THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 15, Number 1/2014, pp. 11–17

EXACT LOOP SOLITONS, CUSPONS, COMPACTONS AND SMOOTH SOLITONS FOR THE BOUSSINESQ-LIKE B(2,2) EQUATION

Lina ZHANG[1], Aiyong CHEN[2]

[1] Huzhou University, School of Science, Huzhou, Zhejiang 313000, China
[2] Guilin University of Electronic Technology, School of Mathematics and Computing Science, Guilin, Guangxi 541004, China
Corresponding author: Lina ZHANG, E-mail: zsdzln@126.com

This paper emphasizes on a study of single peak soliton solutions for the Boussinesq-like $B(2,2)$ equation $u_{tt}+(u^2)_{xx}+(u^2)_{xxxx}=0$. By setting the equation under the boundary condition $\lim_{|x|\to\infty}u=A$, we show that the regular compacton solutions correspond to the case of $A=0$. For the case of $A\neq 0$, both new cusped solitons and new type of smooth solitons, which are expressed in terms of trigonometric and logarithmic functions, are given by virtue of the qualitative theory of differential equations.

Key words: $B(2,2)$ equation, single peak soliton, loop soliton, cuspon, compacton, smooth soliton

1. INRODUCTION

Solitons and integrable systems play an increasingly important role in nonlinear wave phenomena. They have many significant applications in fluid mechanics, nonlinear optics, plasma physics, biochemistry, and optical fibers. Particularly in recent years, the interest inspired by the genuinely nonlinear dispersive equations $K(m,n)$ and its singular *compacton solutions* [1] prompted search for other integrable equations with similar properties. In 2001, Yan [2] introduced a fully Boussinesq equation $B(m,n)$

$$u_{tt}=(u^n)_{xx}+(u^m)_{xxxx}, \tag{1}$$

in order to understand the role of nonlinear dispersion in pattern formation. Solitary wave solutions of $B(1,n)$ equation and compacton solutions of $B(m,m)$ equation are constructed by employing the direct reduction method. New families of solitons with compact support for the Boussinesq-like $B(m,n)$ equation (1) with fully nonlinear dispersion are given by Yan [3]. Zhu [4] investigated the $B(2n,2n)$ equation of the form

$$u_{tt}+(u^{2n})_{xx}+(u^{2n})_{xxxx}=0, \tag{2}$$

and new exact solitary solutions with compact support are developed by the Adomian decomposition method. Zhang [5] introduced a more generalized form of Boussinesq-like $B(m,n)$ equation

$$u_{tt}+a(u^n)_{xx}+b(u^m)_{xxxx}=0, \tag{3}$$

under different parameter conditions, the compactons, peakons, solitary and periodic cusp wave solutions of the generalized $B(m,n)$ equation (3) are obtained by using the integral approach.

In fact, it is very important and interesting to investigate the traveling wave solutions under boundary conditions. Qiao and Zhang [6] discussed the traveling wave solutions for the Camassa-Holm equation under the inhomogeneous boundary condition $\lim_{|x|\to\infty}u=A\neq 0$, and found new soliton solutions both smooth and cusped. Later, Zhang and Qiao [7] explored all possible single peak soliton solutions for the Degasperis-Procesi equation under the boundary condition $\lim_{|x|\to\infty}u=A$. Chen and Li [8] applied the qualitative theory of differential equations to the osmosis $K(2,2)$ equation and obtained smooth, peaked and cusped soliton solutions under inhomogeneous boundary condition.

In this paper, we give all possible single peak soliton solutions of the Boussinesq-like $B(2,2)$ equation

$$u_{tt} + (u^2)_{xx} + (u^2)_{xxxx} = 0 \tag{4}$$

through setting the traveling wave solution under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. Our procedure shows that the regular compacton solutions of the $B(2,2)$ equation correspond to the case of $A = 0$. The most interesting case is $A \neq 0$, and new *cuspon solutions*, *loop soliton solutions*, and *smooth soliton solutions* are obtained in this work.

