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A PERTURBATIVE ANALYSIS OF NONLINEAR CUBIC-QUINTIC DUFFING OSCILLATORS

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Duffing oscillators comprise one of the canonical examples of Hamilton systems. The presence of a quintic term makes the cubic-quintic Duffing oscillator more complex and interesting to study. In this paper, the homotopy analysis method (HAM) is used to obtain the analytical solution for the nonlinear cubic-quintic Duffing oscillators. The HAM helps to obtain the frequency ω in the form of approximation series of a convergence control parameter $\hbar$. The valid region of $\hbar$ is determined by plotting the ω - $\hbar$ curve and afterwards we compared the obtained results with the exact solutions.

Key words: homotopy analysis method, homotopy-Padé technique, oscillator.

1. INTRODUCTION

Duffing equation is used to model the conservative double-well oscillators, which can occur, for example, in magneto-elastic mechanical systems [1]. These systems can be presented in the following form

$$\begin{cases} \dfrac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \alpha u(t) + \beta u^3(t) + \gamma u^5(t) = 0, \\ u(0) = B, \qquad u'(0) = 0, \end{cases} \tag{1}$$

where B is the amplitude of the oscillator. Under the initial conditions mentioned above the nonlinear cubic-quintic Duffing oscillator has the exact frequency [2]:

$$\begin{aligned} \omega_\varepsilon &= \frac{\pi\sqrt{\alpha + \beta \dfrac{B^2}{2} + \gamma \dfrac{B^4}{4}}}{2\displaystyle\int_0^{\frac{\pi}{2}} (1 + M\ \sin^2(t) + N\ \sin^4(t))^{\frac{-1}{2}}\, \mathrm{d}t}, \\ M &= \frac{3\beta B^2 + 2\gamma B^4}{6\alpha + 3\beta B^2 + 2\gamma B^4}, \\ N &= \frac{2\gamma B^4}{6\alpha + 3\beta B^2 + 2\gamma B^4}. \end{aligned} \tag{2}$$

Eq. (1) describes an oscillator with an unknown frequency ω. Under the transformations

$$\tau = \omega t, \quad u(t) = BU(\tau), \tag{3}$$

the original Eq. (1) becomes

$$\begin{cases}\omega^2 \dfrac{\mathrm{d}^2U}{\mathrm{d}\tau^2} + \alpha U(\tau) + B^2\beta U^3(\tau) + B^4\gamma U^5(\tau) = 0,\\ U(0) = 1, \quad U'(0) = 0.\end{cases} \tag{4}$$

In the present paper, we use the homotopy analysis method (HAM) to obtain the periodic solutions of the nonlinear cubic-quintic Duffing oscillators. It is to be noted that Liao [3] employed the basic ideas of the homotopy to overcome the restrictions of traditional techniques [4–18] namely the HAM. Notice that the HAM contains an auxiliary parameter $\hbar$ which provides a convenient way to control the convergence region and the rate of approximation series. Liao investigated the influence of $\hbar$ on the convergence of solution series by means of $\hbar$-curves [3].

2. APPLICATION OF HOMOTOPY ANALYSIS METHOD

The periodic solution of $U(\tau)$ with the frequency ω can be written as

$$U(\tau) = \sum_{m=0}^{\infty} c_m \cos(m\tau), \tag{5}$$

where c_m are coefficients. It is convenient to choose

$$U_0(\tau) = \cos\tau, \tag{6}$$

as the initial guess of $U(\tau)$. Let ω_0 denotes the initial guess of ω, then we choose the auxiliary linear operator

$$L\left[\varphi(\tau;q)\right] = \omega_0^2\left[\frac{\mathrm{d}^2\varphi(\tau;q)}{\mathrm{d}\tau^2} + \varphi(\tau;q)\right], \tag{7}$$

with the property

$$L\left[C_1\cos\tau + C_2\sin\tau\right] = 0, \tag{8}$$

where C_1 and C_2 are coefficients. We define a nonlinear operator

$$N\left[\varphi(\tau;q),\Omega(q)\right] = \Omega^2(q)\frac{\mathrm{d}^2\varphi(\tau;q)}{\mathrm{d}\tau^2} + \alpha\varphi(\tau;q) + B^2\beta\varphi^3(\tau;q) + B^4\gamma\varphi^5(\tau;q). \tag{9}$$

