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ON A CERTAIN PRODUCT OF BANACH ALGEBRAS AND SOME OF ITS PROPERTIES

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In this paper, for two arbitrary Banach algebras A and B and a homomorphism $T: B \to A$, we construct and study a product on the Cartesian product space $A \times B$, and we denote this algebra by $A \times_T B$. Among other things, we characterize the set of all continuous derivations from $A \times_T B$ into its n'th dual space $(A \times_T B)^{(n)}$ and as an application we study the n-weak amenability of $A \times_T B$ and its relation with A and B. Moreover, we obtain characterizations of (bounded) approximate identities and study the ideal structure of these products.

Key words: Banach algebra, Arens regularity, weak amenability.

1. INTRODUCTION

The Lau product of two Banach algebras that are pre-duals of von Neumann algebras, and for which the identity of the dual is a multiplicative linear functional, was introduced and investigated by Lau [13]. This paper initiated a series of subsequent publications [6, 7, 9, 15, 16, 17] and has had a great impact. Extension to arbitrary Banach algebras was proposed by Monfared [14] with the notation $A \times_\theta B$, where θ is a non-zero multiplicative linear functionals on B.

Recently, a new extension of Lau product has been studied by Bhatt and Dabhi [1]. More precisely, for a *commutative* Banach algebra A and arbitrary Banach algebra B, by using the algebra homomorphism $T: B \to A$ they defined T-Lau product on the Cartesian product space $A \times B$ perturbing the pointwise-defined product resulting in a new Banach algebra $A \times_T B$, as follows

$$(a,b)(a',b') = (aa' + T(b)a' + T(b')a, bb'),$$

for all $a, a' \in A$ and $b, b' \in B$ and they investigated the Arens regularity and some notions of amenability of $A \times_T B$. Note that T-Lau product coincides with Lau product and θ-Lau product of Banach algebras. However, routine observations show that the imposed hypothesis on A for the definition of T-Lau product given by [1, p. 1] is too strong. As an evidence for this claim, recall that for two arbitrary Banach algebras A and B and a multiplicative linear functional $\theta: B \to \mathbb{C}$, Monfared [14, Corollary 2.13] proved that the Banach algebra $A \times_\theta B$ is Arens regular if and only if both A and B are Arens regular, but Bhatt and Dabhi [1, Theorem 3.1] proved the same result for $A \times_T B$ only when A is an Arens regular commutative Banach algebra; This is because of, on the one hand the commutativity assumption is needed in the Bhatt and Dabhi's definition, and on the other hand we know that the second dual of a commutative Banach algebra is commutative when it is Arens regular.

This paper continues these investigations. In Section 2, by a change in the Bhatt and Dabhi's definition we show that T-Lau products can be defined for Banach algebras in a fairly general setting; Indeed, we remove the commutativity assumption imposed on the definition of T-Lau products given by [1] and we show that we can recover most of the results obtained in [1] in the general case. Moreover, we study the ideal structure of T-Lau product of Banach algebras and we characterize the existence of the (bounded) approximate identity for this product. In the third part we introduce a bimodule action of $(A\times_T B)$ on $(A\times_T B)^{(n)}$ and, we characterize the set of all continuous derivations from $A\times_T B$ into $(A\times_T B)^{(n)}$, where $(A\times_T B)^{(n)}$ denotes the n'th dual space of $A\times_T B$. Finally, we study the n-weak amenability of $A\times_T B$ and its relation with A and B and then we apply our results to some particular instances of Banach algebras such as quotient space of Banach algebras with a closed ideal, Banach algebras related to locally compact group and unital Banach algebra.

2. THE DEFINITION AND SOME BASIC RESULTS

Let A and B be arbitrary Banach algebras (not necessary commutative) and $T:B\to A$ be an algebra homomorphism. If we replaced the first coordinate of the ordered pair in equation (1) above by "$aa'+aT(b')+T(b)a'$", then $A\times_T B$ will be an associative algebra. So, we are lead to the following definition, where it paves the way to remove the commutativity assumption imposed on A in the definition of $A\times_T B$ given by [1, Page 1].

Definition 2.1. Let A and B be arbitrary Banach algebras and $T:B\to A$ be an algebra homomorphism. The T-Lau product $A\times_T B$ is defined as the Cartesian product $A\times B$ with the algebra product

$$(a,b)(a',b'):=(aa'+aT(b')+T(b)a',bb'),$$

and the norm $\|(a,b)\|=\|a\|+\|b\|$ for all $a,a'\in A$ and $b,b'\in B$.