2. SOME PRELIMINARY RESULTS

Denote $\xi = x - ct$ and let $u(x,t) = u(\xi)$. Substituting it into Eq. (4) it follows that

$$c^2 u'' + (u^2)'' + (u^2)'''' = 0, \tag{5}$$

where " $'$ " is the derivative with respect to ξ. Integrating (5) once and neglecting the integration constant, we have

$$c^2 u' + (u^2)' + (u^2)''' = 0. \tag{6}$$

Taking integral twice on both sides of (6), we obtain

$$c^2 u + u^2 + 2u'^2 + 2uu'' = g_1, \tag{7}$$

$$u'^2 = -\frac{u^2}{4} - \frac{c^2 u}{3} + \frac{g_1}{2} + \frac{g_2}{u^2}, \tag{8}$$

where g_1 and g_2 are two integration constants.

To seek solitary traveling wave solutions for Eq. (8), we impose the boundary condition

$$\lim_{|\xi|\to\infty} u(\xi) = A, \tag{9}$$

where A is a constant. We can figure out the two constants g_1, g_2 through substituting the boundary condition (9) into Eq. (8), which generates the following two constants:

$$g_1 = A(A + c^2), \quad g_2 = \frac{-A^3(3A + 2c^2)}{12}. \tag{10}$$

So Eq. (8) becomes

$$u'^2 = -\frac{(u-A)^2[3u^2 + (6A + 4c^2)u + 3A^2 + 2Ac^2]}{12u^2}. \tag{11}$$

If $3A + 2c^2 \geq 0$, then the ordinary differential equation (11) can be written as

$$u'^2 = -\frac{(u-A)^2(u-B_1)(u-B_2)}{4u^2}, \tag{12}$$

where

$$B_1 = -\frac{3A + 2c^2}{3} + \frac{\sqrt{2c^2(3A + 2c^2)}}{3}, \quad B_2 = -\frac{3A + 2c^2}{3} - \frac{\sqrt{2c^2(3A + 2c^2)}}{3}. \tag{13}$$

Apparently, $B_1 \geq B_2$.

Definition 1. A function $u(\xi)$ is said to be a single peak soliton solution for the $B(2,2)$ equation (4) if $u(\xi)$ satisfies the following conditions:

(A1) $u(\xi)$ is continuous on R and has a unique peak point ξ_0, where $u(\xi)$ attains its global maximum or minimum value;

(A2) $u(\xi) \in C^4(R-\{\xi_0\})$ satisfies (6) on $R-\{\xi_0\}$;

(A3) $\lim_{|\xi|\to\infty} u(\xi) = A$.

Without any loss of generality, we choose the peak point ξ_0 as $\xi_0 = 0$.

THEOREM 1. *Suppose that $u(\xi)$ is a single peak soliton solution for the $B(2,2)$ equation (4) at the peak point $\xi_0 = 0$, then we have $3A+2c^2>0$ and $u(0)=0$ or $u(0)=B_1$ or $u(0)=B_2$.*

Proof. Notice that $3A+2c^2 \le 0$ corresponds to the trivial solution $u \equiv A$. When $3A+2c^2>0$, if $u(0)\neq 0$, from Eq. (7) we know that $u'(0)$ exists. According to the definition of peak point, we have $u'(0)=0$, hence $u(0)=B_1$ or $u(0)=B_2$ since $u(0)=A$ contradicts the fact that 0 is the unique peak point.

3. SMOOTH AND NONSMOOTH SOLITONS OF THE $B(2,2)$ EQUATION

Let us assume that $3A+2c^2>0$ (if $3A+2c^2 \le 0$, then $u\equiv A$). This solution is not a soliton, which is not interesting for us). According to Definition 1 and Theorem 1, any single peak soliton solution of the $B(2,2)$ equation (4) must satisfy the following initial and boundary values problem

$$\begin{cases} u'^2 = -\dfrac{(u-A)^2(u-B_1)(u-B_2)}{4u^2}, \\ u(0)\in\{0,B_1,B_2\}, \\ \lim\limits_{|\xi|\to\infty} u(\xi) = A. \end{cases} \tag{14}$$

