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ON SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR CERTAIN CONVEX FUNCTIONS

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In this paper, some new inequalities of Hermite-Hadamard type for product of different kinds of convex functions and their applications for special means are given.

Key words: Hermite-Hadamard inequality, Chebyshev's inequality, m-convex, (α,m)-convex, s-convex, h-convex functions.

1. INTRODUCTION

The following definition is well known in the literature: Let *I* be an interval in *R*. Then $f: I \to \mathbb{R},\ \varnothing \neq I \subseteq \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0,1]$,

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y). \tag{1}$$

If (1) is strict for all $x \neq y$ and $t \in (0,1)$, then *f* is said to be strictly convex. If the inequality (1) is reversed, then *f* is said to be concave. If it is strict for all $x \neq y$ and $t \in (0,1)$, then *f* is said to be strictly concave.

Geometrically, this means that if *X, Z* and *Y* are three different points on the graph of *f* with *Z* between *X* and *Y*, then *Z* is on or below the chord *XY*.

Let $f: I \subseteq R \to R$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{2}$$

is known in the literature as Hermite-Hadamard inequality for convex mapping. Note that some of the classical inequalities for means can be acquired from (2) for suitable particular choices of the mapping *f*. Both inequalities hold in the reversed direction if *f* is concave.

In [15], Toader introduced the following new definition to the literature:

Definition 1.1. [15] A function $f: [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have:

$$f\left(tx + m(1-t)y\right) \leq tf(x) + m(1-t)f(y). \tag{3}$$

In [10], Miheşan also contributed a new definition to the literature as follows:

Definition 1.2. [10] A function $f: [0,b] \to \mathbb{R}$ is said to be (α,m)-convex, where $(\alpha,m) \in [0,1]^2$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have:

$$f\left(tx + m(1-t)y\right) \leq t^{\alpha} f(x) + m\left(1-t^{\alpha}\right) f(y). \tag{4}$$

Note that for $(\alpha,m)=\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, *m*-convex, convex and α-convex [6, pp. 301].

In [8], Hudzik and Maligranda defined the *s*-convex function in the second sense:

Definition 1.3. [8] Let $s\in(0,1]$ be a fixed real number. A function $f:\mathbb{R}_+\to\mathbb{R}_+$, where $\mathbb{R}_+=[0,\infty)$, is said to be s-convex in the second sense if

$$f\left(tx+(1-t)y\right)\le t^s f(x)+(1-t)^s f(y), \tag{5}$$

for all $x,y\in\mathbb{R}_+$ and $t\in[0,1]$. This class of s-convex functions is usually denoted by K_s^2.

In 1978, Breckner contributed *s*-convex functions as a generalization of convex functions [3]. A number of properties and connections with *s*-convexity in the first sense are discussed in paper [8]. It can be easily checked for $s=1$, *s*-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

In the paper [16] a large class of nonnegative functions appears, called *h*-convex functions. This class contains several well-known classes of functions such as nonnegative convex functions, *s*-convex in the second sense, Godunova-Levin functions and *P*-functions.

Definition 1.4 [16]. *Let $h:[0,1]\to\mathbb{R}$ be a positive function. We say that $f:I\to\mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x,y\in I$, $t\in(0,1)$ we have*

$$f\left(tx+(1-t)y\right)\le h(t)f(x)+h(1-t)f(y). \tag{6}$$

If inequality (6) is reversed, then *f* is said to be *h*-concave, i.e. $f\in SV(h,I)$. Obviously, if $h(t)=t$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(t)=1/t$, then $SX(h,I)=Q(I)$; if $h(t)=1$, then $SX(h,I)\supseteq P(I)$; and if $h(t)=t^s$, where $s\in(0,1)$, then $SX(h,I)\supseteq K_s^2$.

In [5], Dragomir and Fitzpatrick proved a new variety of Hermite-Hadamard inequality which holds for *s*-convex functions in the second sense.

THEOREM 1.5 [5]. *Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ is an s-convex functions in the second sense, where $s\in(0,1)$, and let $a,b\in\mathbb{R}_+$, $a<b$. If $f\in L_1([a,b])$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\le\frac{f(a)+f(b)}{s+1}. \tag{7}$$

In [12], Pachpatte established two Hermite-Hadamard type inequalities for products of convex functions. The analogous result for *s*-convex functions can also be seen in the paper [9].

