THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

*PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,*
*Volume 15, Number 3/2014, pp. 235–240*

# OPTICAL SOLITONS WITH POLYNOMIAL AND TRIPLE-POWER LAW NONLINEARITIES AND SPATIO-TEMPORAL DISPERSION

A. H. BHRAWY [1,2], A. A. ALSHAERY [3], E. M. HILAL [3], Daniela MILOVIC [4], Luminita MORARU [5], Michelle SAVESCU [6,7], Anjan BISWAS [6,1]

[1] King Abdulaziz University, Faculty of Science, Department of Mathematics, Jeddah, Saudi Arabia
[2] Beni-Suef University, Faculty of Science, Department of Mathematics, Beni-Suef, Egypt
[3] King Abdulaziz University, Faculty of Science for Girls, Department of Mathematics, Jeddah, Saudi Arabia
[4] University of Nis, Faculty of Electronic Engineering Department of Telecommunications, Aleksandra Medvedeva 14, 18000 Nis, Serbia
[5] "Dunarea de Jos" University Galați, Physics and Environment, Department of Chemistry, 111 Domneasca Street, 800201 Galati, Romania
[6] Delaware State University, Department of Mathematical Sciences, Dover, DE 19901-2277, USA
[7] Kuztown University of Pennsylvania, Department of Mathematics, 15200 Kutztown Road, Kuztown, PA-19530, USA
E-mail: biswas.anjan@gmail.com

This paper obtains exact traveling wave solutions to the nonlinear Schrödinger's equation that is considered with spatio-temporal dispersion in addition to group-velocity dispersion. The two types of nonlinearity associated with this equation are polynomial and triple-power law. The obtained optical soliton solutions are in terms of elliptic function of third kind.

*Key words*: spatio-temporal dispersion, integrability, optical solitons.

## 1. INTRODUCTION

The theory of optical solitons is one of the most progressive areas of research in applied and theoretical physics [1-32]. This research has lead to several overwhelming and interesting results. The most commonly studied aspects of optical solitons are conservation laws, birefringent fibers, dense wavelength-division multiplexing (DWDM) systems, polarization mode dispersion, Thirring solitons, cascaded systems, timing, frequency and amplitude jitter, four-wave mixing, and several other aspects. The governing equation is the nonlinear Schrödinger's equation (NLSE) that contains the nonlinear term and the group velocity dispersion (GVD), besides the linear evolution term. It is the delicate balance between GVD and nonlinearity that results in the formation of optical solitons. These temporal optical solitons propagate along trans-continental and trans-oceanic distances along the optical fibers.

It was very recently pointed out that NLSE with only GVD and nonlinearity is an ill-posed model [4, 7]. The inclusion of spatio-temporal dispersion (STD) was suggested in 2012 in order to make NLSE a well posed model [7]. This paper will therefore address NLSE with GVD and STD terms included. The types of nonlinearity that will be studied in this paper are polynomial law as well as triple-power law. It must be noted that the cases of Kerr law, power law, parabolic law, dual-power law as well as the log law nonlinearity are already studied with the so called "improved" NLSE [9]. Also, the cases of polynomial law and triple-power law nonlinearity of NLSE with just GVD, without STD, was studied in 2009 [2]. Additionally in Ref. [2] this problem was studied with only GVD, and the GVD coefficient was a particular constant, namely 1/2. However, this paper will study with GVD and STD, both with arbitrary real-valued coefficients. Therefore, the results of this paper will match the results obtained earlier for special choice of the parameters and the coefficients of the evolution equation. The *traveling wave hypothesis* will be the integration tool that will be adopted here.