Equation (14) implies

$$(u-B_1)(u-B_2)\le 0,\ (A-B_1)(A-B_2)\le 0. \tag{15}$$

Introducing the constant $\alpha = \dfrac{A}{3A+2c^2}$ yields

$$\alpha(\alpha+1)\le 0, \tag{16}$$

which implies:

$$\alpha=-1;\ \ -1<\alpha<0;\ \ \alpha=0. \tag{17}$$

From the standard phase analysis (Fig. 1), we know that if $u(\xi)$ is a single peak soliton solution of the $B(2,2)$ equation (4), then

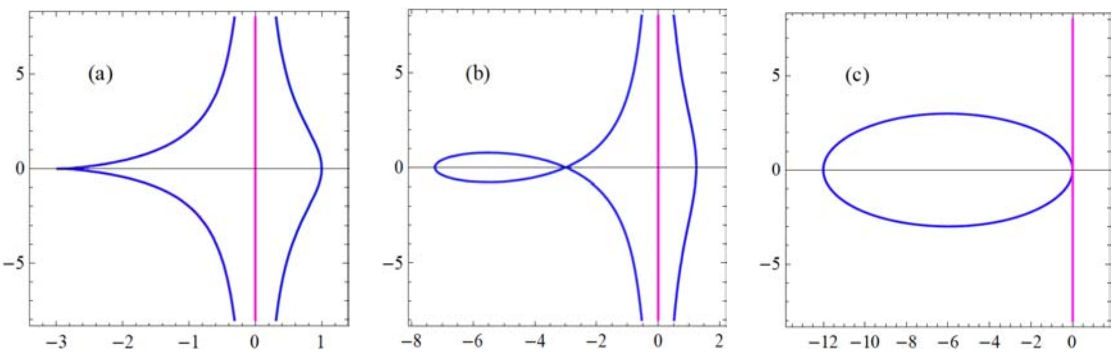

Fig. 1 – Phase portraits of Eq. (4) on the (u, u_ξ) plane: a) $\alpha=-1$; b) $-1<\alpha<0$; (c) $\alpha=0$.

$$u' = -\mathrm{sgn}(A)\frac{u-A}{2u}\sqrt{(B_1-u)(u-B_2)}\mathrm{sgn}(\xi). \quad (18)$$

Taking indefinite integral on both sides of (18) leads to

$$\Phi(u) \equiv \int \phi(u)\mathrm{d}u = \frac{|\xi|}{2} + K, \quad (19)$$

where $\phi(u) = -\mathrm{sgn}(A)\dfrac{u}{(u-A)\sqrt{(B_1-u)(u-B_2)}}$ and K is an integration constant. Thus we obtain the implicit solution $u(\xi)$ defined by

$$\Phi(u) = \frac{|\xi|}{2} + K. \quad (20)$$

In the following, let us discuss the cases of $\alpha = -1$, $-1 < \alpha < 0$, and $\alpha = 0$ separately.

Case I: $\alpha = -1$. In this case, we have $A = B_2 = -\dfrac{c^2}{2}$ and $B_1 = \dfrac{c^2}{6}$.

• (i) $u(0) = 0$. By standard phase portrait analysis (Fig. 1a), there exist two hyperbolic sectors of (11), to the cusp point $(A, 0)$. Choosing $u(0) = 0$ as the initial value, integrating (18) from 0 to u, we arrive at

$$|\xi| = 2\arcsin\left(\frac{A+B_1-2u}{B_1-A}\right) - 3\sqrt{\frac{B_1-u}{u-A}} - \beta_0, \quad u \in (A, 0], \quad (21)$$

where $\beta_0 = -\dfrac{\pi}{3} - \sqrt{3}$. Applying (21) to draw the graphs of $u(\xi)$, we get the cusped soliton solution which is shown in Fig. 2a.