THEOREM 1.6 [12]. *Let $f,g:[a,b]\to[0,\infty)$ be convex functions on $[a,b]\subset R$, $a<b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x\le\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{8}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{9}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

THEOREM 1.7 [9]. *Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that g and fg are in $L_1([a,b])$. If f is convex and nonnegative on $[a,b]$, and if g is s-convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{M(a,b)}{(s+1)(s+2)}+\frac{N(a,b)}{s+2}. \tag{10}$$

We give here definition of Beta function of Euler type which will be helpful in our next discussion, which is for $u,v>0$ defined as

$$\beta(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t.$$

In the result section, we establish the new inequalities for product of some different kinds of convex functions as discussed in the above given theorems. In the application section we give applications of the results from section 2 for special means to positive real numbers.

2. MAIN RESULTS

THEOREM 2.1. *Let $h:[0,1]\to R$ be a positive function, $a,b\in[0,\infty)$, $a<b$, $f,g:[a,b]\to\mathbb{R}$ functions and $fg\in L_1([a,b])$, $h\in L_1([0,1])$. If f is h-convex and $g\in K_s^2([a,b])$ for some fixed $s\in(0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le M(a,b)\int_0^1 h(t)t^s\mathrm{d}t+N(a,b)\int_0^1 h(1-t)t^s\mathrm{d}t, \tag{11}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since *f* is *h*-convex on $[a,b]$ and $g\in K_s^2([a,b])$, we know that

$$f(ta+(1-t)b)\le h(t)f(a)+h(1-t)f(b)$$
$$g(ta+(1-t)b)\le t^s g(a)+(1-t)^s g(b),$$

for all $t\in[0,1]$. Since *f* and *g* are nonnegative, we can write the following inequality,

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b)\le\\&\le\left[h(t)f(a)+h(1-t)f(b)\right]\left[t^s g(a)+(1-t)^s g(b)\right]=\\&=h(t)t^s f(a)g(a)+h(t)(1-t)^s f(a)g(b)+\\&+h(1-t)t^s f(b)g(a)+h(1-t)(1-t)^s f(b)g(b).\end{aligned}$$

Now integrating both sides of the inequality over *t* on $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t=\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\\&\le f(a)g(a)\int_0^1 h(t)t^s\mathrm{d}t+f(b)g(b)\int_0^1 h(1-t)(1-t)^s\,\mathrm{d}t+\\&+f(b)g(a)\int_0^1 h(1-t)t^s\mathrm{d}t+f(a)g(b)\int_0^1 h(t)(1-t)^s\,\mathrm{d}t.\end{aligned} \tag{12}$$

And the following equality can be written,

$$\begin{aligned}\int_0^1 h(t)t^s dt&=\int_0^1 h(1-t)(1-t)^s\,\mathrm{d}t\\\int_0^1 h(1-t)t^s dt&=\int_0^1 h(t)(1-t)^s\,\mathrm{d}t.\end{aligned} \tag{13}$$

The combination of (12) and (13) gives the inequality (11), then the proof is complete. □

***Remark* 2.2.** a) In Theorem 2.1., if we choose $h(t)=t$ and $g(x)=1$, for some fixed $s\in(0,1]$, then (11) is reduced to right side of (7). So, we obtain

b) For $h(t)=t^s$, $s_{1,2}\in(0,1)$, i.e. if *f* is an s_1-convex function in the second sense and *g* is an s_2-convex function in the second sense, then we have a result of Theorem 6 from [9]

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le M(a,b)\int_0^1 h(t)t^{s_2}\,\mathrm{d}t + N(a,b)\int_0^1 h(1-t)t^{s_2}\,\mathrm{d}t =$$
$$= \frac{M(a,b)}{s_1+s_2+1} + \beta(s_1+1,s_2+1)N(a,b).$$

