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ANALYTICAL METHOD FOR FITTING THE RAMBERG-OSGOOD MODEL TO GIVEN HYSTERESIS LOOPS

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The paper presents a simple and efficient algorithm for fitting the Ramberg-Osgood model to experimental hysteresis loops. In the proposed approached, only four parameters measured on the experimental curve are needed for the fitting algorithm. The skeleton curve can be determined even in the case when it is not specified within the experimental data. The efficiency of this algorithm is proved by applying this method for identification of Ramberg-Osgood model from hysteresis loops measured for a Buckling Restrained Axial Damper, used for seismic protection of buildings.

Key words: hysteresis, Ramberg-Osgood, BRAD anti-seismic devices.

1. INTRODUCTION

The well-known formula, firstly proposed in 1943 by Ramberg and Osgood [1], describes the stress-strain relations of softening type in terms of three parameters (Young's modulus and two secant yielding strengths) for monotonic loading starting from the origin. It was used by Jennings [2] as backbone curve to portray the hysteresis force-deflection loops of yielding type by a model with four parameters.

The Ramberg-Osgood equation describes the nonlinear hysteretic relation of the one-dimensional elasto-plastic behavior of many materials [3, 4]. It belongs to the class of algebraic models based on the skeleton curve and has been used to model the dynamic soil behavior [5, 6, 7]. This model combines kinematic and isotropic hardening in a linear manner by the introducing a weighting coefficient. Due to the arrangement of the variables in the Ramberg-Osgood equation, it is not a simple and straightforward task to obtain the values of these parameters which give the best fit of the given data. Computational procedures were developed for obtaining the values of parameters in the Ramberg-Osgood elasto-plastic model based on experimental data of shear modulus and damping ratio at various shear strains [8]. The response of Ramberg-Osgood type single-degree-of-freedom structure to strong motion earthquake was studied in [9]. A family of output spectral curves was presented for this system, where the Ramberg-Osgood exponent, restoring force amplitude, the ductility and the energy ratio were the main parameters considered. A simple model based on the Ramberg-Osgood model using to simulate the hysteretic behavior of shear panels made of low-yield steel which exhibits significant strain-hardening under load reversals was discussed in [10].

In this paper is presented a simple and efficient analytical method for fitting the Ramberg-Osgood to given yielding hysteresis loops. The algorithm is presented considering a virtual experiment with the target hysteric loop obtained by Ramberg-Osgood model for a given set of parameters. It is shown that by applying the proposed identification method, based only on four measured values on the target curve, there are obtained, with a very good accuracy, the same parameter values used for simulation of the target loop. The method efficiency for practical application is assessed through identification of Ramberg-Osgood model from experimental hysteresis loops obtained by laboratory tests of Buckling Restrained Axial Damper.

2. ANALYTICAL DESCRIPTION OF FITTING ALGORITHM

The Ramberg-Osgood (RO) model is used to describe load-displacement hysteresis curves $F(D)$ displaying an elastic branch up to the yield displacement D_y and the corresponding yield force F_y, followed by a transition curve which leads to a plastic branch as shown in figure 1. Since the transition from elastic to plastic behavior is gradual and the elastic range generally extends slightly beyond the proportional limit, the location of the "yield point" (D_y, F_y) is rather a matter of convention. In what follows the hysteresis loop shown in figure 1 will be considered as a target loop that must be fitted by RO model through an algorithm based on the measured values D_0, F_0, D_1, F_1.

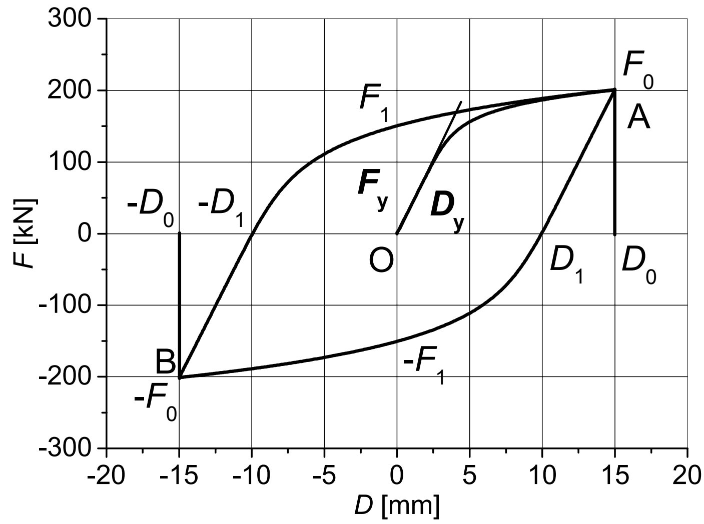

Fig. 1 – Target hysteresis loop to be fit by RO model:
D_0 =15mm, D_1 =10mm, F_0 =200daN, F_1 =150daN.