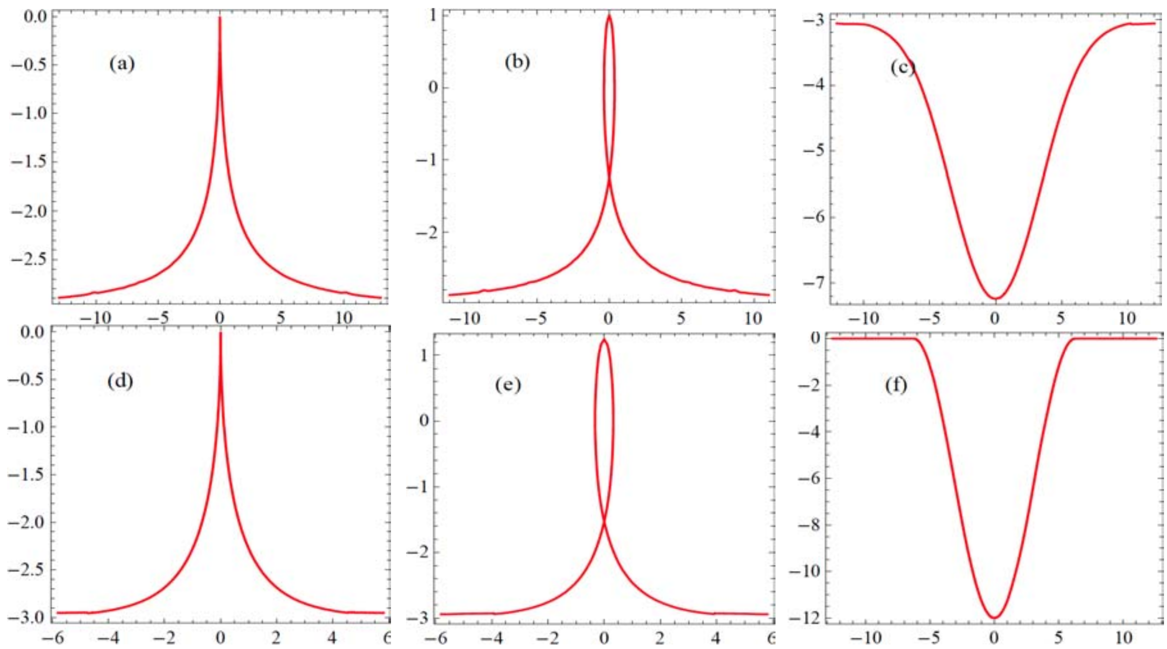

Fig. 2 – Soliton solutions; a) cuspon; b) loop soliton; c) smooth soliton; d) cuspon; e) loop soliton; f) compacton.

• (ii) $u(0) = B_1$. Integrating (18) along the two hyperbolic sectors of (11), approaching the straight line $u = 0$, to the cusp point $(A, 0)$ and the open-end curve passing through the point $(B_1, 0)$, respectively, we get the implicit representations of three breaking wave solutions of Eq. (4), as follows:

$$|\xi| = \int_{A/2}^{u} 2\phi(u)\mathrm{d}u = 3\sqrt{\frac{B_1 - u}{u - A}} - 2\arcsin\left(\frac{A + B_1 - 2u}{B_1 - A}\right) - \beta_1, \quad u \in (A, 0], \tag{22}$$

$$|\xi| = \int_{u}^{B_1} 2\phi(u)\mathrm{d}u = 2\arcsin\left(\frac{A + B_1 - 2u}{B_1 - A}\right) - 3\sqrt{\frac{B_1 - u}{u - A}} + \pi, \quad u \in [0, B_1], \tag{23}$$

where $\beta_1 = \sqrt{15} - 2\arcsin\frac{1}{4}$. The graphs, corresponding to the formulas of (22) and (23), look like a loop Fig. 2b soliton solution of Eq. (4), for more details.

Case II: $-1 < \alpha < 0$. If $-1 < \alpha < 0$, then $-\frac{c^2}{2} < A < 0$, $B_2 < A < 0 < B_1$. After a lengthy calculation of the integral, we obtain

$$\int \phi(u)\mathrm{d}u = -I_1(u) - \sqrt{\frac{-A}{4A + 2c^2}} I_2(u), \tag{24}$$

with

$$I_1(u) = \arcsin\left(\frac{B_1 + B_2 - 2u}{B_1 - B_2}\right), \tag{25}$$

$$I_2(u) = \ln\left|\frac{(B_1 - A)(u - B_2) + (B_1 - u)(A - B_2) - 2\sqrt{(B_1 - A)(A - B_2)(B_1 - u)(u - B_2)}}{u - A}\right|. \tag{26}$$