The reader will remark that, routine computations show that if $\|T\|\le 1$, then $A\times_T B$ is a Banach algebra. In what follows A and B are arbitrary Banach algebras and $T:B\to A$ is an algebra homomorphism, with $\|T\|\le 1$.

We now proceed to show that we can recover most of the results obtained in [1] in the general case. First, by an argument similar to the proof of [1, Theorem 2.1, Theorem 4.1], we can prove the following improvement of those. Note that in Theorems 2.1 and 4.1 of [1] the commutativity of A is needed only to conclude that $A\times_T B$ is Banach algebra; whereas by Definition 2.1 above this Banach spaces always is Banach algebra without any further assumptions on A. Moreover, the weak amenability of the Banach algebra $A\times_T B$ is considered in Corollary 3.6 below as a special case of the Propositions 3.4 and 3.5 in section 3 below.

THEOREM 2.2. *The following assertions hold.*

(i) *Let the Gelfand space* $\Delta(B)$ *is non-empty and* $E=\{(\lambda,\lambda\circ T):\lambda\in\Delta(A)\}$, $F=\{(0,\phi):\phi\in\Delta(B)\}$. *Set* $E=\varnothing$ *if* $\Delta(A)$ *is emptyt. Then* $\Delta(A\times_T B)$ *is equal to the union of its closed disjoint set* E *and* F.

(ii) $A\times_T B$ *is amenable if and only if both* A *and* B *are amenable;*

(iii) $A\times_T B$ *is* (*approximately*) *cyclic amenable if and only if both* A *and* B *are* (*approximately*) *cyclic amenable.*

In order to state our next results we need to set some terminology. In what follows for a Banach space X the notation $X^{(n)}$ is used to denote the n'th dual space of X and we always consider X as naturally embedded into $X^{(2)}$. For $x\in X$ and $f\in X^{(1)}$, by $\langle x,f\rangle$ (and also $\langle f,x\rangle$) we denote the natural duality between X and $X^{(1)}$. Now let A be a Banach algebra. On $A^{(2)}$ there exists two natural products extending the one on A, known as the first and second Arens products of $A^{(2)}$. For a,b in A, f in $A^{(1)}$ and Φ,Ψ in $A^{(2)}$, the elements fa and Φf of $A^{(1)}$ and $\Psi\bullet\Phi$ of $A^{(2)}$ are defined, respectively, as follows.

$$\langle fa,b\rangle=\langle f,ab\rangle,\qquad \langle\Phi f,a\rangle=\langle\Phi,fa\rangle\qquad\text{and}\qquad\langle\Psi\bullet\Phi,f\rangle=\langle\Psi,\Phi f\rangle.$$

Also, the second Arens product "$\lozenge$" is defined using symmetry. Equipped with these products, $A^{(2)}$ is a Banach algebra and A is a subalgebra of it. These products are in general not separately weak* to weak* continuous on $A^{(2)}$. In general, the first and the second Arens products do not coincide on $A^{(2)}$ and A is said to be Arens regular if these products coincide on $A^{(2)}$. The first topological center $Z_t^1(A^{(2)})$ of $A^{(2)}$ is the set of all $\Phi\in A^{(2)}$ such that $\Phi\bullet\Psi=\Phi\lozenge\Psi$, for all $\Psi\in A^{(2)}$.

In [1, Theorem 3.1], Bhatt and Dahbi proved that if A is commutative and Arens regular Banach algebra, then

(i) $(A\times_T B)^{(2)}$ is isometrically isomorphic to $A^{(2)}\times_{T[2]} B^{(2)}$, when the algebras are equipped with the same either of its Arens products; (ii) $Z_t^1((A\times_T B)^{(2)})=A^{(2)}\times_{T[2]} Z_t^1(B^{(2)})$;

(iii) $A\times_T B$ is Arens regular if and only if B is Arens regular,

where $T^{[2]}$, the second adjoint of T, is a continuous homomorphisms in a natural way. Now, by an argument similar to the proof of [1, Theorem 3.1], we can prove the following improvement of it; Indeed, we remove the following assumptions from the Bhatt-Dahbi's result for Arens regularity of $A\times_T B$:

- The commutativity assumption on A,
- The Arens regularity assumption on A.