Let $\hbar$ denotes a nonzero auxiliary parameter and $H(\tau)$ a nonzero auxiliary function. Then, we construct the zero-order deformation equation

$$(1-q)L[\varphi(\tau;q)] = q\hbar H(\tau)N[\varphi(\tau;q),\Omega(q)], \quad q\in[0,1], \tag{10}$$

such that

$$\varphi(0,q) = 1, \qquad \frac{\partial\varphi(\tau;q)}{\partial\tau}\Big|_{\tau=0} = 0. \tag{11}$$

When $q\in[0,1]$, the solution $\varphi(\tau;q)$ varies from $U_0(\tau)$ to $U(\tau)$ so does $\Omega(q)$ from ω_0 to ω. The Taylor's series with respect to q can be constructed for $\varphi(\tau;q)$ and $\Omega(q)$ and if these two series are convergent at $q=1$, we have:

$$U(\tau) = U_0(\tau) + \sum_{j=0}^{\infty} U_j(\tau), \quad \omega = \omega_0 + \sum_{j=0}^{\infty}\omega_j, \tag{12}$$

where

$$U_m(\tau) = \frac{\partial^m\varphi(\tau;q)}{m!\partial q^m}\Big|_{q=0}. \tag{13}$$

and

$$\omega_m = \frac{\partial^m\Omega(q)}{m!\partial q^m}\Big|_{q=0}. \tag{14}$$

Differentiating the zero-order deformation Eq. (10) and Eq. (11) m times with respect to embedding parameter q and then dividing them by $m!$ and finally setting $q=0$, we have the so-called mth order deformation equation

$$L\left[U_m(\tau)-\chi_m U_{m-1}(\tau)\right]=\hbar R_m(U_0,\omega_0,\ldots,U_{m-1},\omega_{m-1}), \tag{15}$$

$$U_m(0)=0, \qquad U'_m(0)=0, \quad m\geq 1, \tag{16}$$

where

$$R_m(U_0,\omega_0,\ldots,U_{m-1},\omega_{m-1})=\frac{\partial^{m-1}N\left[\varphi(\tau;q),\Omega(q)\right]}{(m-1)!\partial q^{m-1}}\Big|_{q=0}. \tag{17}$$

Note that U_m,ω_{m-1} are all unknown, but we have only Eq. (15) for U_m, thus an additional algebraic equation is required for determining ω_{m-1}. It is found that the right-hand side of the m^{th} order deformation Eq. (15) is expressed by

$$R_m(U_0,\omega_0,\ldots,U_{m-1},\omega_{m-1})=\sum_{k=0}^{\psi(m)} c_{m,k}(\omega_{m-1})\cos((2k+1)\tau), \tag{18}$$

where $c_{m,k}$ is a coefficient and $\psi(m)$ is a positive integer dependent on order m.

If $R_m(U_0,\omega_0,\ldots,U_{m-1},\omega_{m-1})$ contains the term $\cos\tau$ then the solution of Eq. (15) involves the so-called secular term $\tau\cos\tau$ that this disobeys the rules of solutions expression, thus coefficient $c_{m,0}$ must be enforced to be zero. This provides with the additional algebraic equation for determining ω_{m-1}

$$c_{m,0}(\omega_{m-1})=0. \tag{19}$$

Consequently, we obtain

$$U_m(\tau)=\chi_m U_{m-1}(\tau)+\frac{\hbar}{\omega_0^2}\sum_{j=1}^{\psi(m)}\frac{c_{m,j}(\omega_{m-1})}{1-(2j+1)^2}\cos\left[(2j+1)\tau\right]+C_1\cos\tau+C_2\sin\tau, \tag{20}$$

where C_1,C_2 are two coefficients and to be determined by conditions $U_m(0)=0$ and $U'_m(0)=0$.