## 2. DYNAMICAL MODEL

In non-Kerr law media, the dynamics of soliton propagation for long distances is governed by dimensionless NLSE with GVD and STD as follows

$$iq_t + aq_{xx} + bq_{xt} + F(|q|^2)\ q = 0, \tag{1}$$

where $q(x,t)$ represents the complex-valued wave profile. Here $x$ and $t$ represent spatial and temporal variables, respectively. The coefficients $a$ and $b$ are GVD and STD, respectively. The first term is the linear evolution of the pulses in nonlinear optical fibers. The last term represents non-Kerr law nonlinearity. Here, $F(|q|^2)q : C \to C$. If the complex plane $C$ is a two-dimensional linear space $R^2$, $F(|q|^2)q$ is $k$ times continuously differentiable, then

$$F(|q|^2)q \in \bigcup_{m,n-1}^{\infty} C^k\left((-n,\ n)\times(-m,\ m); R^2\right).$$

The current trend is to study the dynamics of solitons with GVD and nonlinearity. The STD term is newly proposed so that equation (1) is a well-posed problem [7]. Solitons are the outcome of a delicate balance between nonlinearity and dispersion, in this case between GVD and STD. In this paper we will integrate equation (1) by traveling wave hypothesis for two forms of nonlinear optical media. They are polynomial law nonlinearity and triple-power law nonlinearity.

In order to integrate (1), the traveling wave hypothesis given by [2, 9]

$$q(x,t) = g(s)\mathrm{e}^{\mathrm{i}\phi(x,\ t)} \tag{2}$$

is selected. Here

$$s = x - vt, \tag{3}$$

where $v$ is the speed of the soliton and the phase is

$$\phi(x,\ t) = -\kappa x + \omega t + \theta. \tag{4}$$

The frequency is denoted by κ while the wave number of the soliton is denoted by ω. Substituting (2) into (1) and decomposing into real and imaginary parts leads to

$$v = \frac{\omega b - 2a\kappa}{1 - b\kappa} \tag{5}$$

$$(a - bv)g'' = (\omega + a\kappa^2 - b\omega\kappa)g - F(g^2)g \tag{6}$$

respectively. Here the notations $g' = \mathrm{d}g/\mathrm{d}s$ and $g'' = \mathrm{d}^2g/\mathrm{d}s^2$ are introduced. Thus, the imaginary part leads to the speed of the soliton irrespective of the type of nonlinearity in question. Multiplying both sides of (6) by $g'$ and integrating leads to

$$(a - bv)(g')^2 = (\omega + a\kappa^2 - b\omega\kappa)g^2 - 2\int_g F(h^2)hh'\mathrm{d}h \tag{7}$$

after choosing integration constant to be zero, since the search is for a localized soliton solution. Next, separating variables and integrating leads to

$$x - vt = \int \frac{\sqrt{a - bv}\mathrm{d}g}{\left[(\omega + a\kappa^2 - b\omega\kappa)g^2 - 2\int_g F(h^2)hh'\mathrm{d}h\right]^{1/2}}. \tag{8}$$

Equation (8) will now be separately analyzed for the two laws of nonlinearity in the next two subsections.

### 2.1. Polynomial law nonlinearity

Polynomial law nonlinearity is an extension of parabolic law nonlinearity. Therefore this type of nonlinear optical medium takes the form

$$F(s) = k_1 s + k_2 s^2 + k_3 s^3 \tag{9}$$

so that the NLSE (1) transforms to

$$iq_t + aq_{xx} + bq_{xt} + (k_1|q|^2 + k_2|q|^4 + k_3|q|^6)\, q = 0\,. \tag{10}$$

Equation (8) therefore reduces to

$$x - vt = \int \frac{2\sqrt{3(a-bv)}\, dg}{g\sqrt{12(\omega + a\kappa^2 - b\omega\kappa) - g^2(6k_1 + 4k_2 g^2 + 3k_3 g^4)}}\,. \tag{11}$$