The stiffness corresponding to the elastic branch is given by

$$k_y = \left.\frac{\mathrm{d}F_{\mathrm{OA}}}{\mathrm{d}D}\right|_{D=0} = \frac{F_y}{D_y}. \tag{1}$$

The equations of RO model portraying this hysteretic behavior are [2]

$$\mathrm{OA}: \frac{D}{D_y} = \frac{F}{F_y}\left(1 + \eta\left|\frac{F}{F_y}\right|^{\gamma-1}\right), \quad \gamma > 1, \ \eta > 0;$$

$$\mathrm{AB}: \frac{D - D_0}{2D_y} = \frac{F - F_0}{2F_y}\left(1 + \eta\left|\frac{F - F_0}{2F_y}\right|^{\gamma-1}\right), \ \mathrm{BA}: \frac{D + D_0}{2D_y} = \frac{F + F_0}{2F_y}\left(1 + \eta\left|\frac{F + F_0}{2F_y}\right|^{\gamma-1}\right). \tag{2}$$

Introducing the following dimensionless parameters

$$k_y = \frac{F_y}{D_y}, \quad k_0 = \frac{F_0}{D_0}, \quad k_1 = \frac{F_1}{D_1}, \quad \lambda = \frac{k_0}{k_y}, \quad \beta = \frac{k_1}{k_0},$$

$$\delta = \frac{D_1}{D_0}, \quad \xi = \frac{D}{D_y}, \quad \xi_0 = \frac{D_0}{D_y}, \quad \xi_1 = \frac{D_1}{D_y}, \quad \Phi = \frac{F}{F_y}, \quad \Phi_0 = \Phi(\xi_0) = \frac{F_0}{F_y}, \quad \Phi_1 = \frac{F_1}{F_y} \tag{3}$$

yields

$$\begin{gathered} 0<\lambda<1, \quad \beta>1, \quad 0<\delta<1, \quad \xi_0>1, \quad \xi_1>0, \quad \Phi_0>1, \quad \Phi_1>0 \\ \Phi_0=\lambda\xi_0<\xi_0, \quad \Phi_1=\lambda\beta\delta\xi_0<\xi_0, \quad \xi_1=\delta\xi_0. \end{gathered} \tag{4}$$

Therefore, equations (2) can be rewritten as follows

$$\begin{gathered} \mathrm{OA}: \xi=\Phi\left(1+\eta|\Phi|^{\gamma-1}\right), \\ \mathrm{AB}: \xi-\xi_0=\left(\Phi-\lambda\xi_0\right)\left(1+\eta\left|\frac{\Phi-\lambda\xi_0}{2}\right|^{\gamma-1}\right), \quad \mathrm{BA}: \xi+\xi_0=\left(\Phi+\lambda\xi_0\right)\left(1+\eta\left|\frac{\Phi+\lambda\xi_0}{2}\right|^{\gamma-1}\right). \end{gathered} \tag{5}$$

Using relation (4), for $\xi=\xi_0$ the first equation (5) yields

$$\eta=\frac{(1-\lambda)}{\lambda^{\gamma}\xi_0^{\gamma-1}}. \tag{6}$$

Relation (6) implies that for given values of λ and γ the Jenning's parameter η is dependent only on the chosen value of yield displacement D_{y}. This can be proved by taking in (6) two different values $\xi_0'=D_0/D_{\mathrm{y}}'$ and $\xi_0''=D_0/D_{\mathrm{y}}''$ for the same values of parameters λ and γ. Then one obtains

$$\frac{\eta''}{\eta'}=\left(\frac{D_y''}{D_y'}\right)^{\gamma-1}. \tag{7}$$

In what follows the model parameters λ, γ and $D_{1\mathrm{y}}$ will be assessed by taking $\eta=1$. In this case, relation (6) becomes

$$\xi_{10}=\left(\frac{1-\lambda}{\lambda^{\gamma}}\right)^{\frac{1}{\gamma-1}}. \tag{8}$$