Note that, the Banach algebra $A^{(2)}$ is commutative, if the underlying Banach algebra is commutative and Arens regular, so in Theorem 3.1 of [1] the commutativity and Arens regularity of A are needed only to conclude that $A\times_T B$ and $A^{(2)}\times_{T[2]} B^{(2)}$ are Banach algebras; whereas by Definition 2.1 above these Banach spaces always are Banach algebras without any further assumptions on A.

THEOREM 2.3. T*he following assertions hold.*

(i) $(A\times_T B)^{(2)}$ *is isometrically isomorphic to the* $A^{(2)}\times_{T[2]} B^{(2)}$*;*

(ii) $Z_t^1((A\times_T B)^{(2)})=Z_t^1(A^{(2)})\times_{T[2]} Z_t^1(B^{(2)})$;

(iii) $A\times_T B$ *is Arens regular if and only if both* A *and* B *are Arens regular*.

We end this section by the following three results, which are of interest in its own right.

PROPOSITION 2.4. *Let* I *be a left ideal of* A *and* J *be a left ideal of* B. *Then the following assertions hold.*

(i) *If* $T(J)\subseteq I$. *Then* $I\times_T J$ *is a left ideal of* $A\times_T B$.

(ii) *If* I *is a prime left ideal and* $I\times_T J$ *is a left ideal of* $A\times_T B$, *then* $T(J)\subseteq I$.

(iii) *If* A *is a unital Banach algebra. Then* $I\times_T J$ *is a left ideal of* $A\times_T B$ *if and only if* $T(J)\subseteq I$.

(iv) *If* I *is a close left ideal of* A *and, suppose that* A *has a left approximate identity. Then* $I\times_T J$ *is a left ideal of* $A\times_T B$ *if and only if* $T(J)\subseteq I$.

Proof. The proof of (i) is routine and is omitted. For the proof of (ii) we need only note that if I is a prime ideal of A and $I\times_T J$ is an ideal of $A\times_T B$, then $T(J)\subseteq I$. To this end, assume that there exists $b'\in J$ such that $T(b')\notin I$. Then

$$(a,0)(a',b')=(aa'+aT(b'),0)\in I\times_T J,$$

for all $a'\in I$ and $a\in A\setminus I$, which contradicts with assumption.

(iv) Let $(a_\alpha)_\alpha$ be a left approximate identity for A, and suppose that $a' \in I$ and $b' \in J$. Then

$$(a_\alpha, 0)(a', b') = (a_\alpha a' + a_\alpha T(b'), 0) \in I \times_T J,$$

for all α. Since I is closed it follows that $a' + T(b') \in I$. Consequently, $T(J) \subseteq I$. Finally, note that one can prove the assertion (iii) by the same manner as in the proof of (iv).□

Next we consider the converse.

PROPOSITION 2.5. *Let N be a left ideal of $A \times_T B$ and*
$I = \{a \in A : (a,b) \in N$ *for some* $b \in B\}$, $J = \{b \in B : (a,b) \in N$ *for some* $a \in A\}$.
Then the following assertions hold.
(i) *J is a left ideal in B.*
(ii) *If T is onto, then I is a left ideal of A.*
(iii) *If N is closed and A has a left approximate identity, then a necessary condition for the equality $N = I \times_T J$ is that $T(J) \subseteq I$.*

Proof. For briefness we only give the proof for (ii) and (iii). Let $c \in I$ and $a \in A$. Then there exists $b \in B$ such that $T(b) = a$. Therefore, $(a,b)(c,0) \in N$ and hence $ac \in I$. Now, we give a proof for (iii). To this end, let $N = I \times_T J$ and $j \in J$. Then there is $i \in I$ such that $(i, j) \in N$. If now $(a_\alpha)_\alpha$ is a left approximate identity for A, then $(a_\alpha, 0)(i, j) \in N$. Hence $i + T(j) \in I$ and this implies that $T(j) \in I$.□

PROPOSITION 2.6. *The Banach algebra $A \times_T B$ has a bounded* (*resp. unbounded*) *left* (*resp. right, or two-sided*) *approximate identity if and only if A and B have the same approximate identity.*