Performing the integration in (11) gives the implicit solution

$$\begin{aligned} &\frac{x - vt}{2\sqrt{3(a-bv)}} g_3 \sqrt{12a\kappa^2 - 6g^2 k_1 - 4g^4 k_2 - 3g^6 k_3 + 12\omega(1-b\kappa)} = \\ &= -\prod\left(1 - \frac{g_2}{g_3};\ \sin^{-1}\sqrt{\frac{g^2 - g_3}{g_2 - g_3}},\ \frac{g_2 - g_3}{g_1 - g_3}\right)\sqrt{\frac{(g^2 - g_1)(g^2 - g_2)(g^2 - g_3)}{g_1 - g_3}}, \end{aligned} \tag{12}$$

where Π is the incomplete elliptic integral of third kind that is defined as

$$\prod(n;\phi,\alpha) = \int_0^{\phi} \frac{\mathrm{d}\theta}{(a - n\sin^2\theta)\sqrt{1 - \sin^2\alpha\sin^2\theta}}. \tag{13}$$

Here

$$g_1 = \frac{1}{9k_3}\left[4k_2 - \frac{2(8k_2^2 - 27k_1k_3)}{(2R_1)^{1/3}} - (2R_1)^{1/3}\right], \tag{14}$$

$$g_2 = -\frac{1}{9k_3}\left[4k_2 + (8k_2^2 - 27k_1k_3)\left(-\frac{4}{R_1}\right)^{1/3} + \left(-\frac{R_1}{4}\right)^{1/3}\left(1 - \mathrm{i}\sqrt{3}\right)\right], \tag{15}$$

$$g_3 = -\frac{1}{9k_3}\left[4k_2 - (8k_2^2 - 27k_1k_3)\left(-\frac{4}{R_1}\right)^{1/3} + \left(-\frac{R_1}{4}\right)^{1/3}\left(1 + \mathrm{i}\sqrt{3}\right)\right], \tag{16}$$

and

$$R_1 = r_1 + \frac{1}{2}\left[\left(-16k_2^2 + 54k_1k_3\right)^3 + 4r_1\right]^{1/2}, \tag{17}$$

with

$$r_1 = -32k_2^3 + 162k_1k_2k_3 + 729k_3^2\left[a\kappa^2 + \omega(1-b\kappa)\right]. \tag{18}$$

Thus, relation (18) prompts the constraint condition

$$\left(-16k_2^2 + 54k_1k_3\right)^3 + 4r_1 > 0 \tag{19}$$

in order for the solution to exist.

### 2.2. Triple-power law nonlinearity

This is a generalization of polynomial law nonlinearity and an extension of dual-power law nonlinearity. In this case,

$$F(s)=k_4 s^n+k_5 s^{2n}+k_6 s^{3n} \tag{20}$$

so that the NLSE (1) transforms to

$$iq_t+aq_{xx}+bq_{xt}+(k_4|q|^{2n}+k_5|q|^{4n}+k_6|q|^{6n})\ \ q=0\,. \tag{21}$$

Equation (8) therefore reduces to

$$x-vt=\sqrt{(n+1)(2n+1)(3n+1)(a-bv)}\int\frac{\mathrm{d}g}{g\sqrt{G(g)}}\,. \tag{22}$$

Here

$$\begin{aligned}G(g)&=(n+1)(2n+1)(3n+1)(\omega+a\kappa^2-\omega b\kappa)-\\&-\left[(2n+1)(3n+1)k_4 g^{2n}+(n+1)(3n+1)k_5 g^{4n}+(n+1)(2n+1)k_6 g^{6n}\right].\end{aligned} \tag{23}$$

Performing the integration in (22) leads to the solution

$$\begin{aligned}&\frac{x-vt}{\sqrt{(n+1)(2n+1)(3n+1)(a-bv)}}g_3 n\sqrt{P}=\\&=-\prod\left(1-\frac{g_5}{g_6};\ \sin^{-1}\sqrt{\frac{g^{2n}-g_6}{g_5-g_6}},\ \frac{g_5-g_6}{g_4-g_6}\right)\sqrt{\frac{(g^{2n}-g_4)(g^{2n}-g_5)(g^{2n}-g_6)}{g_4-g_6}}.\end{aligned} \tag{24}$$