It is worth mentioning that the value of parameter D_{1y}, obtained for $\eta=1$ is not necessarily the "true" yield displacement. If the value D_y of yield displacement can be assessed empirically, then the corresponding value of parameter η_y is straightforward obtained from equation (7). Therefore, the RO models corresponding to the parameters sets $\{\lambda, \gamma, \eta_y, D_y\}$, $\{\lambda, \gamma, \eta_1, D_{1y}\}$ will provide the same approximation of target hysteresis loop.

Using relations (5) for $\eta=1$, the intersections of the hysteresis loop with dimensionless displacement and force coordinate axes are obtained from

$$\xi_{10}=\Phi_{10}\left(1+\Phi_{10}^{\gamma-1}\right), \; \xi_{11}=\xi_{10}-\left(\Phi_{10}+\frac{\Phi_{10}^{\gamma}}{2^{\gamma-1}}\right), \xi_{10}=\left(\Phi_{10}+\Phi_{11}\right)\left[1+\left(\frac{\Phi_{10}+\Phi_{11}}{2}\right)^{\gamma-1}\right]. \tag{9}$$

Introducing (4) in (9) and solving the resulting equations for parameter γ, yields a system of transcendental algebraic equations for parameters λ and γ:

$$\gamma=1+\frac{1}{\ln 2}\ln\frac{1-\lambda}{1-\lambda-\delta}=\frac{\ln 2+\ln(1-\lambda)-\ln(1-\lambda-\lambda\beta\delta)}{\ln 2-\ln(1+\beta\delta)}. \tag{10}$$

It can be shown that the necessary and sufficient condition for the existence of a unique solution $0<\lambda<1, \; \gamma>1$ is

$$\beta\delta>2^{\frac{\ln(1-\delta)}{\ln(1-\delta)-\ln 2}}-1. \tag{11}$$

The inequality (11) can be rewritten in terms of experimentally determined values as follows

$$\frac{F_1}{F_0} > 2^{\frac{\ln(1-D_1/D_0)}{\ln(1-D_1/D_0)-\ln 2}} - 1\,. \tag{12}$$

The system consisting of equations (8) and (10) allow assessing the parameters D_y, F_y, γ of Ramberg-Osgood model (2) that fits a target hysteresis loop by using the measured values D_0, F_0, D_1, F_1. Thus, the parameters λ and γ are obtained from equations (10). Next, the value of ξ_{10} is given by (8). Finally, the remaining parameters D_{1y} and F_{1y} of RO model (2) are determined from

$$D_{1y} = \frac{D_0}{\xi_{10}}, \quad F_{1y} = k_y D_{1y} = \frac{k_0 D_0}{\lambda \xi_{10}}\,. \tag{13}$$

The hysteretic loop described by the identified RO model (2), can be plotted by considering a harmonic loading

$$F(t) = F_0 \sin 2\pi \frac{t}{T}, \quad 0 \le t \le \frac{5T}{4} \tag{14}$$

and using relations (2) to determine numerically the corresponding time evolution of displacement $D(t)$. Next, the elimination of parameter t yields the hysteresis loop $F(D)$.

By applying the described fitting algorithm for $\eta = 1$ and the measured values of D_0, D_1, F_0, F_1, specified in figure 1, the values $\lambda = 0.33, \gamma = 8.65$, and $k_y = 40.4\,\text{kN/mm}$ were obtained for the invariant parameters with respect to the chosen value of η and the values $D_{1y} = 4.5\,\text{mm}$, $F_{1y} = k_y D_{1y} = 182.3\,\text{kN}$ for "yield displacement and force". Figure 2 shows that there is no difference between the target hysteresis loop and its analytical fit by RO model (2).