THEOREM 2.3. *Let* $h:[0,1]\to\mathbb{R}$ *be a positive function,* $a,b\in[0,\infty)$, $a<b$, $f,g:[a,b]\to\mathbb{R}$ *functions and* $fg\in L_1([a,b])$, $h\in L_1([0,1])$. *If f is h-convex on* $[a,b]$ *and g is s-convex in the second sense on* $[a,b]$ *for some fixed* $s\in(0,1]$, *then*

$$\frac{2^{s-1}}{h(1/2)} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le$$
$$\le M(a,b)\int_0^1 h(1-t)t^s\,\mathrm{d}t + N(a,b)\int_0^1 h(t)t^s\,\mathrm{d}t, \tag{14}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since *f* is *h*-convex on $[a,b]$ and $g\in K_s^2([a,b])$, we can write

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \le h\left(\frac{1}{2}\right)\left[f(ta+(1-t)b)+f((1-t)a+tb)\right],$$
$$g\left(\frac{a+b}{2}\right) = g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \le \frac{1}{2^s}\left[g(ta+(1-t)b)+g((1-t)a+tb)\right],$$

for $\frac{a+b}{2}=\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}$. Since *f* and *g* are nonnegative, we get

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le$$
$$\le h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]\times\left[g(ta+(1-t)b)+g((1-t)a+tb)\right] =$$
$$= h\left(\frac{1}{2}\right)\frac{1}{2^s}\{f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)+$$
$$+f(ta+(1-t)b)g((1-t)a+tb)+f((1-t)a+tb)g(ta+(1-t)b)\} \le$$
$$\le h\left(\frac{1}{2}\right)\frac{1}{2^s}\{f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)+$$
$$+\left[h(t)f(a)+h(1-t)f(b)\right]\left[(1-t)^s g(a)+t^s g(b)\right]+$$
$$+\left[h(1-t)f(a)+h(t)f(b)\right]\left[t^s g(a)+(1-t)^s g(b)\right]\} =$$
$$= h\left(\frac{1}{2}\right)\frac{1}{2^s}\{f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)+$$

$$+\left[h(t)(1-t)^s+h(1-t)t^s\right]\left[f(a)g(a)+f(b)g(b)\right]+$$
$$+\left[h(t)t^s+h(1-t)(1-t)^s\right]\left[f(a)g(b)+f(b)g(a)\right]\Big\}.$$

Thus, we obtain

$$\frac{2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{h(1/2)}-\left[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\right]\le$$
$$\le\left[h(t)(1-t)^s+h(1-t)t^s\right]M(a,b)+\left[h(t)t^s+h(1-t)(1-t)^s\right]N(a,b).$$

Integrating both sides of the above inequality over t on $[0,1]$, we establish:

$$\frac{2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{h(1/2)}-\frac{2}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le$$
$$\le M(a,b)\left\{\int_0^1 h(t)(1-t)^s\,\mathrm{d}t+\int_0^1 h(1-t)t^s\mathrm{d}t\right\}+N(a,b)\left\{\int_0^1 h(t)t^s\mathrm{d}t+\int_0^1 h(1-t)(1-t)^s\,\mathrm{d}t\right\}$$

and

$$\int_0^1 h(t)t^s\mathrm{d}t=\int_0^1 h(1-t)(1-t)^s\,\mathrm{d}t$$
$$\int_0^1 h(1-t)t^s\mathrm{d}t=\int_0^1 h(t)(1-t)^s\,\mathrm{d}t.$$

When above equalities are considered, the proof is complete.

Remark **2.4. a)** In Theorem 2.3, if we choose $h(t)=t$ and $f:[a,b]\to\mathbb{R}$ defined as $f(x)=1$ for all $x\in[a,b]$, we obtain

$$2^s g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\le$$
$$\le M(a,b)\int_0^1(1-t)t^s dt+N(a,b)\int_0^1 tt^s\mathrm{d}t=$$
$$=\left[g(a)+g(b)\right]\int_0^1\left(t^s-t^{s+1}\right)\mathrm{d}t+\left[g(a)+g(b)\right]\int_0^1 t^{s+1}\mathrm{d}t=$$
$$=\frac{g(a)+g(b)}{(s+1)(s+2)}+\frac{g(a)+g(b)}{s+2}=\frac{g(a)+g(b)}{s+1},$$

which is a result of Remark 4 from [9].