Thus the N th order approximation can be given by

$$U(\tau)=U_0(\tau)+\sum_{j=0}^{N}U_j(\tau), \tag{21}$$

and

$$\omega=\omega_0+\sum_{j=0}^{N}\omega_j. \tag{22}$$

3. NUMERICAL RESULTS AND DISCUSSION

For given $\alpha=\beta=\gamma=1$ the amplitude ω_{m-1} can be determined by the analytic approach mentioned above. For $m=1,2$ we have:

$$\begin{cases}\omega_0=\dfrac{1}{\sqrt{10B^4+12B^2+16}},\\[2ex] \omega_1=\dfrac{1}{96}\dfrac{B^2\hbar(48+100B^2+65B^6+96B^4)}{\sqrt{10B^4+12B^2+16}(5B^4+6B^2+8)}.\end{cases} \tag{23}$$

Note. The obtained results contain the auxiliary parameter $\hbar$. It is found that convergence regions of the approximation series are dependent upon $\hbar$ [3]. For example, consider cases $B = 0.1, 0.15$. We plotted the $\omega - \hbar$ curve to determine the so-called valid region of $\hbar$, as shown in Fig. 1. Obviously, the valid regions of $\hbar$ for $B = 0.1, 0.15$ are $-4 < \hbar < 4$ and $-3.5 < \hbar < 3.5$, respectively, for instance for $B = 0.1$ we have the result $\omega = 1.003733$ as shown in Table 1.

Also, we consider cases $B = 1, 1.1$. We plotted the $\omega - \hbar$ curve to determine the so-called valid region of $\hbar$ as shown in Fig. 2. It is shown that the valid regions of $\hbar$ for $B = 1, 1.1$ are $-1 < \hbar < 0.9$ and $-0.9 < \hbar < 0.8$, respectively. Furthermore for $B = 1$, we have result $\omega = 1.538669$ as shown in Table 1. In Table 2, we compared the 14$^{\text{th}}$ order approximations of HAM with the exact solutions.

Table 1

Approximation of ω for $B = 0.1, 1$ and comparison with exact frequency ω_e

M	$B = 0.1(\omega_e = 1.003770)$	$B = 1(\omega_e = 1.523590)$
7	1.003753	1.538663
8	1.003728	1.538669
9	1.003733	1.538669
10	1.003732	1.538695
11	1.003733	1.538669
12	1.003733	1.538669
13	1.003733	1.538669
14	1.003733	1.538669

Table 2

Comparison of the 14$^{\text{th}}$-order approximations of HAM with the exact solutions

B	HAM	Exact	Error
0.3	1.035516	1.035540	2.32×10^{-5}
0.5	1.107092	1.106540	4.99×10^{-4}
3	7.636706	7.268630	5.06×10^{-2}
5	20.256660	19.181500	5.60×10^{-2}
8	51.078098	48.294600	5.75×10^{-2}
10	79.536112	75.177400	5.79×10^{-2}
20	316.703160	299.223000	5.84×10^{-2}
50	1976.897968	1867.570000	5.85×10^{-2}

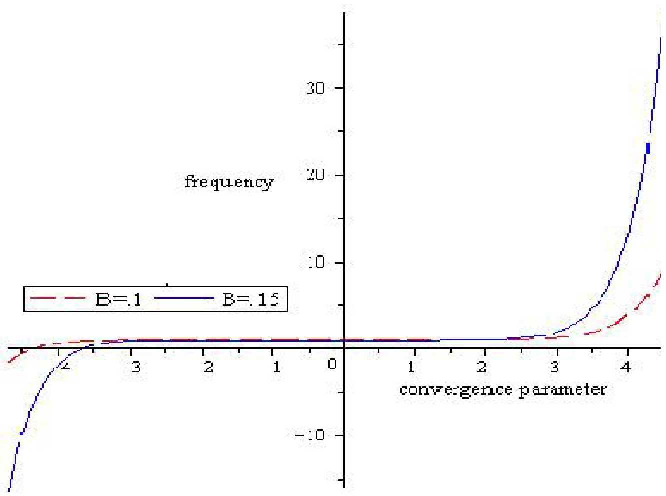

Fig. 1 – The $\omega - \hbar$ curve for $B = 0.1, 0.15$.