Obviously,

$$I_1(B_1) = -I_1(B_2) = -\frac{\pi}{2}, \tag{27}$$

$$I_2(B_1) = I_2(B_2) = \ln|B_1 - B_2|. \tag{28}$$

Hence for $u(0) = B_1$ or B_2, the constant $K_0 = \Phi(u(0))$ is defined by

$$K_0 = \pm\frac{\pi}{2} - \sqrt{\frac{-A}{4A + 2c^2}} \ln|B_1 - B_2|, \tag{29}$$

and for $u(0) = 0$,

$$K_0 = -I_1(0) - \sqrt{\frac{-A}{4A + 2c^2}} I_2(0). \tag{30}$$

• (i) $u(0) = B_2$. If $u(0) = B_2$, then $B_2 \le u < A$. From $\phi(u) < 0$, we know that $\Phi(u)$ is strictly decreasing on the interval $[B_2, A)$.

$$\Phi_1(u) = \Phi_{[B_2, A)}(u) \tag{31}$$

has its inverse function denoted by $u_1(\xi) = \Phi_1^{-1}\left(\frac{|\xi|}{2} + K_0\right)$ and $u_1(\xi)$ has the following characteristic features:

$$u_1(0) = B_2, \quad \lim_{|\xi| \to \infty} u_1(\xi) = A, \quad u_1'(0) = 0. \tag{32}$$

So, $u_1(\xi)$ is a smooth soliton solution for Eq. (4); see the graphs of $u_1(\xi)$ in Fig. 2c.

- (ii) $u(0)=0$. In this case, we have $A<u\le 0$. Similarly, we know that $\Phi(u)$ is strictly decreasing on the interval $(A,0]$.

$$\Phi_2(u)=\Phi_{(A,0]}(u) \tag{33}$$

gives a unique cusped soliton solution $u_2(\xi)=\Phi_2^{-1}(\frac{|\xi|}{2}+K_0)$ which has the following properties:

$$u_2(0)=0,\ \lim_{|\xi|\to\infty}u_2(\xi)=A,\ \ u_2'(0+)=-\infty,\ \ u_2'(0-)=+\infty. \tag{34}$$

The graphs of $u_2(\xi)$ are plotted in Fig. 2(d).

- (iii) $u(0)=B_1$. By doing integrations along the stable and unstable manifolds, approaching the straight line $u=0$, to the saddle point $(A,0)$, and the open-end curve passing through the point $(B_1,0)$ (Fig. 1b), we get the following implicit breaking wave solutions, respectively,

$$|\xi|=\int_{A/2}^{u}2\phi(u)\mathrm{d}u=-2I_1(u)-\sqrt{\frac{-2A}{2A+c^2}}I_2(u)-\beta_2,\ \ u\in(A,0], \tag{35}$$

$$|\xi|=\int_{B_1}^{u}2\phi(u)\mathrm{d}u=2I_1(u)+\sqrt{\frac{-2A}{2A+c^2}}I_2(u)-\beta_3,\ \ u\in[0,B_1], \tag{36}$$

where

$$\begin{aligned}\beta_2&=-2I_1(\frac{A}{2})-\sqrt{\frac{-2A}{2A+c^2}}I_2(\frac{A}{2}),\\ \beta_3&=-2I_1(B_1)-\sqrt{\frac{-2A}{2A+c^2}}I_2(B_1),\end{aligned} \tag{37}$$

Here $I_1(u)$ and $I_2(u)$ are defined by Eqs. (25) and (26), respectively. The plots of $u(\xi)$ are drawn in Fig. 2e by employing the formulas (35) and (36).

Case III: $\alpha=0$. If $\alpha=0$, then $A=B_1=0$, $u(0)=B_2=-\frac{4c^2}{3}$. Integrating both sides of (18) on the interval $[-\frac{4c^2}{3},0)$ leads to a *compacton solution* with compact support

$$u(\xi)=\begin{cases}-\frac{4c^2}{3}\cos^2(\frac{|\xi|}{4}), & \text{if } |\xi|\le 2\pi,\\ 0, & \text{otherwise.}\end{cases} \tag{38}$$

The graph of the compacton is shown in Fig. 2f.