Proof. First assume that the nets $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ are bounded left approximate identities for A and B, respectively. Then the net $((a_\alpha - T(b_\alpha), b_\alpha))_\alpha$ is a bounded left approximate identity for $A \times_T B$; Indeed, for every $(a,b) \in A \times_T B$

$$(a_\alpha - T(b_\alpha), b_\alpha)(a,b) = ((a_\alpha - T(b_\alpha))a + T(b_\alpha)a + (a_\alpha - T(b_\alpha))T(b), b_\alpha b).$$

Conversely, if $((a_\alpha, b_\alpha))_\alpha$ is a bounded left approximate identity for $A \times_T B$, then the nets $(a_\alpha + T(b_\alpha))_\alpha$ and $(b_\alpha)_\alpha$ are bounded. Also for every $a \in A$ and $b \in B$ we must have

$$\|b_\alpha b - b\| \le \|(a_\alpha, b_\alpha)(0,b) - (0,b)\| \to 0, \quad \|(a_\alpha + T(b_\alpha))a - a\| = \|(a_\alpha, b_\alpha)(a,0) - (a,0)\| \to 0.$$

Since a and b are arbitrary we conclude that the nets $(a_\alpha + T(b_\alpha))_\alpha$ and $(b_\alpha)_\alpha$ $(b_\alpha)_\alpha$ are bounded left approximate identities for A and B, respectively. This completes the proof of this proposition, since one can obtain a proof for other cases by similar argument. □

3. DERIVATIONS INTO ITERATED DUALS

We commence this section with some notations. In this section A and B are two arbitrary Banach algebras and $T : B \to A$ be an algebra homomorphism with norm at most 1. Then $(A \times_T B)^{(1)}$, the first dual space of $A \times_T B$, can be identified with $A^{(1)} \times B^{(1)}$ in the natural way

$$\langle (a^{(1)}, b^{(1)}), (a,b) \rangle = \langle a^{(1)}, a \rangle + \langle b^{(1)}, b \rangle,$$

for all $a \in A, b \in B, a^{(1)} \in A^{(1)}$ and $b^{(1)} \in B^{(1)}$. The dual norm on $A^{(1)} \times B^{(1)}$ is of course the maximum norm $\|(a^{(1)}, b^{(1)})\| = \max\left\{\|a^{(1)}\|, \|b^{(1)}\|\right\}$. Now for integer $n \ge 0$, take $A^{(n)} \times B^{(n)}$ as the underlying space of

$(A\times_T B)^{(n)}$. From induction, we can find that the $(A\times_T B)$-bimodule actions on $(A\times_T B)^{(n)}$ are formulated as follows:

$$(a,b)\cdot(a^{(n)},b^{(n)})=\begin{cases}\left((a+T(b))\cdot a^{(n)}+a\cdot T^{[n]}(b^{(n)}),b\cdot b^{(n)}\right) & \text{if } n \text{ is even}\\[2ex] \left((a+T(b))\cdot a^{(n)},T^{[n]}(a\cdot a^{(n)})+b\cdot b^{(n)}\right) & \text{if } n \text{ is odd}\end{cases}$$

and

$$(a^{(n)},b^{(n)})\cdot(a,b)=\begin{cases}\left(a^{(n)}\cdot(a+T(b))+T^{[n]}(b^{(n)})\cdot a,b^{(n)}\cdot b\right) & \text{if } n \text{ is even}\\[2ex] \left(a^{(n)}\cdot(a+T(b)),T^{[n]}(a^{(n)}\cdot a)+b^{(n)}\cdot b\right) & \text{if } n \text{ is odd,}\end{cases}$$

for all $(a,b)\in A\times_T B$ and $(a^{(n)},b^{(n)})\in A^{(n)}\times B^{(n)}$, where $T^{[n]}$ denotes the n'th adjoint of T.