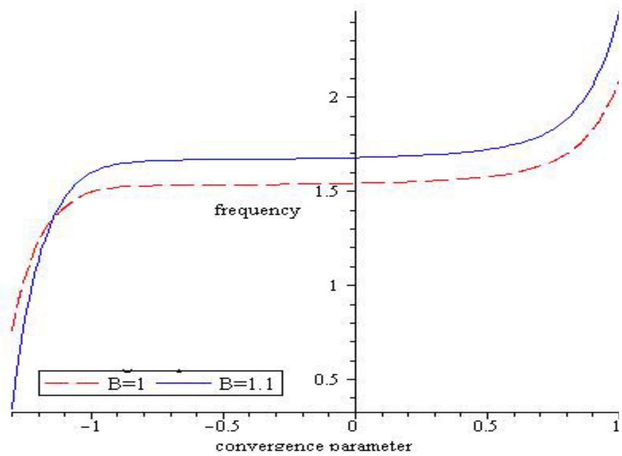

Fig. 2 – The $\omega - \hbar$ curve for $B = 1, 1.1$.

3.1. Square residual error

We obtained the constant $\hbar$ using the least square method. In theory, at the M th-order of approximation, we can define the exact square residual error

$$\Delta_M(\hbar) = \int_D \left(N\left(\sum_{k=0}^{M} U_k(\tau), \sum_{k=0}^{M} \omega_k\right)\right)^2 d\tau. \tag{24}$$

Clearly, the more rapidly $\Delta_M(\hbar)$ decreases to zero, the faster the approximation series converges.

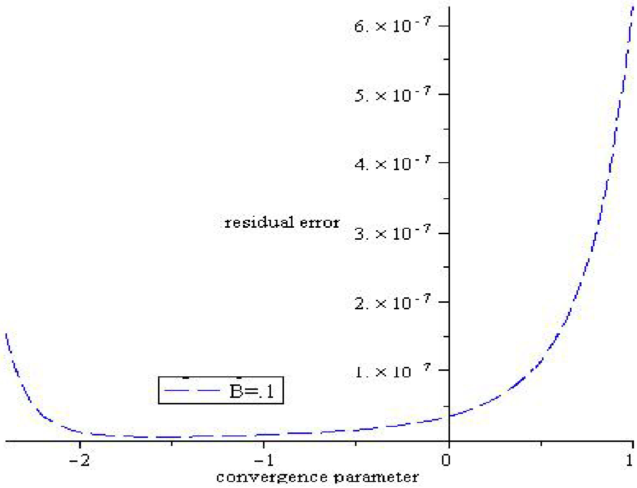

Fig. 3 – The $\Delta_{10}(\hbar)$ - $\hbar$ curve for $B = 0.1$.

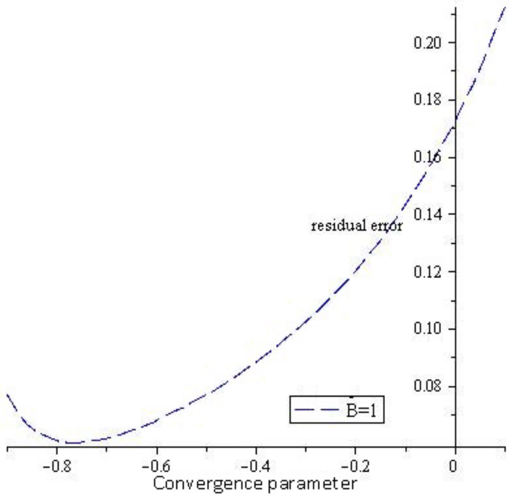

Fig. 4 – The $\Delta_{10}(\hbar) - \hbar$ curve for $B = 1$.