***Remark* 1.** The loop soliton solution is a solitary wave solution with singularity, which possesses infinite derivatives at certain points. The loop soliton solutions illustrated in Figs. 2b and 2e may be regarded as the compound-type solutions, i.e., they consist of three independent branches of solutions.

***Remark* 2.** The compacton solution (38) is different from the well-known smooth soliton that it is a weak solution which has one-order continuous derivative, however the second-order derivative on the two points $\xi=\pm\pi$ do not exist. In other words, this compacon is a non-smooth single peak soliton of the $B(2,2)$ equation (4).

4. CONCLUSIONS

In this paper, we provide an approach different from the sine-cosine and the extended tanh methods [9,10], the extended Jacobi elliptic function expansion method [11,12], and the Adomian method [3,4]. It is easy to establish that the results obtained in this paper include not only some previous results reported in Refs. [2-5], but some new types of soliton solutions. Actually, this approach can be also applied to other types of nonlinear partial differential equations. What we are interested in is to find new solutions of nonlinear equations regardless of their integrability. We are applying this method to another Boussinesq-like $B(2,2)$ equation: $u_{tt}+(u^2)_{xx}-(u^2)_{xxxx}=0$, and some new soliton solutions of this later equation will be reported elsewhere.

ACKNOWLEDGEMENTS

This research is supported by the National Natural Science Foundation of China (Nos. 11071222 and 11071205) and the Natural Science Foundation of Huzhou University (No. KX21061).

REFERENCES

1. P. ROSENAU, J.M. HYMAN, *Compactons: solitons with finite wavelength*, Phys. Rev. Lett., **70**, pp. 564–567, 1993.
2. Z. YAN, *New similarity reductions and compacton solutions for Boussinesq-like equations with fully nonlinear dispersion*, Commun. Theor. Phys., **36**, pp. 385–390, 2001.
3. Z. YAN, *New families of solitons with compact support for Boussinesq-like $B(m,n)$ equations with fully nonlinear dispersion*, Chaos. Soliton. Fract., **14**, pp. 1151–1158, 2002.
4. Y. ZHU, *New solitary solutions with compact support for Boussinesq-like $B(m,n)$ equations with fully nonlinear dispersion*, Chaos. Soliton. Fract., **32**, pp. 768–772, 2007.
5. L. ZHANG, *New exact compacton, peakon and solitary solutions of the generalized Boussinesq-like $B(m,n)$ equations with nonlinear dispersion*, Anal. Appl., **67**, pp. 3276–3282, 2007.
6. Z. QIAO, G. ZHANG, *On peaked and smooth solitons for the Camassa-Holm equation*, Europhys. Lett., **73**, pp. 657–663, 2006.
7. G. ZHANG, Z. QIAO, *Cuspons and smooth solitons of the Degasperis-Procesi equation under inhomogeneous boundary condition*, Math. Phys. Anal. Geom., **10**, pp. 205–225, 2007.
8. A. CHEN, J. LI, *Single peak solitary wave solutions for the osmosis $K(2,2)$ equation under inhomogeneous boundary condition*, J. Math. Anal. Appl., **369**, pp. 758–766, 2010.
9. A. WAZWAZ, *A sine-cosine method for handling nonlinear wave equations*, Math. Comput. Model., **40**, pp. 99–508, 2004.
10. A. WAZWAZ, *The tanh method and the sine-cosine method for solving the KP-MEW equation*, Int. J. Comput. Math., **82**, pp. 235–246, 2005.
11. H. ZHANG, *Extended Jacobi elliptic function expansion method and its applications*, Commun. Nonlinear Sci. Numer. Simul., **12**, pp. 627–635, 2007.
12. Y. ZHOU, M. WANG, Y. WANG, *Periodic wave solutions to coupled KdV equations with variable coefficients*, Phys. Lett., A **308**, pp. 31–36, 2003.

Received August 1. 2013