Let A be a Banach algebra and X be a Banach A-bimodule. Then a linear map $D:A\to X$ is called a derivation if $D(ab)=a\cdot D(b)+D(a)\cdot b,$ for all $a,b\in A$. For $x\in X$, we define $d_x:A\to X$ as follows $d_x(a)=a\cdot x-x\cdot a,$ for all $a,b\in A$. It is easy to show that d_x is a derivation; such derivations are called inner derivations. A derivation $D:A\to X$ is called inner if there exist $x\in X$ such that $D=d_x$. We denote the set of all continuous derivations from A into X by $Z^1(A,X)$. The first cohomology group $H^1(A,X)$ is the quotient of the space of continuous derivations by the inner derivations, and in many situations triviality of this space is of considerable importance. In particular, A is called contractible if $H^1(A,X)=0$ for every Banach A-bimodule X, A is called amenable if $H^1(A,X^{(1)})=0$ for every Banach A-bimodule X, for every $n\geq 0$, A is called n-weakly amenable if $H^1(A,A^{(n)})=0$, and weakly amenable if A is 1-weakly amenable, where $A^{(n)}$ is the n'th dual module of A when $n\geq 1$, and is A itself when $n=0$. Let us mention that, the concept of weak amenability was first introduced and intensively studied by Bade, Curtis and Dales [2] for commutative Banach algebras, and then by Johnson [12] for a general Banach algebra. Dales, Ghahramani and Grønbæk [4] initiated and intensively developed the study of n-weak amenability of Banach algebras; see also [8, 10, 11, 18].

The following result characterize the set of all continuous derivations from $A\times_T B$ into $(A\times_T B)$-bimodule $(A\times_T B)^{(2n+1)}$. In the sequel the notation M is used to denote the T-Lau product of the Banach algebras A and B.

THEOREM 3.1. *Let* $M=A\times_T B$ *and* $n\geq 0$. *Then* $D\in Z^1(M,M^{(2n+1)})$ *if and only if there exist* $d_1\in Z^1(A,A^{(2n+1)})$, $d_2\in Z^1(B,B^{(2n+1)})$, $d_3\in Z^1(B,A^{(2n+1)})$ *and a bounded linear map* $S:A\to B^{(2n+1)}$, *such that for each* $a,a'\in A$ *and* $b\in B$;

(i) $D((a,b))=(d_1(a)+d_3(b),S(a)+d_2(b))$,

(ii) $d_1(T(b)a)=d_3(b)a+T(b)d_1(a)$,

(iii) $d_1(aT(b))=ad_3(b)+d_1(a)T(b)$,

(iv) $S(T(b)a)=T^{[2n+1]}(d_3(b)a)+bS(a)$,

(v) $S(aT(b))=T^{[2n+1]}(a\ \ d_3(b))+S(a)b$,

(vi) $S(aa')=T^{[2n+1]}(d_1(aa'))$.

Proof. Suppose that $D\in Z^1(M,M^{(2n+1)})$. Then there exist bounded linear maps $D_1:A\times B\to A^{(2n+1)}$ and $D_2:A\times B\to B^{(2n+1)}$ such that $D=(D_1,D_2)$. Let $d_1(a)=D_1((a,0))$, $d_2(b)=D_2((0,b))$, $d_3(b)=D_1((0,b))$ and $S(a)=D_2((a,0))$ for all $a\in A$ and $b\in B$. Then trivially d_1, d_2, d_3 and S are linear maps satisfying (i). Moreover for every $a,a'\in A$ and $b,b'\in B$ we have

$$D((a,b)(a',b')) = D((aa' + T(b)a' + aT(b'), bb')) = \\ \left(d_1(aa' + T(b)a' + aT(b')) + d_3(bb'), d_2(bb') + S(aa' + T(b)a' + aT(b'))\right), \tag{1}$$

and

$$D((a,b))(a',b') + (a,b)D((a',b')) = \\ = \left(d_1(a) + d_3(b), S(a) + d_2(b))(a',b') + (a',b')(d_1(a') + d_3(b'), S(a') + d_2(b')\right) = \\ = \left((d_1(a) + d_3(b))a' + (d_1(a) + d_3(b))T(b'), T^{[2n+1]}((d_1(a) + d_3(b))a') + (S(a) + d_2(b))b'\right) = \\ + \left(a(d_1(a') + d_3(b')) + T(b)(d_1(a') + d_3(b')), T^{[2n+1]}(a(d_1(a') + d_3(b'))) + b(S(a') + d_2(b'))\right). \tag{2}$$

It follows that $D \in Z^1(M, M^{(2n+1)})$ if and only if (1) and (2) coincide. Thus

$$d_1(aa' + T(b)a' + aT(b')) + d_3(bb') = d_1(a)a' + d_3(b)a' + d_1(a)T(b') + d_3(b)T(b') \\ + ad_1(a') + ad_3(b') + T(b)d_1(a') + T(b)d_3(b'), \tag{3}$$

and

$$S\left(aa' + T(b)a' + aT(b')\right) + d_2(bb') = T^{[2n+1]}\left(d_1(a)a' + d_3(b)a'\right) + S(a)b' \\ + d_2(b)b' + T^{[2n+1]}\left(ad_1(a') + ad_3(b')\right) + bS(a') + bd_2(b'). \tag{4}$$