Here

$$\begin{aligned}P&=(n+1)(2n+1)(3n+1)[a\kappa^2+\omega(1-b\kappa)]-\\&-\left[(2n+1)(3n+1)k_4 g^{2n}-[(n+1)(3n+1)]k_5 g^{4n}-(n+1)(2n+1)k_6 g^{6n}\right].\end{aligned} \tag{25}$$

Also,

$$\begin{aligned}g_4&=\frac{1}{3k_6(n+1)(2n+1)}\Big\{-k_5(n+1)(3n+1)+\\&+(n+1)(3n+1)\left(\frac{2}{R_2}\right)^{1/3}\left[k_5^2(n+1)(3n+1)-3k_4k_6(2n+1)^2\right]+\left(\frac{R_2}{2}\right)^{1/3}\Big\},\end{aligned} \tag{26}$$

$$\begin{aligned}g_5&=\frac{1}{3k_6(n+1)(2n+1)}\Big\{-k_5(n+1)(3n+1)-\\&-(n+1)(3n+1)(1+\mathrm{i}\sqrt{3})\left(\frac{1}{4R_2}\right)^{1/3}\left[k_5^2(n+1)(3n+1)-3k_4k_6(2n+1)^2\right]+(-1+\mathrm{i}\sqrt{3})\left(\frac{R_2}{16}\right)^{1/3}\Big\},\end{aligned} \tag{27}$$

$$\begin{aligned}g_6&=\frac{1}{3k_6(n+1)(2n+1)}\Big\{k_5(n+1)(3n+1)+\\&+(n+1)(3n+1)(1-\mathrm{i}\sqrt{3})\left(\frac{1}{4R_2}\right)^{1/3}\left[k_5^2(n+1)(3n+1)-3k_4k_6(2n+1)^2\right]+(1+\mathrm{i}\sqrt{3})\left(\frac{R_2}{16}\right)^{1/3}\Big\},\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
R_2 = {} & -2k_5^3(n+1)^3(3n+1)^3 + 9k_4k_5k_6(1+6n+11n^2+6n^3)^2 + \\
& +27k_6^2(n+1)^3(2n+1)^3(3n+1)[a\kappa^2+\omega(1-b\kappa)] + \\
& +[(n+1)^3(3n+1)^3[(n+1)\{27k_6^2(n+1)(2n+1)^3[a\kappa^2+\omega(1-b\kappa)]+9k_4k_5k_6(2n+1)^2(3n+1) - \\
& -2k_5^3(n+1)(3n+1)^2]^2\} + 4(3n+1)\{3k_4k_6(2n+1)^2 - k_5^2(n+1)^3(3n+1)^3]^{1/2}.
\end{aligned}
\tag{29}
$$

The relation for $R_2$ given by (29) prompts the constraint condition

$$
\begin{aligned}
& (n+1)^4(3n+1)[27k_6^2(n+1)(2n+1)^3[a\kappa^2+\omega(1-b\kappa)]+9k_4k_5k_6(2n+1)^2(3n+1) - \\
& -2k_5^3(n+1)(3n+1)^2]^2 + 4\{3k_4k_6(2n+1)^2 - k_5^2(n+1)^3(3n+1)^3\} > 0,
\end{aligned}
\tag{30}
$$

that must remain valid in order for the soliton solution to exist.

## 3. CONCLUSIONS

This paper studied optical solitons with polynomial and triple-power law nonlinearity. The traveling wave hypothesis revealed exact solutions that are in terms of incomplete elliptic functions of third kind. These exact solutions, which are rather complicated ones, are however important in nonlinear optics studies. One can easily employ these solutions to address several important aspects of optical solitons that were enumerated in introduction. The inclusion of perturbation terms such as inter-modal dispersion, multi-photon absorption, and several others will give an additional flavor to this research area. Those results, along with several other interesting aspects will be reported elsewhere.