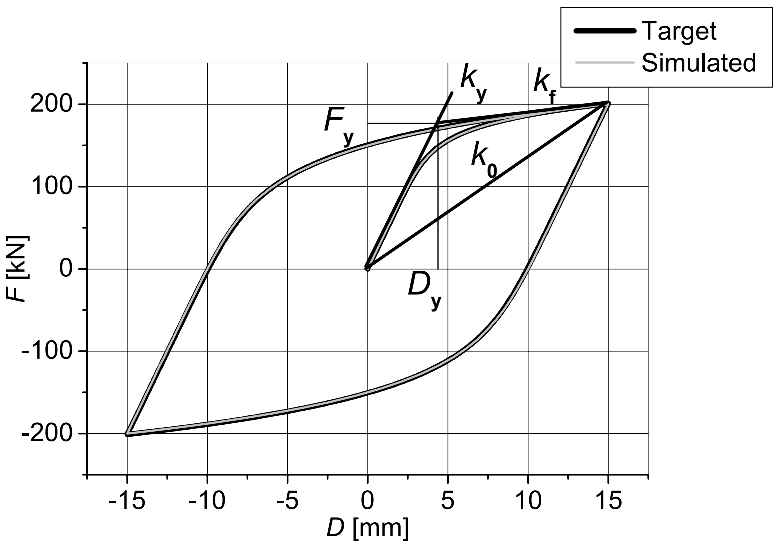

Fig. 2 – Target hysteresis loop and its analytical fit by Ramberg-Osgood model.

According to a well establish practice [11], the reference values of Ramberg Osgood model are chosen so that the "yield point" to be the intersection of the tangent line in (0,0) having the slope k_y (pre-yield stiffness) with the left tangent line in (D_0, F_0) having the slope $k_f = \alpha_0 k_y, \alpha_0 < 1$ (post-yield stiffness). In this case, the coordinates D_y, F_y of this new "yield point" (shown in Fig.2) can be determined by using the model parameters obtained for $\eta = 1$. The slopes of aforementioned tangents are given by

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right|_{\xi=0} = 1, \ \left.\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right|_{\substack{\xi=\xi_{10} \\ \xi<\xi_{10}}} = \frac{1}{1+\gamma\Phi_{10}^{\gamma-1}} = \frac{\lambda}{\lambda+\gamma(1-\lambda)} = \frac{k_0}{k_0+\gamma\left(k_y - k_0\right)} = \alpha_0 \,. \tag{15}$$

Since $\Phi_{10} = \lambda\xi_{10}$, the intersection coordinates ξ_y, Φ_y of these two tangents can be expressed as

$$\xi_y = \Phi_y = \frac{\lambda\left(1+\gamma\Phi_{10}^{\gamma-1}\right)-1}{\lambda\gamma\Phi_{10}^{\gamma-2}} \,. \tag{16}$$

On the other hand one can write

$$\Phi_y = \frac{F_y}{F_{1y}}, \ \xi_y = \frac{D_y}{D_{1y}} = \frac{k_y}{F_{1y}} D_y \quad . \tag{17}$$

Therefore, in view of (4), (6), (9) and (17), the coordinates D_y, F_y of the considered "yield point" can be expressed as

$$D_y = \frac{\lambda(\gamma-1)}{\gamma} D_0, \qquad F_y = \frac{\gamma-1}{\gamma} F_0 \,. \tag{18}$$

The value η_y corresponding to "yielding displacement" D_y given by (18) is obtained from (7) as follows

$$\eta_y = \left[\frac{\lambda(\gamma-1)D_0}{\gamma D_{1y}}\right]^{\gamma-1} . \tag{19}$$

Table 1 shows the values obtained for parameters of Ramberg-Osgood model that fits the hysteresis loop plotted in Fig. 2 for this definition of yield point.

Table 1

D_y [mm]	F_y [kN]	k_y [kN/mm]	λ	γ	η_y
4.38	176.9	40.4	0.33	8.65	0.79

The dissipated energy per cycle E_d is equal with the area captured by the hysteresis loop depicted in Fig. 1. Denoting by Σ_d the area enclosed by the dimensionless hysteresis loop shown in Fig. 2, the dissipated energy per cycle E_d is given by

$$E_\mathrm{d} = F_y D_y \Sigma_\mathrm{d} \,. \tag{20}$$

The area Σ_d can be calculated straightforward by using the last two equations (5), where the dimensionless force Φ is considered the independent variable of function $\xi(\Phi)$. Due to the symmetry of hysteresis loop one can write

$$\Sigma_\mathrm{d} = 4\lambda(1-\lambda)\frac{\gamma-1}{\gamma+1}\xi_0^2 \,. \tag{21}$$

Using (3), (4) in (20), the dissipated energy per cycle can be expressed as

$$E_\mathrm{d} = 4(1-\lambda)\frac{\gamma-1}{\gamma+1}F_0 D_0 \,. \tag{22}$$

For the considered hysteresis loop, the dissipated energy per cycle is $E_d = 6\ 374$ J .