Remark 1. The curve of $\Delta_{10}(\hbar)$ *versus* $\hbar$ at $B = 0.1$ is shown in Fig. 3, which indicates that the optimal values of $\hbar$ is about -1.5.

Remark 2.The curve of $\Delta_{10}(\hbar)$ *versus* $\hbar$ at $B = 1$ is shown in Fig. 4, which indicates that the optimal values of $\hbar$ is about -0.78.

4. HOMOTOPY-PADÉ TECHNIQUE

The Padé technique is widely applied to enlarge the convergence region and convergence rate of given series. The so-called homotopy-Padé technique was suggested by means of combining the Padé technique with HAM.

For a given series

$$S_k(q) = \sum_{j=0}^{k} \omega_j q^j, \tag{25}$$

the corresponding $[n, m]$ Padé approximate is expressed by

$$S_{n+m}(q) = \frac{C_{n,m}(q)}{D_{n,m}(q)} = \frac{\sum_{j=0}^{n} c_j q^j}{1 + \sum_{j=1}^{m} d_j q^j}, \tag{26}$$

where c_j, d_j can be determined by solving the linear system:

$$\begin{cases} \omega_i + \sum_{j=0}^{i-1} \omega_j d_{i-j} = c_i, \ i = 0,1,2,...,n, \\ \omega_i + \sum_{j=i-m}^{i-1} \omega_j d_{i-j} = 0, \quad i = n+1,...,n+m. \end{cases} \tag{27}$$

Setting $q = 1$ provides the $[n,m]$ homotopy-Padé approximation

$$\omega^{[n,m]} = S_{n+m}(1) = \sum_{j=0}^{n+m} \omega_j = \frac{C_{n,m}(1)}{D_{n,m}(1)} = \frac{\sum_{j=0}^{n} c_j}{1 + \sum_{j=1}^{m} d_j}, \tag{28}$$

which accelerate the convergence rate of solution series of HAM. We have applied the homotopy-Padé technique to accelerate the convergence rate of M th-order approximations of HAM. In Table 3 we compared the approximations of homotopy-Padé technique with exact solutions.

Table 3

Comparison the exact solution with approximation of homotopy-Padé technique

	$B = 0.1$	$B = 0.3$	$B = 1$	$B = 3$
$\omega^{[1,1]}/\omega_e$	1.00037	1.00482	1.01757	1.01842
$\omega^{[2,2]}/\omega_e$	1.00015	1.00326	1.01317	1.01674
$\omega^{[3,3]}/\omega_e$	1.00011	1.00232	1.01219	1.01655
$\omega^{[4,4]}/\omega_e$	1.00013	1.00236	1.01087	1.01638
$\omega^{[5,5]}/\omega_e$	1.00010	1.00219	1.01058	1.01550
$\omega^{[6,6]}/\omega_e$	1.00010	1.00168	1.00876	1.01537
$\omega^{[7,7]}/\omega_e$	1.00010	1.00143	1.00781	1.01531

5. CONCLUSIONS

In this paper, the HAM is presented to calculate the frequency and the solution of the nonlinear cubic-quintic Duffing oscillators. According to obtained results, the HAM and homotopy-Padé technique could give efficient frequency approximations for the nonlinear cubic-quintic Duffing oscillators. It is worth mentioning that nonlinear cubic-quintic oscillator models arise in many areas of nonlinear science, e.g., in the study of optical solitons [19]-[20]; thus in nonlinear optics the cubic-quintic Ginzburg-Landau partial differential equation is a generic nonlinear dynamical model describing optical soliton propagation in laser cavities, see, for example, Ref. [21].

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