## ACKNOWLEDGMENTS

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under grant No. (14-130/1433 HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

## REFERENCES

1. A. BISWAS, S. KONAR, *Introduction to non-Kerr law optical solitons,* CRC Press Boca Raton, FL., USA, 2006.
2. A. BISWAS, D. MILOVIC, *Traveling wave solutions of the nonlinear Schrödinger's equation in non-Kerr law media*, Communications in Nonlinear Science and Numerical Simulation, **15**, pp. 1993–1998, 2009.
3. A. S. FOKAS, *On a class of physically important integral equations*, Physica D, **87**, pp. 145–150, 1995.
4. X. GENG, Y. LV, *Darboux transformation for an integrable generalization of the nonlinear Schrödinger equation*, Nonlinear Dynamics, **69**, pp. 1621–1630, 2012.
5. H. LEBLOND, H. TRIKI, D. MIHALACHE, *Theoretical studies of ultrashort-soliton propagation in nonlinear optical media from a general quantum model,* Romanian Reports in Physics, **65**, pp. 925–942, 2013.
6. H. LEBLOND, D. MIHALACHE, *Models of few optical cycle solitons beyond the slowly varying envelope approximation*, Physics Reports, **523**, pp. 61–126, 2013.
7. S. KUMAR, K. SINGH, R. K. GUPTA, *Coupled Higgs field equation and Hamiltonian amplitude equation: Lie classical approach and G'/G- expansion method*, Pramana, **79**, pp. 41–60, 2012.
8. L. LENELLS, A. S. FOKAS, *Dressing for a novel integrable generalization of the nonlinear Schrödinger's equation*, J. Nonlinear Sci., **20**, pp. 709–722, 2010.
9. M. SAVESCU, K. R. KHAN, P. NARUKA, H. JAFARI, L. MORARU, A. BISWAS, *Optical solitons in photonic nanowaveguides with an improved nonlinear Schrödinger's equation*, Journal of Computational and Theoretical Nanoscience, **10**, pp. 1182–1191, 2013.
10. M. WADATI, H. SEGUR, M. J. ABLOWITZ, *A new Hamiltonian amplitude equation governing modulated wave instabilities*, Journal of the Physical Society of Japan, **61**, pp. 1187–1193, 1992.
11. H. TRIKI, S.CRUTCHER, A. YILDIRIM, T. HAYAT, O. M. ALDOSSARY, A. BISWAS, *Bright and dark solitons of the modified complex Ginzburg-Landau equation with parabolic and dual-power law nonlinearity*, Romanian Reports in Physics, **64**, pp. 367–380, 2012.
12. G. EBADI, A.YILDIRIM, A. BISWAS, *Chiral solitons with Bohm potential using G'/G method and exp-function method*, Romanian Reports in Physics, **64**, pp. 357–366, 2012.

