac{2k_{1}e^{k_{1}x-k_{1}^{3}t}}{1-e^{2k_{1}x-2k_{1}^{3}t}} & \text{for } j=1, \\ (-1)^{j} \frac{8k_{1}e^{k_{1}x-k_{1}^{3}t}}{1-e^{2k_{1}x-2k_{1}^{3}t}} & \text{for } 2 \le j \le n. \end{cases}$$
(35)

In other words we obtain two sets of single singular soliton solutions with distinct amplitudes for  $u_r, r \ge 2$ . For the two soliton solutions we set the auxiliary functions by

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2},$$
  

$$g(x,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}.$$
(36)

Using (36) in (30) and substituting the result in (26), we obtain the following phase shift coefficient

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2},\tag{37}$$

and hence we set

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \le i < j \le 3.$$
(38)

Combining the obtained results gives two sets of two singular soliton solutions.

For the three soliton solutions, we set

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1 + \theta_2} + a_{13}e^{\theta_1 + \theta_3} + a_{23}e^{\theta_2 + \theta_3} + b_{123}e^{\theta_1 + \theta_2 + \theta_3},$$
  

$$g(x,t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1 + \theta_2} + a_{13}e^{\theta_1 + \theta_3} + a_{23}e^{\theta_2 + \theta_3} - b_{123}e^{\theta_1 + \theta_2 + \theta_3}.$$
(39)

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{40}$$

This in turn gives two sets of three singular soliton and the three singular anti-soliton solutions are obtained by substituting (39) into (30).

### 4. DISCUSSION

In this work, two kinds of couplings of the mKdV equation were developed by using the algebra of coupled scalars. Multiple soliton and multiple singular solutions are derived for the both couplings of the mKdV equation. In fact two sets of soliton solutions were derived for each coupling. We showed that each equation of the two couplings possesses the same properties as the mKdV equation: the same phase variable, the same phase shift, and the non resonance phenomena. However, the only difference is that the amplitudes are distinct for each equation of the couplings.

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