Therefore $D \in Z^1(M, M^{(2n+1)})$ if and only if the equations (3) and (4) are satisfied. Now a straightforward verification shows that if d_1, d_2 and d_3 are derivations and the equalities (ii), (iii), (iv), (v), (vi) are satisfied, then (3) and (4) are valid. Applying (3) and (4) for suitable values of a, a', b, b' shows that d_1, d_2 and d_3 are derivations and the equalities (ii), (iii), (iv), (v), (vi) are also satisfied, as claimed. □

As an immediate corollary we have the following result, which characterize the set of all inner derivations from $A \times_T B$ into $(A \times_T B)$-bimodule $(A \times_T B)^{(2n+1)}$.

COROLLARY 3.2. *Let* $M = A \times_T B$ *and* $n \geq 0$. *Then* $D \in Z^1(M, M^{(2n+1)})$ *and* $D = d_{\left(a^{(2n+1)}, b^{(2n+1)}\right)}$ *if and only if* $D((a,b)) = \left(d_1(a) + d_3(b), S(a) + d_2(b)\right)$, *for all* $a \in A$ *and* $b \in B$, *where* $d_1 = d_{a^{(2n+1)}}$, $d_2 = d_{b^{(2n+1)}}$, $d_3 = d_{a^{(2n+1)}}$, $S = \delta_{a^{(2n+1)}}$ *and* $\delta_{a^{(2n+1)}}(a) = T^{[2n+1]}(d_{a^{(2n+1)}}(a))$ *for all* $a \in A$.

Proof. For the proof we need only note that if $D = d_{\left(a^{(2n+1)}, b^{(2n+1)}\right)}$ for some $a^{(2n+1)} \in A^{(2n+1)}$ and $b^{(2n+1)} \in B^{(2n+1)}$, then

$$\left(d_1(a), S(a)\right) = D((a,0)) = d_{\left(a^{(2n+1)}, b^{(2n+1)}\right)}(a,0) = \left(d_{a^{(2n+1)}}(a), T^{[2n+1]}(d_{a^{(2n+1)}}(a))\right),$$

for all $a \in A$. It follows that $d_1 = d_{a^{(2n+1)}}$ and $S = \delta_{a^{(2n+1)}}$. Similarly $d_2 = d_{b^{(2n+1)}}$ and $d_3 = d_{a^{(2n+1)}}$, and this completes the proof. □

By replacing $(2n+1)$ by $(2n)$ in Theorem 3.1 and Corollary 3.2 and using a similar argument, one can obtain the following result.

THEOREM 3.3. *Let* $M = A \times_T B$ *and* $n \geq 0$. *Then* $D \in Z^1(M, M^{(2n)})$ *if and only if there exist* $d_2 \in Z^1(B, B^{(2n)})$, $d_3 \in Z^1(B, A^{(2n)})$ *and bounded linear maps* $S : A \to B^{(2n)}$ *and* $R : A \to A^{(2n)}$ *such that for each* $a, a' \in A$ *and* $b \in B$ *;*

(i) $D((a,b)) = (R(a) + d_3(b), S(a) + d_2(b))$,

(ii) $R(aa') = R(a)a' + T^{[2n]}(S(a))a' + aR(a') + aT^{[2n]}(S(a'))$,

(iii) $R(T(b)a) = d_3(b)a + T^{[2n]}(d_2(b))a + T(b)R(a)$,

(iv) $R(aT(b)) = ad_3(b) + aT^{[2n]}(d_2(b)) + R(a)T(b)$,

(v) $S(T(b)a) = bS(a)$,

(vi) $S(aT(b)) = S(a)b$,

(vii) $S(aa') = 0$.

Moreover, $D = d_{\left(a^{(2n)}, b^{(2n)}\right)}$ *if and only if* $R = d_{a^{(2n)} + T^{[2n]}(b^{(2n)})}$, $S = 0$, $d_2 = d_{b^{(2n)}}$ *and* $d_3 = d_{a^{(2n)}}$.