b) For $h(t)=t^{s_1}$, $s_{1,2}\in(0,1)$, i.e. if f is an s_1-convex function in the second sense and g is an s_2-convex function in the second sense, we obtain the following inequality

$$2^{s_1+s_2-1}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le$$
$$\le M(a,b)\int_0^1(1-t)^{s_1}\,t^{s_2}\mathrm{d}t+N(a,b)\int_0^1 t^{s_1}t^{s_2}\,\mathrm{d}t=$$
$$=M(a,b)\beta(s_1+1,s_2+1)+\frac{N(a,b)}{s_1+s_2+1}.$$

THEOREM 2.5. *Let* $a,b\in[0,\infty)$, $a<b$ *such that* $f,g:[a,b]\to R$ *be integrable, nonnegative and either synchronously increasing or decreasing functions,* $f,g\in L_1([a,b])$. *If* f *is* m_1*-convex and* g *is* (α,m_2)*-convex on* $[a,b]$ *for* $m_1\in[0,1]$ *and* $(\alpha,m_2)\in[0,1]^2$*, then*

$$\begin{aligned}&\frac{1}{m_1b-a}\int_a^{m_1b}f(x)\mathrm{d}x\cdot\frac{1}{m_2b-a}\int_a^{m_2b}g(x)\mathrm{d}x\le\\&\le Ef(a)g(a)+Lf(a)g(b)+If(b)g(a)+Ff(b)g(b),\end{aligned}\tag{15}$$

where $E=\dfrac{1}{\alpha+2}$, $L=\dfrac{m_2\alpha}{2(\alpha+2)}$, $I=\dfrac{m_1}{(\alpha+1)(\alpha+2)}$, $F=\dfrac{m_1m_2\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}$.

Proof. Since *f* is m_1-convex and *g* is (α,m_2)-convex on $[a,b]$, we have

$$\begin{aligned}&f(ta+m_1(1-t)b)\le tf(a)+m_1(1-t)f(b),\\&g(ta+m_2(1-t)b)\le t^\alpha g(a)+m_2(1-t^\alpha)g(b),\end{aligned}$$

for $m_1\in[0,1]$ and $(\alpha,m_2)\in[0,1]^2$. Since *f* and *g* are nonnegative, we get

$$\begin{aligned}&f(ta+m_1(1-t)b).g(ta+m_2(1-t)b)\le\\&\le f(a)g(a)t^{\alpha+1}+m_2f(a)g(b)t(1-t^\alpha)+\\&\quad+m_1f(b)g(a)t^\alpha(1-t)+m_1m_2f(b)g(b)(1-t)(1-t^\alpha).\end{aligned}$$

Integrating both sides of the above inequality over *t* on $[0,1]$, and by using Chebychev's integral inequality, we obtain

$$\begin{aligned}&\frac{1}{m_1b-a}\int_a^{m_1b}f(x)dx.\frac{1}{m_2b-a}\int_a^{m_2b}g(x)\mathrm{d}x\le\\&\le\int_0^1f(ta+m_1(1-t)b)\mathrm{d}t\int_0^1g(ta+m_2(1-t)b)\mathrm{d}t\le\\&\le\int_0^1f(ta+m_1(1-t)b)g(ta+m_2(1-t)b)\mathrm{d}t\le\\&\le f(a)g(a)\int_0^1t^{\alpha+1}\mathrm{d}t+m_2f(a)g(b)\int_0^1t(1-t^\alpha)\mathrm{d}t\le\\&\quad+m_1f(b)g(a)\int_0^1t^\alpha(1-t)\mathrm{d}t+m_1m_2f(b)g(b)\int_0^1(1-t)(1-t^\alpha)\mathrm{d}t+\\&=\frac{f(a)g(a)}{\alpha+2}+\frac{m_2\alpha f(a)g(b)}{2(\alpha+2)}+\frac{m_1f(b)g(a)}{(\alpha+1)(\alpha+2)}+\frac{m_1m_2\alpha(\alpha+3)f(b)g(b)}{2(\alpha+1)(\alpha+2)}.\end{aligned}$$

The proof is completed.