3. FITTING THE RAMBERG-OSGOOD MODEL TO EXPERIMENTAL HYSTERESIS LOOPS OF BRAD SEISMIC DEVICE

Consider the hysteresis loops (Fig. 3) obtained by laboratory testing of a bracing device of type BRAD (Buckling Restrained Axial Damper), manufactured by FIPP Industriale (Italy) for seismic protection of buildings [12]. As one can see from these plots, the first branch is not shown. This presentation of experimental data is often encountered in many technical papers since the hysteresis loops are usually recorded after completion of several loading cycles in order to ensure the stabilization and repeatability of the force output. Therefore, in many cases the relation (1) cannot be used for the assessment of device initial stiffness k_y. Nevertheless, the proposed fitting algorithm allows determining with good accuracy this model parameters only by measuring the intersection coordinates of experimental hysteresis loops with displacement and force axes. The device was tested for three amplitudes of imposed cyclic motion: $D_{01} = 5\text{mm}, D_{02} = 10\text{mm}, D_{03} = 15\text{mm}$. The following values where assessed for the maximum force developed by the device and for the intercepts of hysteresis loops with force and displacement axes:

$$\begin{aligned} &F_{01} = 165\text{kN},\ D_{11} = 3.5\text{mm},\ F_{11} = 140\text{kN} \\ &F_{02} = 180\text{kN},\ D_{12} = 8.3\text{mm},\ F_{12} = 172\text{kN} \\ &F_{03} = 200\text{kN},\ D_{13} = 13\text{mm},\ \ F_{13} = 182\text{kN}. \end{aligned} \tag{23}$$

By applying the described fitting algorithm for $\eta = 1$ and the measured values (23), were obtained the following values of model parameters

$$\begin{aligned} &\lambda_1 = 0.3, \quad \gamma_1 = 14.5,\ k_{1y1} = 111\,\text{kN/mm},\ \ D_{1y1} = 1.4\,\text{mm},\ F_{1y1} = 155\,\text{kN} \\ &\lambda_2 = 0.17, \ \ \gamma_2 = 23.5,\ k_{1y2} = 108\,\text{kN/mm},\ \ D_{1y2} = 1.6\,\text{mm},\ F_{1y2} = 173\,\text{kN} \\ &\lambda_3 = 0.13, \ \ \gamma_3 = 18.2,\ k_{1y3} = 100\,\text{kN/mm},\ \ D_{1y3} = 1.8\,\text{mm},\ F_{1y3} = 180\,\text{kN}. \end{aligned} \tag{24}$$

The values of dissipated energy per cycle, obtained from area enclosed by experimental hysteresis loops, $E_{d,exp}$ and those of $E_{d,sim}$, given by introducing the corresponding model parameters (23) and (24) in relation (22), are presented in Table 2.

Table 2

D_0 [mm]	5	10	15
$E_{d,exp}$ [Nm]	2042	5524	9378
$E_{d,sim}$ [Nm]	2012	5489	9358
Relative error [%]	1.5	0.63	0.21

In Fig. 3 are shown comparatively the hysteresis loops obtained by testing the BRAD device and their analytical fit by RO model with parameters (24) plotted against the target.

The obtained results prove the effectiveness of the proposed algorithm for fitting the RO model to hysteresis loops. From relations (24), one can see a certain degradation of the device initial stiffness k_y , due to the repeated tests required to obtain the hysteresis loops for different increasing amplitudes of imposed cyclic motion. The results specified in (24) show that is not always possible fitting through the described algorithm the RO model defined by the same set of parameters in equations (2) to all experimental hysteresis loops, obtained for the same device at different amplitudes of imposed cyclic motion. Nevertheless, if the target hysteresis loop is measured for the maximum admissible amplitude of imposed displacement D_0 or applied loading F_0 , as so to obtain a good estimation for the ratio α_0 of post-yield to pre-yield stiffness, one can achieve a sufficiently good approximation of experimental hysteresis loops measured for different amplitudes by using the RO model parameters determined for the maximum imposed displacement.