Now we apply these results to study the n-weak amenability of $A \times_T B$. Our starting point is the following proposition which gives a necessary condition for the $(2n+1)$-weak amenability of $A \times_T B$.

PROPOSITION 3.4. *Let* $M = A \times_T B$ *be* $(2n+1)$*-weakly amenable for some* $n \geq 0$. *Then* A *and* B *are* $(2n+1)$*-weakly amenable.*

Proof. Suppose that $d_1 \in Z^1(A, A^{(2n+1)})$ and $d_2 \in Z^1(B, B^{(2n+1)})$. By Theorem 3.1 we conclude that the map $D: A \times_T B \to A^{(2n+1)} \times B^{(2n+1)}$ defined by

$$D((a,b)) = \left(d_1(a) + d_3(b), S(a) + d_2(b)\right) \ \left((a,b) \in A \times_T B\right),$$

is a derivation, where $d_3 = d_1 \circ T$ and $S = T^{[2n+1]} \circ d_1$. Therefore, there exists $\left(a^{(2n+1)}, b^{(2n+1)}\right) \in A^{(2n+1)} \times B^{(2n+1)}$ such that $D = d_{\left(a^{(2n+1)}, b^{(2n+1)}\right)}$. Hence, $d_1 = d_{a^{(2n+1)}}$ and $d_2 = d_{b^{(2n+1)}}$ by Corollary 3.2.□

In the following proposition, for $n \geq 0$ we denote by $X_{A^{(2n)}}$ the closed linear span of the set $\left(A \cdot A^{(2n)} \cup A^{(2n)} \cdot A\right)$ in $A^{(2n)}$. The following result, with a suitable condition, gives a sufficient condition for the $(2n+1)$-weak amenability of $A \times_T B$.

PROPOSITION 3.5. *Let* n *be a positive integer number such that* $X_{A^{(2n)}} = A^{(2n)}$. *If* A *and* B *are* $(2n+1)$*-weakly amenable, then* M *is* $(2n+1)$*-weakly amenable.*

Proof. Suppose that $D \in Z^1(M, M^{(2n+1)})$. Then there exist $d_1 \in Z^1(A, A^{(2n+1)})$, $d_2 \in Z^1(B, B^{(2n+1)})$, $d_3 \in Z^1(B, A^{(2n+1)})$ and a bounded linear map $S: A \to B^{(2n+1)}$ such that

$$D((a,b)) = (d_1(a) + d_3(b), S(a) + d_2(b)),$$

for all $(a,b) \in A \times B$. By assumption there exist $a^{(2n+1)} \in A^{(2n+1)}$ and $b^{(2n+1)} \in B^{(2n+1)}$ such that $d_1 = d_{a^{(2n+1)}}$ and $d_2 = d_{b^{(2n+1)}}$. Thus, by Theorem 3.1, we have $S(aa') = T^{[2n+1]}(d_1(aa'))$, for all $a, a' \in A$. Since A is $(2n+1)$-weak amenable, it follows that $\overline{A^2} = A$. Hence, we conclude that $S = T^{[2n+1]} \circ d_1 = \delta_{a^{(2n+1)}}$. On the other hand by Theorem 3.1 and Corollary 3.2, we have

$$d_{a^{(2n+1)}}(T(b)a) = d_3(b)a + T(b)d_{a^{(2n+1)}}(a),$$

for all $a \in A$ and $b \in B$. Therefore, $\left(d_3(b) - d_{a^{(2n+1)}}(T(b))\right)a = 0$ for all $a \in A$ and $b \in B$. By replacing d_3' with $(d_3 - d_{a^{(2n+1)}})$ we conclude that $d_3'(b)|_{\overline{A.A^{(2n)}}} = 0$ for all $b \in B$. Similarly we can show that $d_3'(b)|_{\overline{A^{(2n)}.A}} = 0$ for all $b \in B$. It follows from assumption that $d_3 = d_{a^{(2n+1)}}$. Hence, $D = d_{\left(a^{(2n+1)}, b^{(2n+1)}\right)}$ by Theorem 3.1.□

We recall from [4, Proposition 1.2] that if A is weakly amenable, then $\overline{A^2} = A$. Thus by Propositions 3.4 and 3.5 we have the following result.