13. L. GIRGIS, D. MILOVIC, S. KONAR, A. YILDIRIM, H. JAFARI, A. BISWAS, *Optical gaussons in birefringent fibers and DWDM systems with inter-modal dispersion*, Romanian Reports in Physics, **64**, pp. 663–671, 2012.
14. H. TRIKI, A. YILDIRIM, T. HAYAT, O. M. ALDOSSARY, A. BISWAS, *Topological and non-topological solitons of a generalized derivative nonlinear Schrödinger's equation with perturbation terms*, Romanian Reports in Physics, **64**, pp. 672–684, 2012.
15. A. H. BHRAWY, M. A. ABDELKAWY, S. KUMAR, A. BISWAS, *Solitons and other solutions to Kadomtsev-Petviashvili equation of B-type*, Romanian Journal of Physics, **58**, pp. 729–748, 2013.
16. B. S. AHMED, A. BISWAS, E. V. KRISHNAN, S. KUMAR, *Solitons and other solutions to the generalized Maccari system*, Romanian Reports in Physics, **65**, pp. 1138–1154, 2013.
17. A. G. JOHNPILLAI, A.YILDIRIM, A. BISWAS, *Chiral solitons with Bohm potential by Lie group analysis and traveling wave hypothesis*, Romanian Journal of Physics, **57**, pp. 545–554, 2012.
18. H. TRIKI, A. YILDIRIM, T. HAYAT, O. M. ALDOSSARY, A. BISWAS, *Shock wave solution of the Benney-Luke equation*, Romanian Journal of Physics, **57**, pp. 1029–1034, 2012.
19. G. EBADI, N. YOUSEFZADEH, H. TRIKI, A. YILDIRIM, A. BISWAS, *Envelope solitons, periodic waves and other solutions to Boussinesq-Burgers equation*, Romanian Reports in Physics, **64**, pp. 915–932, 2012.
20. G. EBADI, A. MOJAVER, H. TRIKI, A. YILDIRIM, A. BISWAS, *Topological solitons and other solutions of the Rosenau-KdV equation with power law nonlinearity*, Romanian Journal of Physics, **58**, pp. 3–14, 2013.
21. A. H. KARA, H. TRIKI, A. BISWAS, *Conservation laws of the Bretherton equation*, Applied Mathematics and Information Sciences, **7**, pp. 877–879, 2013.
22. A. BISWAS, S. KUMAR, E. V. KRISHNAN, B. AHMED, A. STRONG, S. JOHNSON, A. YILDIRIM, *Topological solitons and other solutions to potential KdV equation,* Romanian Reports in Physics, **65**, pp. 1125–1137, 2013.
23. S. CRUTCHER, A. OSEO, A. YILDIRIM, A. BISWAS, *Oscillatory parabolic law optical spatial solitons*, Journal of Optoelectronics and Advanced Materials, **14**, pp. 29–40, 2012.
24. A. BISWAS, K. KHAN, A. RAHMAN, A.YILDIRIM, T. HAYAT, O. M. ALDOSSARY, *Bright and dark optical solitons in birefringent fibers with Hamiltonian perturbations and Kerr law nonlinearity*, Journal of Optoelectronics and Advanced Materials, **14**, pp. 571–576, 2012.
25. L. CHERBI, A. BELLALIA, M. UTHMAN, A. BISWAS, *Modeling of two rings-photonic crystal fibers with the scalar finite element method*, Journal of Optoelectronics and Advanced Materials, **15**, pp. 1385–1391, 2013.
26. B. AHMED, A. BISWAS, *Solitons, kinks and singular solitons of coupled Korteweg-de Vries equation*, Proc. Romanian Acad. A, **14**, pp. 111–120, 2013.
27. A. BISWAS, A. YILDIRIM, O. M. ALDOSSARY, R. SASSAMAN, *Soliton perturbation theory for the generalized Klein Gordon equation with full nonlinearity*, Proc. Romanian Acad. A, **13**, pp. 32–41, 2012.
28. G. EBADI, A. H. KARA, M. D. PETKOVIC, A. YILDIRIM, A. BISWAS, *Solitons and conserved quantities of the Ito equation*, Proc. Romanian Acad. A, **13**, pp. 215–224, 2012.
29. D. MIHALACHE, *Linear and nonlinear light bullets: Recent theoretical and experimental studies*, Romanian Journal of Physics, **57**, pp. 352–371, 2012.
30. G. YANG et al., *Control of high power pulses extracted from the maximally compressed pulse in a nonlinear optical fiber*, Romanian Reports in Physics, **65**, pp. 902–914, 2013.
31. G. YANG *et al*., *High power pulses extracted from the Peregrine rogue wave*, Romanian Reports in Physics, **65**, pp. 391–400, 2013.
32. A. M. WAZWAZ, *Multiple soliton solution for two integrable couplings of the modified Korteweg-de Vries equation*, Proc. Romanian Acad. A, **14**, pp. 219–225, 2013.

*Received December 16, 2013*