The experimental hysteresis loops and the output of RO_3 model with parameters $D_{03}, D_{y3}, F_{03}, F_{y3}, \gamma_3$ are presented comparatively in Fig. 4.

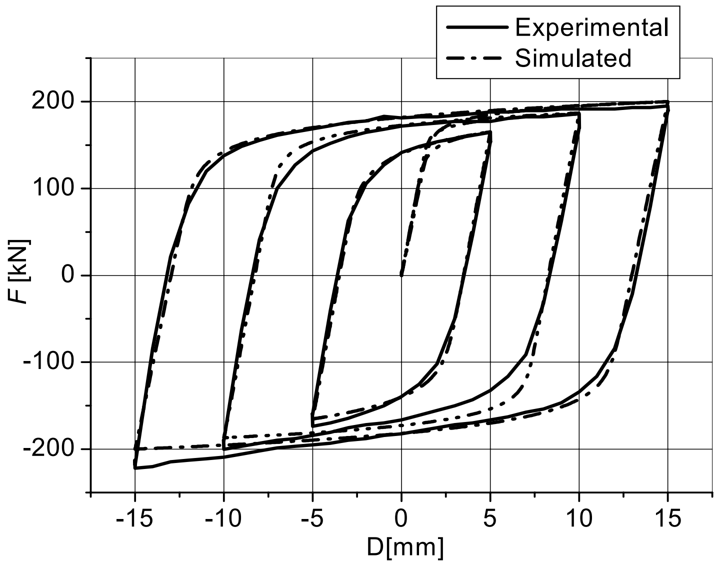

Fig. 3 – Experimental hysteresis loops and their individual analytical fit by RO model.

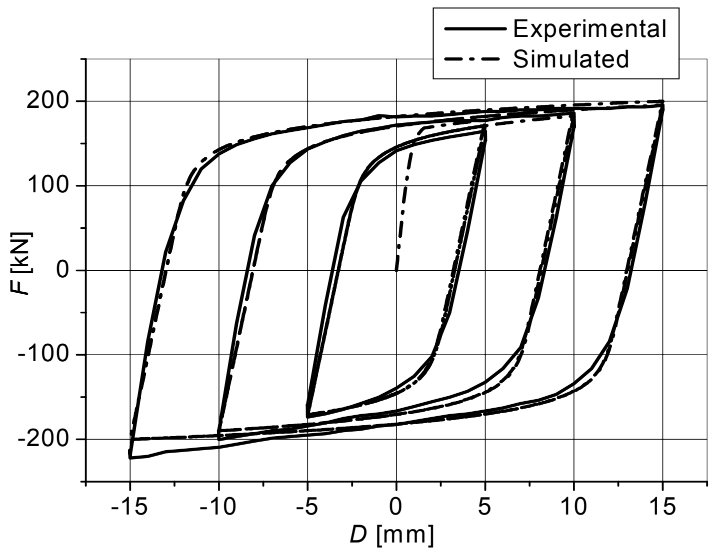

Fig. 4 – Experimental hysteresis loops and their analytical fit by RO_3 model.

4. CONCLUSIONS

The paper presents a simple and efficient analytical method for fitting the Ramberg-Osgood to given yielding hysteresis loops. The algorithm input data are only the maximum displacement and force values and hysteresis loops intercepts with force and displacement axes. The algorithm is developed for unit value of Jennings parameter η. A simple analytical relationship allows the assessment of this parameter for any assumed value of yielding displacement.

The proposed method is applied to fit the RO model to hysteresis loops obtained by laboratory testing of a bracing device BRAD (Buckling Restrained Axial Damper), manufactured by FIPP Industriale (Italy) for seismic protection of buildings. It is shown that the RO model identified for the maximum design displacement can be used to approximate with good accuracy the hysteresis loops measured for smaller amplitudes of imposed cyclic displacement.

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