COROLLARY 3.6. *A and B are weakly amenable if and only if M is weak amenable.*

Below we observe that *2n*-weak amenability of A is necessary for $2n$-weak amenability of M.

PROPOSITION 3.7. *Let $M = A\times_T B$ be $2n$-weakly amenable for some $n\geq 0$. Then A is $2n$-weakly amenable and all derivations $d_2\in Z^1(B, B^{(2n)})$ with $d_2(B)\subseteq \ker T^{[2n]}$ are inner.*

Proof. Suppose that $d_1 \in Z^1(A, A^{(2n+1)})$ and $d_2\in Z^1(B, B^{(2n)})$ such that $d_2(B)\subseteq \ker T^{[2n]}$. By Theorem 3.3, we conclude that the map $D: A\times_T B \to A^{(2n)}\times_{T[2n]} B^{(2n)}$ defined by

$$D((a,b)) = \left(d_1(a) + d_1(T(b)), d_2(b)\right) \ \left((a,b)\in A\times_T B\right),$$

is a derivation. Therefore, there exists $(a^{(2n)}, b^{(2n)})\in A^{(2n)}\times_{T[2n]} B^{(2n)}$ such that $D = d_{\left(a^{(2n)},b^{(2n)}\right)}$. It follows that $d_1 = d_{a^{(2n)}+T^{[2n]}(b^{(2n)})}$ and $d_2 = d_{b^{(2n)}}$.■

Let I be a closed ideal in A. Then it is trivial that the natural quotient map $q: A\to A/I$ is an algebra homomorphism with $\|q\|\leq 1$. In particular $\ker q = I$ and $\ker q^{[2n]}\cong I^{(2n)}$ for all $n\geq 0$.

COROLLARY 3.8. *Let I be a closed ideal in A and $n\geq 0$. Suppose that $A/I\times_q A$ is $2n$-weakly amenable. Then A/I is $2n$-weakly amenable and each derivation from A into $I^{(2n)}$ is inner.*

As two applications of Theorems 3.1 and 3.3, for the case that A is unital we have the following characterizations for elements in $Z^1(A\times_T B, (A\times_T B)^{(n)})$.

COROLLARY 3.9. *Let A be unital and $n\geq 0$. Then $D\in Z^1(M, M^{(2n+1)})$ if and only if*

$$D((a,b)) = \left(d_1(a) + d_3(b), S(a) + d_2(b)\right),$$

where $d_1\in Z^1(A, A^{(2n+1)})$, $d_2\in Z^1(B, B^{(2n+1)})$, $d_3\in Z^1(B, A^{(2n+1)})$ and $S: A\to B^{(2n+1)}$ is a bounded linear map such that $d_3 = d_1\circ T$ and $S = T^{[2n+1]}\circ d_1$. Moreover, D is inner if and only if both d_1 and d_2 are inner.

COROLLARY 3.10. *Let A be unital and $n\geq 0$. Then $D\in Z^1(M, M^{(2n)})$ if and only if*

$$D((a,b)) = \left(d_1(a) + d_3(b), d_2(b)\right),$$

where $d_1\in Z^1(A, A^{(2n)})$, $d_2\in Z^1(B, B^{(2n)})$ and $d_3\in Z^1(B, A^{(2n)})$ such that $d_3 = d_1\circ T - T^{[2n]}\circ d_2$. Moreover, D is inner if and only if both d_1 and d_2 are inner.

As an immediate corollary we have the following result. The details are omitted.

PROPOSITION 3.11. *Let A be unital and $n\geq 0$. Then M is n-weakly amenable if and only if A and B are n-weakly amenable.*

We end this work with the following example.

EXAMPLE 3.12. (*i*) It is well known that for any locally compact group G, the group algebra $L^1(G)$ is n-weakly amenable for all $n\in N$. Now, suppose that G_1 and G_2 are two locally compact groups and let

$T: L^1(G_1) \to L^1(G_2)$ be an algebra homomorphism with $\|T\| \le 1$. Then $L^1(G_1) \times_T L^1(G_2)$ is weakly amenable by Corollary 3.6.

(*ii*) Let $T: M(G_1) \to L^1(G_2)$ be an algebra homomorphism with $\|T\| \le 1$. Then [5, Theorem 1.2] and Proposition 3.11 implies that G_1 is discrete if and only if $M(G_1) \times_T L^1(G_2)$ is n-weakly amenable for all $n \in N$.

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