***Remark* 2.6.** In Theorem 2.5, if we choose $m_1=m_2=\alpha=1$, then (15) is reduced to (8). So, we obtain

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^bf(x)\mathrm{d}x\int_a^bg(x)\mathrm{d}x\le\frac{1}{b-a}\int_a^bf(x)g(x)\mathrm{d}x\le\\&\le\frac{f(a)g(a)}{3}+\frac{f(a)g(b)}{6}+\frac{f(b)g(a)}{6}+\frac{f(b)g(b)}{3}=\frac{M(a,b)}{3}+\frac{N(a,b)}{6}.\end{aligned}$$

3. APPLICATIONS TO SOME SPECIAL MEANS

In the papers [1,14], the following result is given.

Let $g: I \to I_1 \subseteq [0,\infty)$ be a nonnegative convex function on *I*. Then $g^s(x)$ is *s*-convex on *I*, $0 < s < 1$.

For arbitrary positive real numbers a,b $(a \neq b)$, we shall consider the following special means:

- the arithmetic mean: $A = A(a,b) = \dfrac{a+b}{2}$,
- the geometric mean: $G = G(a,b) = \sqrt{ab}$,
- the identric mean: $I = I(a,b) = \begin{cases} a & \text{if } a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b; \end{cases}$
- the p-logarithmic mean: $L_p = L_p(a,b) = \begin{cases} a & \text{if } a = b \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } a \neq b \end{cases}$, $p \in R - \{-1,0\}$.

In the paper [14], some new properties have been written for the above means. Let $f:[a,b] \to R$, $(0 < a < b)$, $f(x) = x^s$, $s \in (0,1]$. Then

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = L_s^s(a,b), \quad \frac{f(a)+f(b)}{2} = A(a^s,b^s), \quad f\left(\frac{a+b}{2}\right) = A^s(a,b).$$

Now, using the results of Section 2, some new inequalities are derived for the above means. It will be taken $h(t) = t^s$ in each proposition.

PROPOSITION 3.1. *Let* $f,g:[a,b] \to R$, $(0 < a < b)$, $f(x) = g(x) = x^s$ *be an s-convex in the second sense on* $[a,b]$, $0 < s < 1$. *Then from* Theorem 2.1., *we obtain*

$$L_{2s}^s(a,b) \leq \frac{A(a^{2s},b^{2s})}{s+1/2} + 2\beta(s+1,s+1)G^{2s}(a,b). \tag{16}$$

PROPOSITION 3.2. *Let* $f,g:[a,b] \to R$, $(0 < a < b)$, $f(x) = g(x) = x^s$, $s \in (0,1]$. *Then from* Theorem 2.3., *we obtain*

$$2^{2s}A^s(a,b) - L_{2s}^s(a,b) \leq 2\beta(s+1,s+1)A(a^{2s},b^{2s}) + \frac{2G^{2s}(a,b)}{2s+1}. \tag{17}$$

Let $f:[a,b] \to R$, $(0 < a < b)$, $f(x) = -\ln x^s$, $s \in (0,1]$. Then, we can write

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = -s\ln I(a,b)$$

$$\frac{f(a)+f(b)}{2} = -s\ln G(a,b)$$

$$f\left(\frac{a+b}{2}\right) = -s\ln A(a,b).$$

PROPOSITION 3.3. *Let* $f,g:[a,b]\to R$, $(0<a<b)$, $f(x)=g(x)=-\sqrt{\ln x^s}$ *be an s-convex in the second sense on* $[a,b]$, $0<s<1$. *Then from* Theorem 2.1., *we obtain*

$$-s\ln I(a,b)\leq\frac{2s\ln G^2(a,b)}{2s+1}+2s^2G^2(\ln a,\ln b)\beta(s+1,s+1). \quad (18)$$

PROPOSITION 3.4. *Let* $f,g:[a,b]\to R$, $(0<a<b)$, $f(x)=g(x)=-\sqrt{\ln x^s}$, $s\in(0,1]$. *Then from* Theorem 2.3., *we obtain*

$$2^{2s-1}\ln A^s(a,b)-s\ln I(a,b)\leq s\ln G^2(a,b)\beta(s+1,s+1)+\frac{2sG(\ln a,\ln b)}{2s+1}. \quad (19)$$

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